

The metric dimension of the circulant graph with $2k$ generators can be less than k

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Abstract

Circulant graphs are useful networks because of their symmetries. For $k \geq 2$ and $n \geq 2k + 1$, the circulant graph $C_n(1, 2, \dots, k)$ consists of the vertices $v_0, v_1, v_2, \dots, v_{n-1}$ and the edges $v_i v_{i+1}, v_i v_{i+2}, \dots, v_i v_{i+k}$, where $i = 0, 1, 2, \dots, n-1$, and the subscripts are taken modulo n . The metric dimension $\beta(C_n(1, 2, \dots, k))$ of the circulant graphs $C_n(1, 2, \dots, k)$ for general k (and n) has been studied in several papers. In 2017, Chau and Gosselin proved that $\beta(C_n(1, 2, \dots, k)) \geq k$ for every k , and they conjectured that if $n = 2k + r$, where k is even and $3 \leq r \leq k - 1$, then $\beta(C_n(1, 2, \dots, k)) = k$. We disprove both by showing that for every $k \geq 9$, there exists an $n \in [2k + 5, 2k + 8] \subset [2k + 3, 3k - 1]$ such that $\beta(C_n(1, 2, \dots, k)) \leq \lceil \frac{2k}{3} \rceil + 2$. We conjecture that for $k \geq 6$, $\beta(C_n(1, 2, \dots, k))$ cannot be less than $\lceil \frac{2k}{3} \rceil + 2$.

Keywords: metric dimension, resolving set, distance, Cayley graph, circulant graph.

1 Introduction

The metric dimension is an invariant which has applications in pharmaceutical chemistry (see [4]), pattern recognition and image processing (see [16]), robot navigation (see [14]), and Sonar and coastal guard Loran (see [18]).

In a graph G having vertex set $V(G)$, the number of edges in a shortest path connecting two vertices $u, v \in V(G)$ is the distance $d(u, v)$ between u and v . If $d(u, w) \neq d(v, w)$, then a vertex w resolves two vertices u and v . For an ordered set $W = \{w_1, w_2, \dots, w_z\}$, the ordered z -tuple

$$r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_z))$$

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is the representation of distances of v in terms of W . If all the vertices of G have different representations, then $W \subset V(G)$ is a resolving set of G . The metric dimension $\beta(G)$ is the number of vertices in a smallest resolving set.

Circulant graphs are very useful networks because of their symmetries. For $k \geq 2$ and $n \geq 2k + 1$, the circulant graph $C_n(1, 2, \dots, k)$ has the vertices $v_0, v_1, v_2, \dots, v_{n-1}$ and the edges $v_i v_{i+1}, v_i v_{i+2}, \dots, v_i v_{i+k}$, where $i = 0, 1, 2, \dots, n-1$, and the subscripts are taken modulo n . We assume that $n \geq 2k + 1$, because for $n \leq 2k$, $C_n(1, 2, \dots, k)$ would contain multiple edges. So, $\{\pm 1, \pm 2, \dots, \pm k\}$ is the set of generators of $C_n(1, 2, \dots, k)$.

In this paper, we focus on the following question: For a fixed k , find

$$\min\{\beta(C_n(1, 2, \dots, k)) \mid n \geq 2k + 1\}.$$

By [3] and [13], we have

$$\beta(C_n(1, 2)) = \begin{cases} 4 & \text{if } n \equiv 1 \pmod{4}, \\ 3 & \text{if } n \equiv 0, 2, 3 \pmod{4}. \end{cases}$$

By [3] and [9], for $n \geq 8$,

$$\beta(C_n(1, 2, 3)) = \begin{cases} 5 & \text{if } n \equiv 1 \pmod{6}, \\ 4 & \text{if } n \equiv 0, 2, 3, 4, 5 \pmod{6}. \end{cases}$$

