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# A NEW INTEGRAL VERSION OF GENERALIZED OSTROWSKI–GRÜSS TYPE INEQUALITY WITH APPLICATIONS

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Abstract. Our aim is to improve and further generalize the result of integral Ostrowski–Grüss type inequalities involving differentiable functions and then apply these obtained inequalities to probability theory, special means and numerical integration.

## 1. Introduction

In [12], Ostrowski presented an inequality which is now known as "Ostrowski's inequality" stated below:

$$\left|\zeta(z) - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right| \leq \left[\frac{1}{4} + \frac{\left(z - \frac{m+n}{2}\right)^2}{(n-m)^2}\right](n-m)M, \quad z \in [m,n] \tag{1.1}$$

where $\zeta : [m,n] \to \mathbb{R}$ is a differentiable function such that $|\zeta'(z)| \leq M$, for every $z \in [m,n]$.

In present era, a large number of papers has been written about generalizations of Ostrowski's inequality see for example [1, 3, 4, 6, 7, 8, 10, 13, 15]. Ostrowski's inequality has proven to be an important tool for improvement of various branches of mathematical sciences. Very well said [14] "Inequalities involving integrals that create bounds in the physical quantities are of great significance in the sense that these kinds of inequalities are not only used in approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, statistics and probability theory but we may also see their uses in the various fields of biological sciences, engineering and physics".

In the history, an important inequality that "estimate for the difference between the product of the integral of two functionals and the integral of their product" is known as "Grüss inequality". This celebrated integral inequality was proved by Grüss [5] in 1935, is stated below (see also [11, p. 296]),

$$\left|\frac{1}{n-m}\int_m^n \zeta(z)\eta(z)\,dz - \left(\frac{1}{n-m}\int_m^n \zeta(z)\,dz\right)\left(\frac{1}{n-m}\int_m^n \eta(z)\,dz\right)\right| \leq \frac{1}{4}(M_1 - m_1)(N_1 - n_1) \tag{1.2}$$

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provided that $\zeta$ and $\eta$ are integrable functions on $[m,n]$ such that

$$m_1 \leq \zeta(z) \leq M_1, \quad n_1 \leq \eta(z) \leq N_1,$$

$\forall z \in [m,n]$, where $m_1, M_1, n_1, N_1$ are real constants.

By using Grüss inequality, Dragomir and Wang proved an inequality, in the year 1997, which we would refer as "Ostrowski– Grüss inequality" [4] which is stated as follows:

**Proposition 1.1.** *Suppose $\zeta : I \to \mathbb{R}$ be a function differentiable in the interior $I^o$ of $I$, where $I \subseteq \mathbb{R}$, and let $m,n \in I^o$ and $n > m$. If $\gamma \leq \zeta'(z) \leq \Gamma, z \in [m,n]$ for real constants $\gamma, \Gamma$, then*

$$\left|\zeta(z) - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau - \frac{\zeta(n)-\zeta(m)}{n-m}\left(z - \frac{m+n}{2}\right)\right| \leq \frac{1}{4}(n-m)(\Gamma-\gamma) \quad (1.3)$$

*holds, $\forall\ z \in [m,n]$.*

Above inequality gives a relationship between Ostrowski inequality (1.1) and Grüss inequality (1.2).

If $\zeta$ and $g$ belong to $L_2[m,n]$, then the Čebyšev functional $T(\zeta,\eta)$ is defined as

$$T(\zeta,\eta) = \frac{1}{n-m}\int_m^n \zeta(z)\eta(z)dz - \left(\frac{1}{n-m}\int_m^n \zeta(z)dz\right)\left(\frac{1}{n-m}\int_m^n \eta(z)dz\right).$$

From [9] pre-Grüss inequality is given below.

