**Appendix A. Proof of Theorem 1**

For computational convenience, define the following supporting functions from equation (10):

 (A1)

and  (A2)

Subsequently, the company's total cost becomes  .

Now the first two derivatives of  are :

 (A3)

and  (A4)

Since  includes all cost components of the company for a business cycle,  is always nonnegative. Furthermore, from equations (A3) and (A4),  is convex and differentiable in . In addition, the denominator  is straightforwardly positive, affine and differentiable in . Thus, the company's total cost  is pseudo-convex in  and therefore,  holds its global minimum value at . Additionally, the point  is unique. This finishes the proof.

**Appendix B. Proof of Theorem 2**

For computational convenience, define the following supporting functions from equation (23):

(B1)

and  (B2)

To ensure the joint convexity of , it necessary to build the Hessian matrix for  with respect to  and . Thereby, compute the necessary first and second-order partial derivatives of  as follows

 (B3)



 

 

  (B4)

 (B5)



  (B6)



  (B7)

Subsequently, the Hessian matrix for  is given by

  (B8)

The first principal minor is

 

 

As the per unit opportunity cost () is consistently more noteworthy than or equivalent to the unit acquisition cost (), therefore, . In addition, the second principal minor is

 





 [Using ]

Therefore, the Hessian matrix of  is positive definite. Since  includes all cost components of the company for a business cycle,  is always non-negative. Subsequently,  is a non-negative, convex and differentiable in  and . In addition, the denominator  is always positive, differentiable, and affine. As a result, the company's total cost  is jointly pseudo-convex in  and , and therefore,  holds the global minimum value at . Furthermore, the point  is unique. This finishes the proof. 

**Appendix C. Solution procedure for the case study with zero-end**

The best inventory planning for the Monihari Mango Shop under the inventory procedure with zero-end is computed as follow:

Initially set  and *i* = 3.

*Iteration 1:  for *

Exploiting the values of  and , compute  by solving equation (25). Utilizing this derived value of , determine the optimum requisition amount  boxes for the company from equation (8). Substituting  and  into equation (10), the company’s cost is . Since , this solution is infeasible and move to Step 3.

Putting and  in equation (9) and then, solving for the company’s business cycle, one finds 4.173. Substituting 4.173 and  into equation (10), the company’s cost is . Since , set . As , move to Step 2 considering .

*Iteration 2: for *

Utilizing the values of  and , compute  by solving equation (25). Utilizing this derived value of , determine the optimum requisition amount  boxes for the company from equation (8). Substituting  and  into equation (10), the company’s cost is . Since , this solution is feasible. Furthermore,  > . As a result, the company’s minimum cost per unit time is  with the optimal  days and  boxes.

Thus, the optimal inventory planning for the Monihari Mango Shop under the inventory procedure with zero-end is:  days,  boxes and .

**Appendix D. Solution procedure for the case study under a stock-out situation**

The best inventory planning for the Monihari Mango Shop under a stock-out situation is computed as follow:

Initially set  and *i* = 3.

*Iteration 1:  for *

Exploiting the values of  and , compute  and  by solving equations (26) and (27). Utilizing the derived value of  and , from equation (21), determine the optimum requisition amount  boxes. Since , adopting this feasible solution in equation (23), determine the company’s cost .

Furthermore, as  , set . Therefore, the company’s minimum cost per unit time is  with the optimal  days,  days and  boxes. From this purchased quantity amount, the best backordering quantity amount is  boxes, while the stock amount at starting moment of each cycle after backordering is  boxes.

**Appendix E. Solution procedure for Example 1**

The best inventory planning for the company under this rebate scheme to minimize company’s cost is achieved by utilizing algorithm 5.2 as follows:

Initially set  and *i* = 3.

*Iteration 1:  for *

Exploiting the values of  and , compute  by solving equation (25). Utilizing this derived value of , determine the optimum requisition amount  for the company from equation (8). Substituting and  into equation (10), the company’s cost is . Since , this solution is infeasible and move to Step 3.

Putting and  in equation (9) and then, solving for the company’s business cycle, one finds 4.351. Substituting 4.351 and  into equation (10), the company’s cost is . Since , set . As , move to Step 2 considering .

*Iteration 2: for *

Utilizing the values of  and , compute  by solving equation (25). Utilizing this derived value of , determine the optimum requisition amount  for the company from equation (8). Substituting  and  into equation (10), the company’s cost is . Since , this solution is feasible. Furthermore,  > . As a result, the company’s minimum cost per unit time is  with the optimal  and . As seen from Figure 5, the company’s cost per unit of time is convex in  at .

**Appendix F. Solution procedure for Example 2**

The best inventory planning for the company under the rebate scheme to minimize company’s cost is achieved by utilizing the algorithm 5.3 as follows:

Initially set  and *i* = 3.

*Iteration 1:  for *

Exploiting the values of  and , compute  and  by solving equations (26) and (27). Utilizing the derived value of  and , from equation (21), determine the optimum requisition amount  units. Since , adopting this feasible solution in equation (23), determine the company’s cost .

