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Some Topological Properties on C-α-Normality and C-β-Normality

Samirah AlZahrani[1]

Mathematics Department,Taif University, Saudi Arabia.

Abstract

A topological space (Y, τ) is called *C-α-normal (C-β-normal)* if there exist a bijective function g from Y onto α-normal(β-normal) space Z such that the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y. We discuss some relationships between C-α-normal (C-β-normal) and other properties.

Kay words: Normal, α-normal, β-normal, C-normal, epinormal, mildly normal.

Introduction

In 2017 we discuss the topological property *"C-normal"* [2] . In this paper we introduce a new property called *C-α-Normality and C-β-Normality*. We show any α-normal (β-normal) space is C-α-normal (C-β-normal), but the converse is not true in general. And we show that any C-normal, lower compact, epinormal, epi-α-normal and epi-β-normal spaces is C-α-normal (C-β-normal), and the converse is true under some conditions . we prove any locally compact is C-α-normal (C-β-normal) but the converse is not true in general. Also observe that a witness function of C-α-normal (C-β-normal) not necessarily to be continuous in general, but it will be continuous under some conditions.

1 C-α-Normality and C-β-Normality

Recall that a topological space (Y, τ) is called an *α-normal* space [11] if for every two disjoint closed subsets F and E of Y there are two open subsets G and W of Y such that $F \cap G$ is dense in F, $E \cap W$ is dense in E and $G \cap W = \emptyset$, and a topological space (Y, τ) is called a *β-normal* space [11] if for every two disjoint closed subsets F and E of Y there are two open subsets G and W of Y such that $F \cap G$ is dense in F, $E \cap W$ is dense in E, and $\overline{G} \cap \overline{W} = \emptyset$. A topological space (Y, τ) is called *C-normal*

[1] Samirah AlZahrani, Department of Mathematics and Statistics, College of Science, Taif University, , P.O.Box 11099, Taif 21944, Saudi Arabia, mam_1420@hotmail.com, Samar.alz@tu.edu.sa.

[2] if there exist a bijective function g from Y onto a normal space Z such that the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.

Definition 1.1. A topological space (Y,τ) is called C-α-*normal* (C-β-*normal*) if there exist a bijective function g from Y onto α-normal(β-normal) space Z such that the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.

In these definition, we call the space Z a witness of C-α-*normal* (C-β-*normal*) and the function g a witness function.

A topological space (Y,τ) is called α-*regular* [13] if for any $x \in Y$ and a closed subset $A \subset Y$ such that $x \notin A$ there are two disjoint open sets $G, H \subset Y$ such that $x \in G$ and $\overline{A \cap H} = A$. And topological space (Y,τ) is called almost α-*regular* [1] if for any $x \in Y$ and a regular closed subset $A \subset Y$ such that $x \notin A$ there are two disjoint open sets $G, H \subset Y$ such that $x \in G$ and $\overline{A \cap H} = A$.

Lemma 1.2. Any regular space is α-regular.

Proof. Let (Y,τ) be a regular space. Pick $y \in Y$ and $F \subseteq Y$ be a closed set such that $y \notin F$, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $y \in W_1$ and $F \subseteq W_2$, hence $\overline{F \cap W_2} = F$ (note that $\overline{F} = F$ since F is closed), and $W_1 \cap W_2 = \emptyset$, therefore (Y,τ) is α-regular space. □

Lemma 1.3. [1] Any α-regular space is almost α-regular.

From **Lemma 1.2**. and **Lemma 1.3**. we conclude the following corollary

Corollary 1.4. Any regular space is almost α-regular.

Lemma 1.5. Any normal space is α-normal.

Proof. Let Y be a normal space. Pick two disjoint closed sets F_1 and F_2 subsets of Y. Since Y is normal, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $F_1 \subseteq W_1, F_2 \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$. Hence $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$. Therefore Y is α-normal space. □

Lemma 1.6. [11] Any normal space is β-normal.

Proof. The proof is the same as the proof of the previous lemma. It remains to prove $\overline{W_1} \cap \overline{W_2} = \emptyset$. Since $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$, then

$(\overline{F_1 \cap W_1}) \cap (\overline{F_2 \cap W_2}) = F_1 \cap F_2$

$(\overline{F_1 \cap W_1}) \cap (\overline{F_2 \cap W_2}) \subseteq \overline{F_1} \cap \overline{W_1} \cap \overline{F_2} \cap \overline{W_2} = (F_1 \cap F_2) \cap (\overline{W_1} \cap \overline{W_2}) = \emptyset$. Hence $\overline{W_1} \cap \overline{W_2} = \emptyset$. □

So we have the following theorem

Theorem 1.7. Any C-normal space is C-α-normal (C-β-normal)

The converse is true under some conditions, first we mention some definition

A Hausdorff space Y is *extremally disconnected* [8] if the closure of any open set in Y is open. A topological space is called *mildly normal* [14] if any two disjoint regular closed subsets can be separated.