**Proposition 1.2.** *Let $\zeta, \eta : [m,n] \to \mathbb{R}$ be integrable such that $\zeta\eta \in L(m,n)$. If*

$$\gamma \leq \eta(z) \leq \Gamma \quad for \quad z \in [m,n],$$

*then*

$$|T(\zeta,\eta)| \leq \frac{1}{2}(\Gamma-\gamma)\sqrt{T(\zeta,\zeta)}.$$

In the article [8] of year 2000, Matić, Pecarić and Ujević improved inequality (1.1), by using pre–Grüss inequality, which is as follows:

**Proposition 1.3.** *Suppose $\zeta : I \to \mathbb{R}$ be a function differentiable in the interior $I^o$ of $I$, where $I \subseteq \mathbb{R}$, and let $m.n \in I^o$ and $n > m$. If $\gamma \leq \zeta'(z) \leq \Gamma, z \in [m,n]$ for real constants $\gamma, \Gamma$, then*

$$\left|\zeta(z) - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau - \frac{\zeta(n)-\zeta(m)}{n-m}\left(z - \frac{m+n}{2}\right)\right| \leq \frac{1}{4\sqrt{3}}(n-m)(\Gamma-\gamma)$$

*holds, $\forall\ z \in [m,n]$.*

In the article [2], Barnett et al., by using Čebyšev functional, improved the Matić-Pečarić-Ujević result (1.3) in terms of "Euclidean norm" as under:

**Proposition 1.4.** *Let function $\zeta : [m,n] \to \mathbb{R}$ be an absolutely continuous and derivative $\zeta' \in L_2[m,n]$. If $\gamma \leq \zeta'(\tau) \leq \Gamma$ almost everywhere for $\tau \in [m,n]$, then $\forall$ $z \in [m,n]$*

$$\begin{aligned}
&\left|\zeta(z) - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau - \frac{\zeta(n)-\zeta(m)}{n-m}\left(z - \frac{m+n}{2}\right)\right| \\
&\leq \frac{n-m}{2\sqrt{3}}\left[\frac{1}{n-m}\|\zeta'\|_2^2 - \left(\frac{\zeta(n)-\zeta(m)}{n-m}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4\sqrt{3}}(n-m)(\Gamma-\gamma) \qquad (1.4)
\end{aligned}$$

*holds.*

This article is divided into six sections: the 1st section totally based on introduction and preliminaries. In the 2nd section, we would give our main result about generalization of integral Ostrowski–Grüss type inequalities and would discuss its different special cases. In the 3rd, 4th and 5th sections, using the obtained result we would give some applications to probability theory, special means and numerical integration respectively and the 6th concludes the article.

## 2. New Generalization of Integral Ostrowski–Grüss Inequality

Our main theorem of this section is given in the following:

**Theorem 2.1.** *Let $\zeta : [m,n] \to \mathbb{R}$ be a differentiable function whose 1st derivative belongs to $L_2(m,n)$. If $\gamma \leq \zeta'(\tau) \leq \Gamma$ almost everywhere for $\tau \in [m,n]$, then $\forall\, z \in [m + \lambda\frac{n-m}{2}, \frac{m+n}{2}]$ and $\lambda \in [0,1]$*

$$\begin{aligned}
&\left|(1-\lambda)\frac{\zeta(z)+\zeta(m+n-z)}{2} + \lambda\frac{\zeta(m)+\zeta(n)}{2} - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right| \\
&\leq \left[\frac{(n-m)^2}{12}(3\lambda^2-3\lambda+1) + \left(z-\frac{m+n}{2}\right)^2(1-\lambda)\right. \\
&\left.+\frac{(n-m)(1-\lambda)^2}{2}\left(z-\frac{m+n}{2}\right)\right]^{\frac{1}{2}}\left[\frac{1}{n-m}\|\zeta'\|_2^2 - \left(\frac{\zeta(n)-\zeta(m)}{n-m}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{(n-m)^2}{12}(3\lambda^2-3\lambda+1) + \left(z-\frac{m+n}{2}\right)^2(1-\lambda)\right. \\
&\left.+\frac{(n-m)(1-\lambda)^2}{2}\left(z-\frac{m+n}{2}\right)\right]^{\frac{1}{2}} \qquad (2.1)
\end{aligned}$$