Furthermore, as  , set . Therefore, the company’s minimum cost per unit time is  with the optimal ,  and . From this purchased quantity amount, the best backordering quantity amount is , while the stock amount at starting moment of each cycle after backordering is .

**Appendix G.**

**Table 1.** Consequences of system parameters on the best solution for the inventory procedure under a stock-out situation

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Parameter | Original value | % of changes | New values |  |  |  |  |  |  |
|  | 200 | -20 | 160 | 4.178 | 4.373 | 440 | 20 | 460 | 465.178 |
| -10 | 180 | 4.177 | 4.373 | 440 | 20 | 460 | 469.752 |
| +10 | 220 | 4.377 | 4.592 | 463 | 22 | 485 | 478.762 |
| +20 | 240 | 4.534 | 4.764 | 481 | 24 | 505 | 483.037 |
|  | 100 | -20 | 80 | 5.036 | 5.333 | 438 | 22 | 460 | 390.661 |
| -10 | 90 | 4.576 | 4.806 | 436 | 24 | 460 | 432.451 |
| +10 | 110 | 4.069 | 4.257 | 469 | 21 | 490 | 516.162 |
| +20 | 120 | 3.943 | 4.121 | 494 | 22 | 516 | 557.869 |
|  | 1.5 | -20 | 1.2 | 4.242 | 4.445 | 445 | 21 | 466 | 472.349 |
| -10 | 1.35 | 4.227 | 4.429 | 444 | 21 | 465 | 473.336 |
| +10 | 1.65 | 4.198 | 4.397 | 443 | 21 | 464 | 475.304 |
| +20 | 1.8 | 4.183 | 4.381 | 443 | 20 | 463 | 476.286 |
|  | 2 | -20 | 1.6 | 4.123 | 4.346 | 437 | 23 | 460 | 477.559 |
| -10 | 1.8 | 4.15 | 4.360 | 439 | 21 | 460 | 475.882 |
| +10 | 2.2 | 4.281 | 4.472 | 450 | 20 | 470 | 472.866 |
| +20 | 2.4 | 4.353 | 4.536 | 457 | 19 | 476 | 471.516 |
|  | 0.03 | -20 | 0.024 | 4.271 | 4.467 | 451 | 20 | 471 | 473.021 |
| -10 | 0.027 | 4.241 | 4.440 | 447 | 21 | 468 | 473.674 |
| +10 | 0.033 | 4.184 | 4.387 | 441 | 21 | 462 | 475.002 |
| +20 | 0.036 | 4.168 | 4.374 | 439 | 21 | 460 | 475.599 |
|  | 0.05 | -20 | 0.04 | 4.557 | 4.749 | 480 | 20 | 500 | 471.532 |
| -10 | 0.045 | 4.371 | 4.568 | 461 | 20 | 481 | 472.999 |
| +10 | 0.055 | 4.152 | 4.363 | 438 | 22 | 460 | 475.539 |
| +20 | 0.06 | 4.130 | 4.354 | 437 | 23 | 460 | 476.729 |
|  | 0.04 | -20 | 0.032 | 4.212 | 4.413 | 444 | 21 | 465 | 474.318 |
| -10 | 0.036 | 4.212 | 4.413 | 444 | 21 | 465 | 474.319 |
| +10 | 0.044 | 4.212 | 4.413 | 444 | 21 | 465 | 474.321 |
| +20 | 0.048 | 4.212 | 4.413 | 444 | 21 | 465 | 474.323 |
|  | 5 | -20 | 4 | 4.212 | 4.415 | 444 | 21 | 465 | 474.301 |
| -10 | 4.5 | 4.212 | 4.414 | 444 | 21 | 465 | 474.311 |
| +10 | 5.5 | 4.213 | 4.412 | 444 | 20 | 464 | 474.330 |
| +20 | 6 | 4.213 | 4.411 | 444 | 20 | 464 | 474.339 |
|  | 3 | -20 | 2.4 | 4.199 | 4.448 | 442 | 26 | 468 | 473.974 |
| -10 | 2.7 | 4.207 | 4.429 | 443 | 23 | 466 | 474.166 |
| +10 | 3.3 | 4.217 | 4.400 | 444 | 19 | 463 | 474.449 |
| +20 | 3.6 | 4.221 | 4.389 | 445 | 17 | 462 | 474.556 |
|  | 4.75 , 4.5 , 4 | -20 | 3.8, 3.6, 3.2 | 4.549 | 4.741 | 483 | 20 | 503 | 389.817 |
| -10 | 4.275,4.05,3.6 | 4.369 | 4.566 | 462 | 20 | 482 | 432.146 |
| +10 | 5.225,4.95,4.4 | 4.163 | 4.375 | 438 | 22 | 460 | 516.392 |
| +20 | 5.7,5.4,4.8 | 3.954 | 4.162 | 437 | 23 | 460 | 558.444 |