Appendix

MML estimation of the location and scale parameters of GL distribution under RSS

First, likelihood Eqs. in (8) and (9) are rewritten in terms of standardized ordered statistics as given in Eqs. (22) and (23)

$$\frac{\partial \ln L}{\partial \mu_x} = -\frac{1}{\sigma_x}\left(\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(i-1)g_1\left(z_{(i)c}\right) - \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)g_2\left(z_{(i)c}\right) + \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}g_3\left(z_{(i)c}\right)\right) = 0 \tag{22}$$

and

$$\frac{\partial lnL}{\partial \sigma_x} = -\frac{1}{\sigma_x}\left(n_x + \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(i-1)z_{(i)c}g_1\left(z_{(i)c}\right) - \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}(m_x-i)z_{(i)c}g_2\left(z_{(i)c}\right) + \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}z_{(i)c}g_3\left(z_{(i)c}\right)\right) = 0 \tag{23}$$

where, $z_{(i)c} = \dfrac{x_{(i)c}-\mu_x}{\sigma_x}$, $g_1\left(z_{(i)c}\right) = \dfrac{f\left(z_{(i)c}\right)}{F\left(z_{(i)c}\right)}$, $g_2\left(z_{(i)c}\right) = \dfrac{f\left(z_{(i)c}\right)}{1-F\left(z_{(i)c}\right)}$ and $g_3\left(z_{(i)c}\right) = \dfrac{f'\left(z_{(i)c}\right)}{f\left(z_{(i)c}\right)}$. Here, it should also be noted that the complete sums are invariant to ordering, i.e. $\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}z_{ic} = \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}z_{(i)c}$.

Secondly, these functions are linearized around the expected values of standardized ordered statistics, i.e., $t_{(i)c} = E\left(z_{(i)c}\right)$, by using the first two terms of the Taylor series expansion as given in Eq. (24)

$$g_1\left(z_{(i)c}\right) \cong \alpha_{1i} - \beta_{1i}z_{(i)c}, g_2\left(z_{(i)c}\right) \cong \alpha_{2i} + \beta_{2i}z_{(i)c} \quad \textit{and} \quad g_3\left(z_{(i)c}\right) \cong \alpha_{3i} - \beta_{3i}z_{(i)c} \tag{24}$$

where $\alpha_{1i} = \dfrac{f\left(t_{(i)c}\right)}{F\left(t_{(i)c}\right)} + \beta_{1i}t_{(i)c}$, $\beta_{1i} = -\left(\dfrac{f'\left(t_{(i)c}\right)}{F\left(t_{(i)c}\right)} - \dfrac{\left(f\left(t_{(i)c}\right)\right)^2}{\left(F\left(t_{(i)c}\right)\right)^2}\right)$, $\alpha_{2i} = \dfrac{f\left(t_{(i)c}\right)}{1-F\left(t_{(i)c}\right)} - \beta_{2i}t_{(i)c}$, $\beta_{2i} = \left(\dfrac{f'\left(t_{(i)c}\right)}{1-F\left(t_{(i)c}\right)} + \dfrac{\left(f\left(t_{(i)c}\right)\right)^2}{\left(1-F\left(t_{(i)c}\right)\right)^2}\right)$,

$\alpha_{3i} = \dfrac{f'\left(t_{(i)c}\right)}{f\left(t_{(i)c}\right)} + \beta_{3i}t_{(i)c}$ and $\beta_{3i} = -\left(\dfrac{f''\left(t_{(i)c}\right)f\left(t_{(i)c}\right) - \left(f'\left(t_{(i)c}\right)\right)^2}{\left(f\left(t_{(i)c}\right)\right)^2}\right)$.

Noting that z_{ic} in (8) and (9) are inid, expected values of ordered $z_{(i)c}$'s are calculated via Monte Carlo simulation to avoid algebraic complexity. Incorporation of the linearizations in (24) into (22) and (23) produces the modified likelihood equations, whose simultaneous solution yields the MML estimators of the parameters μ_x and σ_x in (14), see Subsection 3.1.