*holds.*

*Proof.* We begin the proof of this theorem by defining the piece–wise continuous function $K : [m,n]^2 \to \mathbb{R}$ for $\lambda \in [0,1]$ as:

$$K(z,\tau;\lambda) = \begin{cases} \tau - m - \lambda\dfrac{(n-m)}{2}, & \text{if } \tau \in [m,z], \\[2ex] \tau - \dfrac{m+n}{2}, & \text{if } \tau \in (z, m+n-z], \\[2ex] \tau - n + \lambda\dfrac{(n-m)}{2}, & \text{if } \tau \in (m+n-z, n], \end{cases}$$

by Korkine's identity

$$T(\zeta,g) := \frac{1}{2(n-m)^2}\int_m^n\int_m^n (\zeta(\tau)-\zeta(s))(g(\tau)-g(s))d\tau ds, \qquad (2.2)$$

we obtain

$$\begin{aligned}
&\frac{1}{n-m}\int_m^n K(z,\tau;\lambda)\zeta'(\tau)d\tau - \frac{1}{n-m}\int_m^n K(z,\tau;\lambda)dt\int_m^n \zeta'(\tau)d\tau \\
&= \frac{1}{2(n-m)^2}\int_m^n\int_m^n (K(z,\tau;\lambda)-K(z,s;\lambda))(\zeta'(\tau)-\zeta'(s))d\tau ds. \qquad (2.3)
\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{n-m}\int_m^n K(z,\tau;\lambda)\zeta'(\tau)d\tau &= (1-\lambda)\frac{\zeta(z)+\zeta(m+n-z)}{2}+\lambda\frac{\zeta(m)+\zeta(n)}{2} \\ &- \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau, \\ \int_m^n K(z,\tau;\lambda)dt &= 0,\end{aligned}$$

then by (2.3) we get the following identity

$$\begin{aligned}(1-\lambda)&\frac{\zeta(z)+\zeta(m+n-z)}{2}+\lambda\frac{\zeta(m)+\zeta(n)}{2}-\frac{1}{n-m}\int_m^n\zeta(\tau)d\tau \\ &= \frac{1}{2(n-m)^2}\int_m^n\int_m^n (K(z,\tau;\lambda)-K(z,s;\lambda))(\zeta'(\tau)-\zeta'(s))d\tau ds, \quad (2.4)\end{aligned}$$

$\forall\, z \in [m+\lambda\frac{n-m}{2}, \frac{m+n}{2}]$ and $\lambda \in [0,1]$.

By applying Cauchy−Schwartz inequality for double integrals, we can write

$$\begin{aligned}&\frac{1}{2(n-m)^2}\left|\int_m^n\int_m^n (K(z,\tau;\lambda)-K(z,s;\lambda))(\zeta'(\tau)-\zeta'(s))d\tau ds\right| \\ &\leq \left(\frac{1}{2(n-m)^2}\int_m^n\int_m^n (K(z,\tau;\lambda)-K(z,s;\lambda))^2 d\tau ds\right)^{\frac{1}{2}} \\ &\times \left(\frac{1}{2(n-m)^2}\int_m^n\int_m^n (\zeta'(\tau)-\zeta'(s))^2 d\tau ds\right)^{\frac{1}{2}}. \quad (2.5)\end{aligned}$$

However,

$$\begin{aligned}&\frac{1}{2(n-m)^2}\int_m^n\int_m^n (K(z,\tau;\lambda)-K(z,s;\lambda))^2 d\tau ds \\ &= \frac{1}{(n-m)}\int_m^n K^2(z,\tau;\lambda)dt - \left(\frac{1}{n-m}\int_m^n K(z,\tau;\lambda)d\tau\right)^2 \\ &= \frac{1}{(n-m)}\left[\frac{2}{3}\left(\left(z-m-\lambda\frac{n-m}{2}\right)^3-\left(z-\frac{m+n}{2}\right)^3\right)+\frac{\lambda^3(n-m)^3}{12}\right]. \quad (2.6)\end{aligned}$$