Since $\beta(C_n(1, 2)) \geq 3$ and $\beta(C_n(1, 2, 3)) \geq 4$, Grigorious et al. [8] assumed and proved that $\beta(C_n(1, 2, \dots, k)) \geq k + 1$. Their result does not hold. In [20] it was proved that $\beta(C_n(1, 2, \dots, k))$ can be k , and that for $n \geq k^2 + 1$ where $k \geq 2$,

$$\beta(C_n(1, 2, \dots, k)) \geq k,$$

and for $n \equiv r \pmod{2k}$ where $k + 2 \leq r \leq 2k + 1$ and $k \geq 2$, we have

$$\beta(C_n(1, 2, \dots, k)) \geq k + 1.$$

Chau and Gosselin [5] obtained several strong results, but their inequality (1) is incorrect. They proved that for $n \equiv r \pmod{2k}$,

$$\beta(C_n(1, 2, \dots, k)) \geq k \tag{1}$$

if $3 \leq r \leq k$, and

$$\beta(C_n(1, 2, \dots, k)) \geq k + 1$$

if $k + 1 \leq r \leq 2k + 2$. They also conjectured that if $n = 2k + r$ where k is even and $3 \leq r \leq k - 1$, then

$$\beta(C_n(1, 2, \dots, k)) = k. \tag{2}$$

We disprove (1) and (2) by showing that for every $k \geq 9$, there exists an $n \in [2k + 5, 2k + 8] \subset [2k + 3, 3k - 1]$ such that

$$\beta(C_n(1, 2, \dots, k)) \leq \left\lceil \frac{2k}{3} \right\rceil + 2.$$

The importance of this result is that it has not been known before that $\beta(C_n(1, 2, \dots, k))$ can be less than k .

Let us note that the graphs $C_n(1, 3)$ were studied in [12], $C_n(1, 4)$ in [2], $C_n(2, 3)$ in [6], $C_n(1, \frac{n}{2})$ for even n in [17], $C_n(1, 2, 4)$ in [11], $C_n(1, 2, 5)$ in [10], $C_n(1, 2, 3, 4)$ in [7] and [21], $C_n(1, 2, \dots, k)$ also in [19], and some interesting networks were investigated also in [1] and [15].

2 New bounds

We show that for every $k \geq 7$, there exists an n such that $\beta(C_n(1, 2, \dots, k)) \leq \lceil \frac{2k}{3} \rceil + 2$. The case $k \equiv 1 \pmod{3}$ is studied in Theorem 1, $k \equiv 2 \pmod{6}$ is studied in Theorem 2, $k \equiv 3 \pmod{6}$ is investigated in Theorem 3, $k \equiv 5 \pmod{6}$ is considered in Theorem 4 and $k \equiv 0 \pmod{6}$ is studied in Theorem 5. Note that the subscripts of the vertices v_i are taken modulo n .

Theorem 1. *Let $n = 2k + 5$ where $k \equiv 1 \pmod{3}$ such that $k \geq 7$. Then*

$$\beta(C_n(1, 2, \dots, k)) \leq \frac{2k+1}{3} + 2.$$

Proof. Let $W = \{v_0\} \cup \{v_i : i = 2, 5, \dots, 2k+3\}$. We have $n = 2k + 5$, so any vertex $v_i \in W$ (where $i = 2, 5, \dots, 2k+3$ or 0), there are 4 vertices $v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}$ at distance 2 from v_i (and $2k$ vertices at distance 1 from v_i). Let $V_i = \{v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}\}$. It follows that the representations of a vertex of $V(C_n(1, 2, \dots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W .