Theorem 1.8. [11] Any α-normal extremally disconnected space is normal.

Proof. Let Y be a α-normal extremally disconnected space . Pick two disjoint closed sets F_1 and F_2 subsets of Y. Since Y is α-normal, then there exist two disjoint open sets W_1 and W_2 subsets of Y where $\overline{F_1 \cap W_1} = \overline{F_1}$ and $\overline{F_2 \cap W_2} = F_2$. Hence $F_1 \subseteq \overline{W_1}$ and $F_2 \subseteq \overline{W_2}$. Since Y is extremally disconnected, Let $\overline{W_1} = W_1$ and $\overline{W_2} = W_2$. Therefore Y is normal space. □

From Theorem 1.8, we have the following.

Theorem 1.9. If Y is C-α-normal (C-β-normal) such that the witness of C-α-normal (C-β-normal) is extremally disconnected , then Y is C-normal.

Theorem 1.10. If Y is C-β-normal such that the witness of C-β-normal is mildly normal , then Y is C-normal.

Proof. Let Y be C-β-normal. Then the codomain Z witness of C-β-normal is β-normal. Let F_1 and F_2 be any disjoint closed subsets of Z. Since Z is β-normal, there exist open subsets W_1 and W_2 of Z where $\overline{W_1} \cap \overline{W_2} = \emptyset$, $\overline{F_1 \cap W_1} = F_1$ and $\overline{F_2 \cap W_2} = F_2$. So $\overline{W_1}$, $\overline{W_2}$ are disjoint regular closed subsets containing F_1 and F_2 respectively. Since Z is mildly normal, there exist disjoint open subsets U_1 and U_2 of Z where $F_1 \subseteq \overline{W_1} \subseteq U_1$ and $F_2 \subseteq \overline{W_2} \subseteq U_2$. Hence Z is normal. □

Lemma 1.11. Any α-normal space satisfying T_1 axiom is Hausdorff.

Proof. Let Y be an α-normal T_1-space. Let y, z be any two distinct elements in Y. Hence $\{y\}$ and $\{z\}$ are disjoint closed subsets of Y, by α-normality, there exist two disjoint open subsets G_1 and G_2 of Y where $\overline{\{y\} \cap G_1} = \{y\}$ and $\overline{\{z\} \cap G_2} = \{z\}$ which implies $y \in G_1$ and $z \in G_2$. Therefore Y is Hausdorff. □

Lemma 1.12. [11] Any β-normal space satisfying T_1 axiom is regular (hence Hausdorff).

By Corollary 1.4. we have the following result.

Corollary 1.13. Any β-normal space satisfying T_1 axiom is almost α-regular.

Lemma 1.14. Any β-normal space satisfying T_1 axiom is α-regular.

Proof. Let (Y, τ) be a β-normal space satisfying T_1 axiom. Pick $y \in Y$ and $F \subseteq Y$ be a closed set where $y \notin F$, $\{y\}$ is closed and disjoint from F then by β-normality there exist two open sets W_1 and W_2 subsets of Y such that $y \in W_1$, $\overline{F \cap W_2} = F$ and

$\overline{W_1} \cap \overline{W_2} = \emptyset$, therefore $y \in W_1$, $\overline{F \cap W_2} = F$ and $W_1 \cap W_2 = \emptyset$. Hence (Y, τ) is α-regular space. ☐

Corollary 1.15. Any α-normal space satisfying T_1 axiom is α-regular.

By Lemma 1.3. we conclude the following corollary.

Corollary 1.16. Any α-normal space satisfying T_1 axiom is almost α-regular

Proposition 1.17. *[12]* Every first countable α-normal Hausdorff space is regular.

Theorem 1.18. Every submetrizable space is C-α-normal (C-β-normal).