Consider above terms in the following and simplifying:

$$\begin{aligned}&\left(z-m-\lambda\frac{n-m}{2}\right)^3-\left(z-\frac{m+n}{2}\right)^3 \\ &= \frac{(n-m)^3}{8}(1-\lambda)^3+\frac{3}{2}\left(z-\frac{m+n}{2}\right)^2(n-m)(1-\lambda) \\ &+\frac{3}{4}(n-m)^2(1-\lambda)^2\left(z-\frac{m+n}{2}\right), \quad (2.7)\end{aligned}$$

and

$$\frac{1}{2(n-m)^2}\int_m^n\int_m^n(\zeta'(\tau)-\zeta'(s))^2 d\tau ds = \frac{1}{(n-m)}\|\zeta'\|_2^2 - \left(\frac{\zeta(n)-\zeta(m)}{n-m}\right)^2. \quad (2.8)$$

Using (2.4), (2.6), (2.7) and (2.8), we get the 1st inequality of (2.1). Since $\gamma \leq \zeta'(\tau) \leq \Gamma$ almost everywhere for $\tau \in [m,n]$, by applying Grüss inequality (1.2) we get

$$0 \le \frac{1}{n-m}\int_m^n (\zeta'(\tau))^2 dt - \left(\frac{1}{n-m}\int_m^n \zeta'(\tau)d\tau\right)^2 \le \frac{1}{4}(\Gamma-\gamma)^2, \tag{2.9}$$

which completes the proof of last inequality of (2.1). □

Following remark (Remark 1 of [2]) is also valid for our main result.

*Remark* 2.2. Since $L_\infty[m,n] \subset L_2[m,n]$ (and the inclusion is strict), then we remark that the inequality (2.1) can be applied also for the mappings $\zeta$ whose derivatives are unbounded on $(m,n)$, but $\zeta' \in L_2[m,n]$.

*Remark* 2.3. Since $3\lambda^2 - 3\lambda + 1 \le 1$, $\forall\, \lambda \in [0,1]$ and this is minimum when $\lambda = \frac{1}{2}$. Therefore, (2.1) captures various special cases of main result which is obtained by authors of article [2] as can be seen in remark given below.

*Remark* 2.4. We can get different special cases of (2.1) by using several values of $\lambda$ by fixing $z = \frac{m+n}{2}$. Under the assumptions of Theorem 2.1 following results (special cases) are valid:

**Special Case I:** For $\lambda = 1$ (2.1) gives trapezoid inequality

$$\begin{aligned}&\left|\frac{\zeta(m)+\zeta(n)}{2} - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right| \\ &\le \frac{1}{2\sqrt{3}}\left[(n-m)\|\zeta'\|_2^2 - (\zeta(n)-\zeta(m))^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(n-m),\end{aligned}$$

which is Remark 3.2 $(i)$ of [14].

**Special Case II:** For $\lambda = 0$ (2.1) gives mid−point ineqality

$$\begin{aligned}&\left|\zeta\left(\frac{m+n}{2}\right) - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right| \\ &\le \frac{1}{2\sqrt{3}}\left[(n-m)\|\zeta'\|_2^2 - (\zeta(n)-\zeta(m))^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(n-m).\end{aligned}$$

which is Corollary 1 of [2] and Remark 3.2 $(ii)$ of [14].