For the vertices of V_i where $i = 5, 8, \dots, 2k$ and the ordered set $\{v_{i-3}, v_{i+3}\} \subset W$, we get

$$\begin{aligned} r(v_{i+k+1} | \{v_{i-3}, v_{i+3}\}) &= (2, 1), \\ r(v_{i+k+2} | \{v_{i-3}, v_{i+3}\}) &= (1, 1), \\ r(v_{i+k+3} | \{v_{i-3}, v_{i+3}\}) &= (1, 1), \\ r(v_{i+k+4} | \{v_{i-3}, v_{i+3}\}) &= (1, 2). \end{aligned}$$

Since $v_{i+k+2} \in W$ for $i = 5, 8, \dots, k+1$, and $v_{i+k+3} \in W$ for $i = k+4, k+7, \dots, 2k$, the vertices of V_i are resolved (for $i = 5, 8, \dots, 2k$). We have

$$V_5 \cup V_8 \cup \dots \cup V_{2k} = \{v_{k+6}, v_{k+7}, \dots, v_{3k+4}\}.$$

Note that $3k+4 = k-1$. Finally, we need to resolve $v_k, v_{k+1}, \dots, v_{k+5}$. We have $v_{k+1}, v_{k+4} \in W$ and

$$\begin{aligned} r(v_k | \{v_{2k+3}, v_0, v_2\}) &= (2, 1, 1), \\ r(v_{k+2} | \{v_{2k+3}, v_0, v_2\}) &= (2, 2, 1), \\ r(v_{k+3} | \{v_{2k+3}, v_0, v_2\}) &= (1, 2, 2), \\ r(v_{k+5} | \{v_{2k+3}, v_0, v_2\}) &= (1, 1, 2). \end{aligned}$$

Thus all the vertices of $C_n(1, 2, \dots, k)$ are resolved. So, W is a resolving set and we obtain $\beta(C_n(1, 2, \dots, k)) \leq |W| = \frac{2k+1}{3} + 2$. \square

Theorem 2. Let $n = 2k + 8$ where $k \equiv 2 \pmod{6}$ such that $k \geq 8$. Then

$$\beta(C_n(1, 2, \dots, k)) \leq \frac{2k+2}{3} + 2.$$

Proof. For $i = 0, 1, 2, \dots, \frac{k+1}{3}$, let $W_i = \{v_{6i}, v_{6i+2}\}$. We show that $W = W_0 \cup W_1 \cup \dots \cup W_{\frac{k+1}{3}}$ resolves $C_n(1, 2, \dots, k)$. We have $n = 2\frac{k+1}{3} + 8$, so for any vertex v_j where $j = 0, 1, \dots, n-1$, there are 7 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}, v_{j+k+6}, v_{j+k+7}$ at distance 2 from v_j (and $2k$ vertices at distance 1 from v_j). Thus there are exactly 7 vertices at distance 2 from $v_{6i} \in W_i$ where $i = 0, 1, 2, \dots, \frac{k+1}{3}$. Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1, 2, \dots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W . For the vertices of V_i and $\{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$\begin{aligned} r(v_{6i+k+1} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (2, 2, 1, 1), \\ r(v_{6i+k+2} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 1, 1), \\ r(v_{6i+k+3} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+4} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+5} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+6} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+7} | \{v_{6i-6}, v_{6i-4}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 2). \end{aligned}$$

The vertices $v_{6i+k+4}, v_{6i+k+6} \in W$, so the vertices of V_i are resolved (where $i = 0, 1, 2, \dots, \frac{k+1}{3}$). Since $V_0 \cup V_1 \cup \dots \cup V_{\frac{k+1}{3}} = V(C_n(1, 2, \dots, k))$, the set W resolves the vertices of $C_n(1, 2, \dots, k)$. Therefore $\beta(C_n(1, 2, \dots, k)) \leq |W| = 2(\frac{k+1}{3} + 1)$. \square