Proof. Let (Y, τ) be a submetrizable space, the there exists a metrizable τ' such that $\tau' \subseteq \tau$. Hence (Y, τ') is α-normal since it is normal, and the identity function id_Y from (Y, τ) onto (Y, τ') is a one-to-one and continuous function. If we take B any compact subspace of (Y, τ), then $id_Y(B)$ is hausdorff, since it is subspace of (Y, τ'), and by [[8],3.1.13], $id_{Y_{|B}}$ is a homeomorphism. ☐

Example 1.19. The Rational Sequence Topology $(\mathbb{R}, \mathcal{RS})$ is submetrizable being finer than $(\mathbb{R}, \mathcal{U})$, so $(\mathbb{R}, \mathcal{RS})$ is C-α-normal (C-β-normal).

The converse of Theorem 1.18. is not true in general, for example $\omega_1 + 1$ is C-α-normal (C-β-normal) which is not submetrizable.

Apparently, any α-normal (β-normal) space is C-α-normal (C-β-normal), to prove this, just by considering $Z = Y$ and g is the identity function.
While in general the converse is not true . Example of this.

Example 1.20.

1. The Half-Disc topological space [15] is C-α-normal(C-β-normal) because it is submetrizable by Theorem 1.18. but it is not α-normal nor β-normal because it is first countable and Hausdorff but not regular, so by Proposition *1.17.* the Half-Disc topological space is not α-normal space, hence not β-normal. in general C-α-normality (C-β-normality) do not imply α-normality (β-normality) even with Hausdorff or first countable property.

2. The Deleted Tychonoff Plank [15] , it is C-α-normal(C-β-normal) since it is locally compact by Theorem *2.7.* but it is not α-normal nor β-normal see [11] .

3. The Dieudonn$\acute{e}$ Plank [2] , in example 1.10 we proved that it is C-normal, hence it is C-α-normal(C-β-normal) by Theorem *1.9.* but it is not α-normal nor β-normal see [11] , also not locally compact, hence this example also shows that the converse of Theorem *2.7.* is not true.

4. The Sorgenfrey line square $\mathcal{S} \times \mathcal{S}$ see [15] is not normal, but it is submetrizable space being it is finer than the usual topology on $\mathbb{R} \times \mathbb{R}$, so by Theorem 1.18. it is C-α-normal (C-β-normal).

Theorem 1.21. If Y is a compact non-α-normal(non-β-normal) space, then Y can not be C-α-normal (C-β-normal).

Proof. Assume Y is a compact non-α-normal(non-β-normal) space. Suppose Y is C-α-normal (C-β-normal), then there exists α-normal(β-normal) space Z and a bijective function $g: Y \to Z$ where the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace K of Y. As Y is compact, then $Y \cong Z$, and we have a contradiction as Z is α-normal(β-normal) while Y is not. Hence Y can not be C-α-normal (C-β-normal). ☐

Observe that a function $g: Y \to Z$ witnessing of C-α-normal (C-β-normal) of Y not necessarily to be continuous in general, and here is an example.

Example 1.22. Let $\mathbb{R}$ with the countable complement topology $\mathcal{CC}$ [15]. We know $(\mathbb{R}, \mathcal{CC})$ is T_1 and the only compact sets are finite, hence the compact subspaces are discrete. If we let $\mathcal{D}$ be the discrete topology on $\mathbb{R}$, then obviously the identity function from $(\mathbb{R}, \mathcal{CC})$ onto $(\mathbb{R}, D)$ is a witnessing of the C-α-normality (C-β-normality) which is not continuous.

But it will be continuous under some conditions as the following theorems

Theorem 1.23. If (Y, τ) is a C-α-normal (C-β-normal) and Fréchet space, then any function witnessing of C-α-normality (C-β-normality) is continuous.

Proof. Let Y be a Fréchet C-α-normal (C-β-normal) space and $g: Y \to Z$ be a witness of the C-α-normality (C-β-normality) of Y. Let $A \subseteq Y$ and pick $z \in g(\overline{A})$. There is a unique $y \in Y$ where $g(y) = z$, thus $y \in \overline{A}$. since Y is Fréchet, then there exists a sequence $(a_n) \subseteq A$ where $a_n \to Y$. As the subspace $K = \{y\} \cup \{a_n : n \in \mathbb{N}\}$ of Y is compact, the induced map $g_{|K} : K \to g(K)$ is a homeomorphism. Let $U \subseteq Z$ be any open neighborhood of z. Then $U \cap g(K)$ is an open neighborhood of z in the subspace $g(K)$. Since $g_{|K}$ is a homeomorphism, then $g^{-1}(U \cap g(K)) = g^{-1}(U) \cap K$ is an open neighborhood of y in K, then there exists $m \in \mathbb{N}$ where $a_n \in g^{-1}(U \cap g(K))$ $\forall n \geq m$, hence $g(a_n) \in (U \cap g(K))$ $\quad \forall n \geq m$, then $U \cap g(A) \neq \emptyset$. Hence $z \in \overline{g(A)}$ and $g(\bar{A}) \subseteq \overline{g(A)}$. Thus g is continuous. ☐

Since any first countable space is Fréchet , we conclude that, In C-α-normality (C-β-normality) first countable space a function g: Y → Z is a witness of the C-α-normality (C-β-normality) of Y is continuous. Also, by theorem [[8],3.3.21], we conclude the following.