**Special Case III:** For $\lambda = \frac{1}{2}$ (2.1) gives averaged mid−point and trapezoid inequality

$$\begin{aligned}&\left|\frac{\zeta(m)+2\zeta\left(\frac{m+n}{2}\right)+\zeta(n)}{4} - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right| \\ &\le \frac{1}{4\sqrt{3}}\left[(n-m)\|\zeta'\|_2^2 - (\zeta(n)-\zeta(m))^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{8\sqrt{3}}(\Gamma-\gamma)(n-m).\end{aligned}$$

which is Remark 3.2 $(iii)$ of [14].

**Special Case IV:** For $\lambda = \frac{1}{3}$ (2.1) gives a variant of Simpson's inequality for differentiable function $\zeta$

$$\left|\frac{\zeta(m) + 4\zeta\left(\frac{m+n}{2}\right) + \zeta(n)}{6} - \frac{1}{n-m}\int_m^n \zeta(\tau)d\tau\right|$$
$$\leq \frac{1}{6}\left[(n-m)\|\zeta'\|_2^2 - (\zeta(n) - \zeta(m))^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{12}(\Gamma - \gamma)(n-m).$$

which is Remark 3.2 $(iv)$ of [14].

## 3. Application to Probability Theory

Suppose random variable 'Z' be continuous with PDF $\zeta : [m,n] \to \mathbb{R}_+$ and CDF $\Phi : [m,n] \to [0,1]$ is defined as

$$\Phi(z) = \int_m^z \zeta(\tau)d\tau, \quad z \in \left[m + \lambda\frac{n-m}{2}, \frac{m+n}{2}\right],$$

and

$$E(Z) = \int_m^n \tau\zeta(\tau)d\tau,$$

is expectation of random variable 'Z' on $[m,n]$. Then we have following result:

**Theorem 3.1.** *Let the suppositions of Theorem* 2.1 *be valid and if PDF* $\zeta \in L_2[m,n]$, *then*

$$\left|(1-\lambda)\frac{\Phi(z) + \Phi(m+n-z)}{2} + \frac{\lambda}{2} - \frac{n - E(Z)}{n-m}\right|$$
$$\leq \frac{1}{n-m}\left[\frac{(n-m)^2}{12}(3\lambda^2 - 3\lambda + 1) + \left(z - \frac{m+n}{2}\right)^2(1-\lambda)\right.$$
$$\left. + \frac{(n-m)(1-\lambda)^2}{2}\left(z - \frac{m+n}{2}\right)\right]^{\frac{1}{2}}\left[(n-m)\|\Phi'\|_2^2 - 1\right]^{\frac{1}{2}}$$
$$\leq \frac{(H-h)}{2}\left[\frac{(n-m)^2}{12}(3\lambda^2 - 3\lambda + 1) + \left(z - \frac{m+n}{2}\right)^2(1-\lambda)\right.$$
$$\left. + \frac{(n-m)(1-\lambda)^2}{2}\left(z - \frac{m+n}{2}\right)\right]^{\frac{1}{2}}, \tag{3.1}$$

*where* $h \leq \Phi'(\tau) \leq H$, $\forall\, \tau \in [m,n]$.

*Proof.* Put $\zeta = \Phi$ in (2.1) we obtain (3.1), by applying the identity

$$\int_m^n \Phi(\tau)d\tau = n - E(Z) \quad \text{where} \quad \Phi(m) = 0,\ \Phi(n) = 1.$$

□

**Corollary 3.2.** *Under the assumptions as stated in Theorem 3.1, if we put* $z = \frac{m+n}{2}$, *then*

$$\left|(1-\lambda)\Phi\left(\frac{m+n}{2}\right)+\frac{\lambda}{2}-\frac{n-E(Z)}{n-m}\right|$$
$$\leq \frac{1}{2\sqrt{3}}(3\lambda^2-3\lambda+1)^{\frac{1}{2}}\left[(n-m)\|\Phi'\|_2^2-1\right]^{\frac{1}{2}}$$
$$\leq \frac{(n-m)}{4\sqrt{3}}(3\lambda^2-3\lambda+1)^{\frac{1}{2}}(H-h)$$

*hold for* $h \leq \Phi'(\tau) \leq H \ \forall \tau \in [m,n]$.