Theorem 3. Let $n = 2k + 6$ where $k \equiv 3 \pmod{6}$ such that $k \geq 9$. Then

$$\beta(C_n(1, 2, \dots, k)) \leq \frac{2k}{3} + 2.$$

Proof. For $i = 0, 1, 2, \dots, \frac{k}{3}$, let $W_i = \{v_{6i}, v_{6i+2}\}$. We show that $W = W_0 \cup W_1 \cup \dots \cup W_{\frac{k}{3}}$ resolves $C_n(1, 2, \dots, k)$. We have $n = 2\frac{k}{3} + 6$, so for any vertex v_j where $j = 0, 1, \dots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j . For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 0, 1, 2, \dots, \frac{k}{3}$, there are 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2} . Those vertices are the vertices of the set

$$V_i = \{v_{6i+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1, 2, \dots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W . For the vertices of V_i and the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$\begin{aligned} r(v_{6i+k+1} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (2, 2, 1, 1), \\ r(v_{6i+k+2} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 1, 1), \\ r(v_{6i+k+3} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+4} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+5} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+6} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+7} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 2). \end{aligned}$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved (where $i = 0, 1, \dots, \frac{k}{3}$). Since $V_0 \cup V_1 \cup \dots \cup V_{\frac{k}{3}} = V(C_n(1, 2, \dots, k))$, W resolves the vertices of $C_n(1, 2, \dots, k)$. Therefore $\beta(C_n(1, 2, \dots, k)) \leq |W| = 2(\frac{k}{3} + 1)$. \square

Theorem 4. Let $n = 2k + 6$ where $k \equiv 5 \pmod{6}$ such that $k \geq 11$. Then

$$\beta(C_n(1, 2, \dots, k)) \leq \frac{2k+2}{3} + 2.$$

Proof. For $i = 0, 1, 2, \dots, \frac{k-5}{6}$, let $W_i = \{v_{6i}, v_{6i+2}\}$ and $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$. Let $W' = \{v_{k-1}, v_{2k+2}\}$ and

$$W = (W_0 \cup W_1 \cup \dots \cup W_{\frac{k-5}{6}}) \cup (W'_0 \cup W'_1 \cup \dots \cup W'_{\frac{k-5}{6}}) \cup W'.$$

Note that $|W| = 2(\frac{k-5}{6} + 1) + 2(\frac{k-5}{6} + 1) + 2 = \frac{2k+2}{3} + 2$. We prove that W resolves $C_n(1, 2, \dots, k)$. We have $n \equiv 2k + 6$, so for any vertex v_j where $j = 0, 1, \dots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j .

For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 0, 1, 2, \dots, \frac{k-5}{6}$, there are exactly 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2} . Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex in $V(C_n(1, 2, \dots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W . For the vertices of V_i where $i = 0, 1, 2, \dots, \frac{k-11}{6}$ in terms of the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$\begin{aligned} r(v_{6i+k+1} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (2, 2, 1, 1), \\ r(v_{6i+k+2} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 1, 1), \\ r(v_{6i+k+3} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+4} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+5} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+6} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+7} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 2). \end{aligned}$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved by W (where $i = 0, 1, 2, \dots, \frac{k-11}{6}$).

Similarly, for each $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$ where $i = 0, 1, 2, \dots, \frac{k-5}{6}$, there are exactly 7 vertices at distance 2 from at least one of v_{6i+k+3}, v_{6i+k+5} . Those vertices are the vertices of the set

$$V'_i = \{v_{6i-2}, v_{6i-1}, v_{6i}, v_{6i+1}, v_{6i+2}, v_{6i+3}, v_{6i+4}\}.$$

So, the representations of a vertex of V'_i and a vertex of $V(C_n(1, 2, \dots, k)) \setminus V'_i$ are not the same in terms of W . \square the vertices of V'_i where $i = 0, 1, 2, \dots, \frac{k-11}{6}$ in terms of the ordered set $\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\} \subset W$, we get

$$\begin{aligned} r(v_{6i-2} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (2, \overline{1}, 1), \\ r(v_{6i-1} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (\overline{2}, 1, 1), \\ r(v_{6i} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (\overline{2}, 2, \overline{1}), \\ r(v_{6i+1} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 2, 2, 1), \\ r(v_{6i+2} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 2, 2, 1), \\ r(v_{6i+3} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 1, 2, 1), \\ r(v_{6i+4} | \{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 1, 2, 2). \end{aligned}$$

The vertices $v_{6i}, v_{6i+2} \in W$, so the vertices of V'_i are resolved (where $i = 0, 1, 2, \dots, \frac{k-11}{6}$).