Corollary 1.24. If Y is a C-α-normal (C-β-normal) *k-space and g is a witness function of the C-α-normality (C-β-normality), then g is continuous.*

For simplicity, let us call a T_1 space which satisfies that the only compact subspaces are the finite subsets *F-compact*. Clearly F-compactness is a topological property.

Theorem 1.25. If Y is *F-compact* , then Y is C-α-normal (C-β-normal).

Proof. Let Y be a *F-compact* . Let $Z = Y$ and let Z with the discrete topology. Hence the identity function from Y onto Z does the job. □

Example 1.26. Consider $(\mathbb{R}, \mathcal{CC})$ where $\mathcal{CC}$ is the countable complement topology [15]. We know $(\mathbb{R}, \mathcal{CC})$ is T_1 and the only compact sets are finite, Therefore, by Theorem 1.25. $(\mathbb{R}, \mathcal{CC})$ is C-α-normal (C-β-normal). This is a fourth example of C-α-normal (C-β-normal) but not α-normal (nor β-normal).

Not that any topology finer than a T_1 topological space is T_1. Also any compact subset of a topological space (Y, τ) is compact in any topology coarser than τ on Y.

Hence any topology finer than F-compact topological space is also F-compact . As an example, $(\mathbb{R}, \tau)$ denotes the Fortissimo topological on $\mathbb{R}$, see [15, Example 25]. We know that $(\mathbb{R}, \tau)$ is finer than $(\mathbb{R}, \mathcal{CC})$ which is F-compact , hence $(\mathbb{R}, \tau)$ F-compact too. Thus, $(\mathbb{R}, \tau)$ is C-α-normal (C-β-normal).

Theorem 1.27. C-α-normality (C-β-normality) is a topological property.

Proof. Let Y be a C-α-normal (C-β-normal) space and let $Y \cong W$. Let Z be a α-normal (β-normal) space and let $g: Y \rightarrow Z$ be a bijective function where the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace $B \subseteq Y$. Let $k: W \rightarrow Y$ be a homeomorphism. Hence Z and $g \circ k: W \rightarrow Z$ satisfy the requirements. □

2 C-α-Normality (C-β-Normality) and Some Other Properties

Definition 2.1. A topological space (Y, τ) is called C-α-*regular* if there exist a bijective function g from Y onto α-regular space Z such that the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y.

Corollary 2.2. If Y is C-α-normal (C-β-normal) space and the witness of the C-α-normal (C-β-normal) of Y is T_1, then Y is C-α-regular.

We prove this corollary by lemma 1.11, Lemma 1.12

Corollary 2.3. If Y is C-β-normal space and the codomain witness of the C-β-normal of Y is T_1, then Y is C-regular.

We prove this corollary by lemma 1.12.

Corollary 2.4. If Y is a C-α-normal (C-β-normal) *Fréchet space and the witness of the C-α-normality (C-β-normality) is* T_1, *then* Y *is* T_2 .

Proof. Let Y is a C-α-normal (C-β-normal) Fréchet space , then there exist α-normal (β-normal) space Z (witness of the C-α-normality (C-β-normality)) and a bijective function $g: Y \to Z$ such that the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of Y, then by Theorem 1.23. g is continuous. Let any $a, b \in Y$ such that $a \neq b$, then $g(a) \neq g(b)$, $g(a), g(b) \in Z$. Since Z is α-normal (β-normal) and T_1, then by Lemma 1.11 (Lemma 1.12) the space Z is T_2 , then there exist W_1 and W_2 are open sets in Z where $g(a) \in W_1$, $g(b) \in W_2$ and $W_1 \cap W_2 = \emptyset$. Since W_1, W_2 are open sets in Z and g *is continuous*, then $g^{-1}(a)$ and $g^{-1}(b)$ are open sets in Y, $a \in g^{-1}(W_1)$, $b \in g^{-1}(W_2)$ and $g^{-1}(W_1) \cap g^{-1}(W_2) = g^{-1}(W_1 \cap W_2) = \emptyset$. Hence Y is T_2 .