*Remark* 3.3. The Corollary 3.2 is in fact Corollary 3.1 of [14].

## 4. Application to Special Means

Before we proceed further we need here some definitions of special means.

**Special Means:** These means can be found in [14].

(a) Arithmetic Mean

$$A = \frac{m+n}{2}; \quad m,n \geq 0.$$

(b) Geometric Mean

$$G = G(m,n) = \sqrt{mn}; \quad m,n \geq 0.$$

(c) Harmonic Mean

$$H = H(m,n) = \frac{2}{\frac{1}{m}+\frac{1}{n}}; \quad m,n > 0.$$

(d) Logarithmic Mean

$$L = L(m,n) = \begin{cases} m, & \text{if } m=n \\ \dfrac{n-m}{\ln n - \ln m}, & \text{if } m \neq n; \end{cases} \quad m,n > 0.$$

(e) Identric Mean

$$I = I(m,n) = \begin{cases} m, & \text{if } m=n \\ \ln\left(\dfrac{\left(\frac{n^n}{m^m}\right)^{\frac{1}{n-m}}}{e}\right), & \text{if } m \neq n; \end{cases} \quad m,n > 0.$$

(f) $p$–Logarithmic Mean

$$L_p = L_p(m,n) = \begin{cases} m, & \text{if } m=n \\ \left(\dfrac{n^{p+1}-m^{p+1}}{(p+1)(n-m)}\right)^{\frac{1}{p}}, & \text{if } m \neq n, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}, m,n > 0$. It is known that "$L_p$ is monotonically increasing over $p \in \mathbb{R}$", "$L_0 = I$" and "$L_{-1} = L$".

*Example* 4.1. Consider $\zeta(z) = z^p$, $p \in \mathbb{R}\backslash\{-1, 0\}$, then for $n > m$

$$\begin{aligned}
\frac{1}{n-m}\int_m^n \zeta(\tau)d\tau &= L_p^p(m,n), \\
\frac{\zeta(n)-\zeta(m)}{n-m} &= pL_{p-1}^{p-1}(m,n), \\
\frac{\zeta(m)+\zeta(n)}{2} &= A(m^p,n^p), \\
\frac{m+n}{2} &= A \\
\text{and} \quad \frac{1}{n-m}\|\zeta'\|_2^2 &= \frac{1}{n-m}\int_m^n |\zeta'(\tau)|^2 dt \\
&= p^2 L_{2(p-1)}^{2(p-1)},
\end{aligned}$$

where $z \in [m + \lambda\frac{n-m}{2}, \frac{m+n}{2}]$.

Therefore, (2.1) becomes

$$\begin{aligned}
&\left|(1-\lambda)\frac{z^p+(m+n-z)^p}{2} + \lambda A(m^p,n^p) - L_p^p\right| \leq \\
&|p|\left[\frac{(n-m)^2}{12}(3\lambda^2-3\lambda+1) + (z-A)^2(1-\lambda) + \frac{(n-m)(1-\lambda)^2}{2}(z-A)\right]^{\frac{1}{2}} \\
&\times\left[L_{2(p-1)}^{2(p-1)} - L_{(p-1)}^{2(p-1)}\right]^{\frac{1}{2}}. \qquad (4.1)
\end{aligned}$$

Choose $z = A$ in (4.1), get

$$\begin{aligned}
&\left|(1-\lambda)A^p + \lambda A(m^p,n^p) - L_p^p\right| \\
&\qquad \leq |p|\frac{n-m}{2\sqrt{3}}(3\lambda^2-3\lambda+1)^{\frac{1}{2}}\left[L_{2(p-1)}^{2(p-1)} - L_{(p-1)}^{2(p-1)}\right]^{\frac{1}{2}},
\end{aligned}$$

which is minimum for $\lambda = \frac{1}{2}$. Moreover for $\lambda = 1$

$$\left|A(m^p,n^p) - L_p^p\right| \leq \frac{n-m}{2\sqrt{3}}|p|\left[L_{2(p-1)}^{2(p-1)} - L_{(p-1)}^{2(p-1)}\right]^{\frac{1}{2}}.$$