We have

$$V_0 \cup V_1 \cup \dots \cup V_{\frac{k-11}{6}} = \{v_{k+1}, v_{k+2}, \dots, v_{2k-4}\}$$

and

$$V'_0 \cup V'_1 \cup \dots \cup V'_{\frac{k-11}{6}} = \{v_{2k+4}, v_{2k+5}, v_0, v_1, \dots, v_{k-7}\}.$$

Finally, we resolve the vertices

$$v_{k-6}, v_{k-5}, \dots, v_k \text{ and } v_{2k-3}, v_{2k-2}, \dots, v_{2k+3}.$$

Note that $v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}, v_{2k+2} \in W$ and

$$\begin{aligned} r(v_{k-6} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (\overline{1}, 1, 1, 2, 1), \\ r(v_{k-4} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (1, 1, 1, 2, 2), \\ r(v_{k-2} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (1, 1, 1, 1, 2), \\ r(v_k | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (\overline{1}, 1, 1, 1, 1), \\ r(v_{2k-3} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (2, 1, 1, 1, 1), \\ r(v_{2k-1} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (2, 2, 1, 1, 1), \\ r(v_{2k+1} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (1, 2, 2, 1, 1), \\ r(v_{2k+3} | \{v_{k-5}, v_{k-3}, v_{k-1}, v_{2k-2}, v_{2k}\}) &= (\overline{1}, 1, 2, 1, 1). \end{aligned}$$

Thus all the vertices of $C_n(1, 2, \dots, k)$ are resolved. So $\beta(C_n(1, 2, \dots, k)) \leq |W| = \frac{2k+2}{3} + 2$. \square

The proof of Theorem 5 is longer than the previous proofs, because the resolving sets used in the proof of Theorem 5 are not as simple as resolving sets used in the previous proofs.

Theorem 5. *Let $n = 2k + 6$ where $k \equiv 0 \pmod{6}$ such that $k \geq 12$. Then*

$$\beta(C_n(1, 2, \dots, k)) \leq \frac{2k}{3} + 2.$$

Proof. For $i = 0, 1, \dots, \frac{k}{6} - 1$, let $W_i = \{v_{6i}, v_{6i+2}\}$. For $i = 1, 2, \dots, \frac{k}{6} - 2$, let $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$. Let $W' = \{v_{k-2}, v_{k+2}, v_{k+5}, v_{2k-3}, v_{2k}, v_{2k+4}\}$ and

$$W = (W_0 \cup W_1 \cup \dots \cup W_{\frac{k}{6}-1}) \cup (W'_1 \cup W'_2 \cup \dots \cup W'_{\frac{k}{6}-2}) \cup W'.$$

Note that $|W| = 2(\frac{k}{6}) + 2(\frac{k}{6} - 2) + 6 = \frac{2k}{3} + 2$. We prove that W resolves $C_n(1, 2, \dots, k)$. We have $n = 2k+6$, so for any vertex v_j where $j = 0, 1, \dots, n-1$, there are 5 vertices $v_{j+k+1}, v_{j+k+2}, v_{j+k+3}, v_{j+k+4}, v_{j+k+5}$ at distance 2 from v_j .