Theorem 2.5. Any C-regular Fréchet Lindelof space is C-α-normal (C-β-normal).

Proof. Let Y be any C-regular Fréchet Lindelof space. Let Z be a regular space and $g: Y \to Z$ be a continuous bijective function see Theorem 1.23. By [[8], 3.8.7] Z is Lindelof. Since any regular Lindelof space is normal [[8], 3.8.2]. Hence Y is C-α-normal (C-β-normal). ☐

C-α-normality (C-β-normality) does not imply C-α-regularity nor C-regular, for example.

Example 2.6. Consider the real numbers set $\mathbb{R}$ with its right ray topology $\mathcal{R}$, where $\mathcal{R} = \{\emptyset, \mathbb{R}\} \cup \{(b, \infty): b \in \mathbb{R}\}$. As any two non-empty closed sets must be intersect in $(\mathbb{R}, \mathcal{R})$, then it is normal, and by Lemma in above, it is α-normal (β-normal), hence C-α-normal (C-β-normal). Now, suppose that $(\mathbb{R}, \mathcal{R})$ is C-α-regular. Take α-regular space Z and a bijective function g from $\mathbb{R}$ onto Z where the restriction map $g_{|B}$ from B onto $g(B)$ is a homeomorphism for any compact subspace B of $\mathbb{R}$. We know that a subspace B of $(\mathbb{R}, \mathcal{R})$ is compact if and only if B has a minimal element. Hence $[1, \infty)$ is compact, then $g_{|[1,\infty)}: [1, \infty) \to g([1, \infty)) \subset Z$ is a homeomorphism, it means $[1, \infty)$ as a subspace of $(\mathbb{R}, \mathcal{R})$ is α-regular which is a contradiction, since $[1,4]$ is closed in subspace $[1, \infty)$ and $4.5 \notin [1,4]$, but any non-empty open sets on $[1, \infty)$ must intersect. Then $(\mathbb{R}, \mathcal{R})$ cannot be C-α-regular (C-regular).

Recall that a topological space (Y, τ) is called *Locally Compact* [2] if (Y, τ) is Hausdorff and for every $y \in Y$ and every open neighborhood V of y there exists an open neighborhood U of y such that $y \in U \subseteq \overline{U} \subseteq V$ and $\overline{U}$ is compact.

Theorem 2.7. Every locally compact space is C-α-normal (C-β-normal).

Proof. Let Y be locally compact space. By [[8], 3.3.D], there exists T_2 compact space Z and hence α-normal (β-normal), and a continuous bijective function $g: Y \to Z$. We have $g_{|K}$ from K onto $g(K)$ is a homeomorphism for any compact subspace K of Y, because continuity ,1-1 and onto are inherited by g, also $g_{|K}$ is closed since K is compact and g(K) is T_2. ☐

Example 2.8. Consider ω_1 , the first uncountable ordinar, we consider ω_1 as an open subspace of its successor $(\omega_1 + 1)$, which is compact and hence is locally compact [15, Example 43]. Thus, ω_1 is locally compact as an open subspace of a locally compact space, see [[8],3.3.8]. Then by Theorem 2.7. ω_1 is C-α-normal (C-β-normal).

The converse of Theorem 2.7. is not true in general. We introduce the following example of C-α-normal (C-β-normal) which is not locally compact.

Example 2.9. Let the quotient space $\mathbb{R}/\mathbb{N}$. Let $Z = (\mathbb{R}\backslash\mathbb{N}) \cup \{i\}$, where $i = \sqrt{-1}$. Define $g: \mathbb{R} \to Z$ as follows:

$$g(a) = \begin{cases} a & for \quad a \in \mathbb{R}\backslash\mathbb{N} \\ i & for \quad a \in \mathbb{N} \end{cases}$$

Now consider $\mathbb{R}$ with the usual topology $\mathcal{U}$. Define the topology $\tau = \{V \subseteq Z : g^{-1}(V) \in \mathcal{U}\}$ on Z. Then $g: (\mathbb{R}, \mathcal{U}) \to (Z, \tau)$ is a closed quotient mapping. We explain the open neighborhoods of any element in Z as follows: The open neighborhoods of each $a \in \mathbb{R}\backslash\mathbb{N}$ are $(a - \varepsilon, a + \varepsilon)\backslash\mathbb{N}$ where ε is a natural number. The open neighborhoods of $i \in Z$ are $(G\backslash\mathbb{N}) \cup \{i\}$, where G is an open set in $(\mathbb{R}, \mathcal{U})$ such that $\mathbb{N} \subseteq G$. It is clear that (Z, τ) is T_3, but it is not locally compact. (Z, τ) is a continuous image of $\mathbb{R}$ with its usual topology, so it is Lindelof and T_3, then (Z, τ) is T_4. Hence it is C-α-normal (C-β-normal).