*Example* 4.2. Consider $\zeta(z) = \frac{1}{z}$, $z \neq 0$, then

$$\begin{aligned}
\frac{1}{n-m}\int_m^n \zeta(\tau)d\tau &= L^{-1}(m,n), \\
\frac{\zeta(n)-\zeta(m)}{n-m} &= -\frac{1}{G^2}, \\
\frac{\zeta(m)+\zeta(n)}{2} &= \frac{A}{G^2}, \\
\frac{1}{n-m}\int_m^n |\zeta'(\tau)|^2 dt &= \frac{m^2+mn+n^2}{3m^3n^3} \\
\text{and} \quad \frac{1}{n-m}\int_m^n |\zeta'(\tau)|^2 dt - \left(\frac{\zeta(n)-\zeta(m)}{n-m}\right)^2 &= \frac{(n-m)^2}{3m^3n^3} = \frac{(n-m)^2}{3G^6},
\end{aligned}$$

where $z \in [m + \lambda\frac{n-m}{2}, \frac{m+n}{2}] \subset (0, \infty)$.

Therefore, (2.1) becomes

$$\left|\frac{(1-\lambda)}{2}\left(\frac{1}{z}+\frac{1}{(m+n-z)}\right)+\lambda\frac{A}{G^2}-\frac{1}{L}\right| \leq \left[\frac{(n-m)^2}{12}(3\lambda^2-3\lambda+1)+(z-A)^2(1-\lambda) + \frac{(n-m)(1-\lambda)^2}{2}(z-A)\right]^{\frac{1}{2}}\frac{(n-m)}{\sqrt{3}G^3}. \quad (4.2)$$

If we choose z = A in (4.2), we get

$$\left|(1-\lambda)\frac{1}{A}+\lambda\frac{A}{G^2}-\frac{1}{L}\right| \leq \frac{(n-m)^2}{6G^3}(3\lambda^2-3\lambda+1)^{\frac{1}{2}}.$$

For $\lambda = 1$

$$\left|\frac{A}{G^2}-\frac{1}{L}\right| \leq \frac{(n-m)^2}{6G^3}.$$

*Example* 4.3. Consider $\zeta(z) = \ln z$ $z > 0$, then

$$\begin{aligned}
\frac{1}{n-m}\int_m^n \zeta(\tau)d\tau &= \ln(I(m,n)),\\
\frac{\zeta(n)-\zeta(m)}{n-m} &= \frac{1}{L},\\
\frac{\zeta(m)+\zeta(n)}{2} &= \ln G,\\
\frac{1}{n-m}\int_m^n |\zeta'(\tau)|^2 dt &= \frac{1}{G^2} \quad \text{and}\\
\frac{1}{n-m}\int_m^n |\zeta'(\tau)|^2 dt - \left(\frac{\zeta(n)-\zeta(m)}{n-m}\right)^2 &= \frac{L^2-G^2}{L^2G^2},
\end{aligned}$$

where $z \in [m+\lambda\frac{n-m}{2}, \frac{m+n}{2}] \subset (0,\infty)$.