For each $W_i = \{v_{6i}, v_{6i+2}\}$ where $i = 1, 2, \dots, \frac{k}{6} - 2$, there are 7 vertices at distance 2 from at least one of v_{6i}, v_{6i+2} . Those vertices are the vertices of the set

$$V_i = \{v_{6i+k+1}, v_{6i+k+2}, v_{6i+k+3}, v_{6i+k+4}, v_{6i+k+5}, v_{6i+k+6}, v_{6i+k+7}\}.$$

So, the representations of a vertex of $V(C_n(1, 2, \dots, k)) \setminus V_i$ and a vertex of V_i are not the same in terms of W . For the vertices in V_i and the ordered set $\{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\} \subset W$, we get

$$\begin{aligned} r(v_{6i+k+1} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (2, 2, 1, 1), \\ r(v_{6i+k+2} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 1, 1), \\ r(v_{6i+k+3} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+4} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+5} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 2, 2, 1), \\ r(v_{6i+k+6} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 1), \\ r(v_{6i+k+7} | \{v_{6i-4}, v_{6i}, v_{6i+2}, v_{6i+6}\}) &= (1, 1, 2, 2). \end{aligned}$$

The vertices $v_{6i+k+3}, v_{6i+k+5} \in W$, so the vertices of V_i are resolved by W (where $i = 1, 2, \dots, \frac{k}{6} - 2$).

Similarly, for each $W'_i = \{v_{6i+k+3}, v_{6i+k+5}\}$ where $i = 1, 2, \dots, \frac{k}{6} - 2$, there are 7 vertices at distance 2 from at least one of v_{6i+k+3}, v_{6i+k+5} . Those vertices are the vertices of the set

$$V'_i = \{v_{6i-2}, v_{6i-1}, v_{6i}, v_{6i+1}, v_{6i+2}, v_{6i+3}, v_{6i+4}\}.$$

For the vertices of V'_i and the ordered set $\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\} \subset W$

W , we get

$$\begin{aligned}
r(v_{6i-2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (2, \overset{1}{1}, 1, 1), \\
r(v_{6i-1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (2, 1, 1), \\
r(v_{6i}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (2, 2, 1), \\
r(v_{6i+1}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 2, 2, 1), \\
r(v_{6i+2}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 2, 2, 1), \\
r(v_{6i+3}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 1, 2, 1), \\
r(v_{6i+4}|\{v_{6i+k-1}, v_{6i+k+3}, v_{6i+k+5}, v_{6i+k+9}\}) &= (1, 1, 2, 2).
\end{aligned}$$

The vertices $v_{6i}, v_{6i+2} \in W$, so the vertices of V'_i are resolved.

We have

$$V_1 \cup V_2 \cup \dots \cup V_{\frac{k}{6}-2} = \{v_{k+7}, v_{k+8}, \dots, v_{2k-5}\}$$

and

$$V'_1 \cup V'_2 \cup \dots \cup V'_{\frac{k}{6}-2} = \{v_4, v_5, \dots, v_{k-8}\}.$$

Finally, we resolve the vertices

$$v_{k-7}, v_{k-6}, \dots, v_{k+6} \text{ and } v_{2k-4}, v_{2k-3}, \dots, v_{2k+5}, v_0, v_1, v_2, v_3$$

if $k \geq 18$, and it remains to resolve all the vertices if $k = 12$. Note that

$$\overset{2}{v_{k-6}}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}, v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2 \in W,$$

so those vertices are resolved. We obtain

$$\begin{aligned}
r(v_{k-7}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (2, 1, 1, 1, 1), \\
r(v_{k-5}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (2, 2, 1, 1, 1), \\
r(v_{k-3}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 2, 1, 1, 1), \\
r(v_{k-1}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 2, 2, 1, 1), \\
r(v_k|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 1, 2, 1, 1), \\
r(v_{k+1}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 1, 2, 2, 1), \\
r(v_{k+3}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 1, 2, 2, 2), \\
r(v_{k+4}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 1, 1, 2, 2), \\
r(v_{k+6}|\{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) &= (1, 1, 1, 1, 2).
\end{aligned}$$

and

$$\begin{aligned}
r(v_{2k-4} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (2, 1, 1, 1), \\
r(v_{2k-2} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (2, 2, 1, 1), \\
r(v_{2k-2} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (2, 2, 2, 1, 1), \\
r(v_{2k+1} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 2, 2, 1, 1), \\
r(v_{2k+2} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 1, 2, 1, 1), \\
r(v_{2k+4} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 1, 2, 2, 1), \\
r(v_{2k+5} | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 1, 1, 2, 1), \\
r(v_1 | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 1, 1, 2, 2), \\
r(v_3 | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) &= (1, 1, 1, 1, 2).
\end{aligned}$$