A topological space (Y, τ) is called *Epi*-α-*normal* [3] if there is a coarse topology τ' on Y such that (Y, τ') is α-normal and T_1. A topological space (Y, τ) is called *Epi*-β-*normal* [3] if there is a coarse topology τ' on Y such that (Y, τ') is β-normal and T_1. By the same argument of Theorem 1.18. we can prove the following corollary.

Corollary 2.10. Every epinormal space is C-α-normal (C-β-normal).

Corollary 2.11. Every epi-α-normal (epi-β-normal)space is C-α-normal(C-β-normal).

Any indiscrete space which has more than one point is an example of a C-α-normal (C-β-normal) space which is not epi-α-normal (epi-β-normal).

The converse of Corollary 2.9 is true with Fréchet property.

Theorem 2.12. Any C-α-normal (C-β-normal) Fréchet space is epi-α-normal (epi-β-normal).

Proof. Let (Y,τ) be any C-α-normal (C-β-normal) Fréchet space. Let (Z,τ') be α-normal (β-normal) and $g:(Y,\tau)\to(Z,\tau')$ be a bijective function. Since Y is Fréchet, g is continuous (see Theorem 1.23). Define $\tau^*=\{g^{-1}(V):V\in\tau'\}$. Obviously, τ^* is a topology on Y coarser than τ such that $g:(Y,\tau^*)\to(Z,\tau')$ is continuous. Also g is open, since if we take $U\in\tau^*$, then $U=g^{-1}(V)$ where $V\in\tau'$. Thus $g(U)=g(g^{-1}(V))=V$ which gives that g is open. Therefore g is homeomorphism. Thus (Y,τ^*) is α-normal (β-normal). Hence (Y,τ) is epi-α-normal (epi-β-normal). ☐

A topological space (Y,τ) is called *lower compact* [9] if there exists a coarser topology τ' on Y such that (Y,τ') is T_2-compact.

Theorem 2.13. Any lower compact space is C-α-normal (C-β-normal).

Proof. Let (Y,τ) is lower compact, then (Y,τ') is T_2-compact, hence normal and the identity function $id_Y:(Y,\tau)\to(Y,\tau')$ is a continuous and bijective. If we take B any compact subspace of (Y,τ), then $id_{Y_{|B}}$ is a homeomorphism by [[8],3.1.13]. ☐

In general, the converse of Theorem 2.13. is not true, for example consider a countable complement topology on an uncountable set, it is C-α-normal (C-β-normal) since it is F-compact , but it is not lower compact because it is not T_2.

Theorem 2.14. If (Y,τ) is C-α-normal compact Fréchet *space and the witness of the C-α-normality is T_1, then (Y,τ) is lower compact.*

Proof. Pick α-normal space (Z,τ^*) and a bijective function $g:(Y,\tau)\to(Z,\tau^*)$ such that $g_{|B}:B\longrightarrow g(B)$ is a homeomorphism for any compact subspace $B\subseteq Y$. Since Y is Fréchet, then g is continuous. Hence (Z,τ^*) is compact. Since (Z,τ^*) is T_1 α-normal space, then by Lemma 1.11. it is Hausdorff. Hence (Z,τ^*) is T_2 compact. Define a topology τ' on Y as follows $\tau'=\{g^{-1}(V):V\in\tau^*\}$. Then τ' is coarser than τ and $g:(Y,\tau')\to(Z,\tau^*)$ is a bijection continuous function. Let any $U\in\tau'$, then U is of the form $g^{-1}(V)$ for some $V\in\tau^*$. Hence $g(U)=g\left(g^{-1}(V)\right)=V$. Thus g is open. Hence g is a homeomorphism. So (Y,τ') is T_2 compact. Therefore (Y,τ) is lower compact.

Theorem 2.15. If (Y,τ) is C-β-normal compact Fréchet *space and the witness of the C-β-normality is T_1, then (Y,τ) is lower compact.*

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