For $i = k - 7, k - 5, k - 3, k - 1, k, k + 1, k + 3, k + 4, k + 6$, we obtain

$$r(v_i | \{v_{k-6}, v_{k-4}, v_{k-2}, v_{k+2}, v_{k+5}\}) = (1, 1, 1, 1, 1),$$

and for $i = 2k - 4, 2k - 2, 2k - 1, 2k + 1, 2k + 2, 2k + 3, 2k + 5, 1, 3$,

$$r(v_i | \{v_{2k-3}, v_{2k}, v_{2k+4}, v_0, v_2\}) = (1, 1, 1, 1, 1).$$

Thus all the vertices of $C_n(1, 2, \dots, k)$ are resolved for $k \geq 18$.

If $k = 12$, it remains to consider v_4 and v_{19} . We have

$$r(v_4 | \{v_{k+5}, v_{2k-3}\}) = r(v_4 | \{v_{17}, v_{21}\}) = (2, 2)$$

and

$$r(v_{19} | \{v_2, v_{k-6}\}) = r(v_{19} | \{v_2, v_6\}) = (2, 2).$$

It can be seen above that no other vertex has such representation in terms of the ordered sets $\{v_{k+5}, v_{2k-3}\}$ and $\{v_2, v_{k-6}\}$. So, W resolves all the vertices of $C_n(1, 2, \dots, k)$. Thus $\beta(C_n(1, 2, \dots, k)) \leq |W| = \frac{2k}{3} + 2$. \square

From Theorems 1, 2, 3, 4 and 5, we obtain the following corollary.

Corollary 1. *For every $k \geq 7$, there exists an $n \in [2k + 5, 2k + 8]$ such that*

$$\beta(C_n(1, 2, \dots, k)) \leq \left\lceil \frac{2k}{3} \right\rceil + 2.$$

Let us note that for $k = 7$ and 8 , we have $\lceil \frac{2k}{3} \rceil + 2 = k$, so Corollary 1 is important for $k \geq 9$.

3 Concluding remarks and further work

We showed that for every $k \geq 7$, there is an $n \in [2k + 5, 2k + 8]$ such that $\beta(C_n(1, 2, \dots, k)) \leq \lceil \frac{2k}{3} \rceil + 2$. The importance of this result is that it has not been known before that $\beta(C_n(1, 2, \dots, k))$ can be less than k . We believe that $\beta(C_n(1, 2, \dots, k))$ cannot be less than $\lceil \frac{2k}{3} \rceil + 2$ for $k \geq 6$, thus we provide Conjecture 1 as a possible future work. Note that $n \geq 2k + 1$, otherwise $C_n(1, 2, \dots, k)$ would contain multiple edges.

Conjecture 1. For every $k \geq 6$ (and $n \geq 2k + 1$),

$$\beta(C_n(1, 2, \dots, k)) \geq \left\lceil \frac{2k}{3} \right\rceil + 2.$$

Conjecture 1 would not hold for $k = 4$ and $k = 5$. For example, for $k = 5$ and $n = 13$, the set $\{v_0, v_1, v_2, v_4, v_5\}$ is one possible resolving set of $C_{13}(1, 2, \dots, 5)$.

10 Declaration of competing interest

The authors have no competing interests and no conflicts of interests to disclose.

Data availability statement

The data used to find the results are included in this paper.

Author contributions

All the authors contributed equally to this work. All the authors read and approved the final version.

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