Therefore, (2.1) becomes

$$\left|\ln\left(\frac{(z(m+n-z))^{\frac{(1-\lambda)}{2}} G^\lambda}{I}\right)\right| \leq \left[\frac{(n-m)^2}{12}(3\lambda^2-3\lambda+1)+(z-A)^2(1-\lambda)+\frac{(n-m)(1-\lambda)^2}{2}(z-A)\right]^{\frac{1}{2}} \times\frac{(L^2-G^2)^{\frac{1}{2}}}{LG}.$$

For $z = A$

$$\left|\ln\left(\frac{A^{(1-\lambda)}G^\lambda}{I}\right)\right| \leq \frac{(n-m)}{2\sqrt{3}LG}\left((3\lambda^2-3\lambda+1)(L^2-G^2)\right)^{\frac{1}{2}}.$$

For $\lambda = 1$

$$\left|\ln\left(\frac{G}{I}\right)\right| \leq \frac{(n-m)}{2\sqrt{3}LG}(L^2-G^2)^{\frac{1}{2}}.$$

## 5. Application to Numerical Integration

To get the composite quadrature rules, we have to let $I_j : m = z_0 < z_1 < \cdots < z_{j-1} < z_j = n$ be the partision of the interval $[m,n]$, $h_j = z_{j+1} - z_j$, $\lambda \in [0,1]$, $z_j + \lambda \frac{h_j}{2} \leq \eta_j \leq \frac{z_j + z_{j+1}}{2}$, $j \in \{0, \ldots, i-1\}$, then the following results hold:

**Theorem 5.1.** *If $\gamma \leq \zeta'(\tau) \leq \Gamma$ almost everywhere for $\tau \in [z_j + \lambda \frac{h_j}{2}, z_{j+1}]$ ($j \in \{0, \ldots, i-1\}$), then Under the assumptions of Theorem 2.1 the following quadrature formula holds*

$$\int_m^n \zeta(\tau) d\tau = Q(\zeta, \zeta', I_j, \eta, \lambda) + R(\zeta, \zeta', I_j, \eta, \lambda), \tag{5.1}$$

*where*

$$\begin{aligned} &Q(\zeta, \zeta', I_j, \eta, \lambda) \\ &= \sum_{j=0}^{i-1} h_j \left[ (1-\lambda) \frac{\zeta(\eta_j) + \zeta(z_j + z_{j+1} - \eta_j)}{2} + \lambda \frac{\zeta(z_j) + \zeta(z_{j+1})}{2} \right] \end{aligned} \tag{5.2}$$

*and remainder $R$ satisfies the estimate*

$$\begin{aligned} |R(\zeta, \zeta', I_j, \eta, \lambda)| &\leq \sum_{j=0}^{i-1} \left[ \frac{h_j^2}{12}(3\lambda^2 - 3\lambda + 1) + \left( \eta_j - \frac{z_j + z_{j+1}}{2} \right)^2 (1-\lambda) \right. \\ &\left. + \frac{h_j (1-\lambda)^2}{2} \left( \eta_j - \frac{z_j + z_{j+1}}{2} \right) \right]^{\frac{1}{2}} \left[ h_j \|\zeta'\|_2^2 - (\zeta(z_{j+1}) - \zeta(z_j))^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{2}(\Gamma - \gamma) \sum_{j=0}^{i-1} h_j \left[ \frac{h_j^2}{12}(3\lambda^2 - 3\lambda + 1) + \left( \eta_j - \frac{z_j + z_{j+1}}{2} \right)^2 (1-\lambda) \right. \\ &\left. + \frac{h_j (1-\lambda)^2}{2} \left( \eta_j - \frac{z_j + z_{j+1}}{2} \right) \right]^{\frac{1}{2}}. \end{aligned} \tag{5.3}$$

*Proof.* By using inequalities (2.4), (2.5) and (2.9) on $z_j + \lambda \frac{h_j}{2} \leq \eta_j \leq \frac{z_j + z_{j+1}}{2}$ and summing over $j$ from 0 to $i-1$, then we get required result. □

By putting several values of $\lambda$ and by fixing $\eta_j = \frac{z_j + z_{j+1}}{2}$, under the assumptions of Theorem 5.1 following results (special cases) are valid.

**Special Case I:** Put $\lambda = 1$ in (5.2) and (5.3), we have

$$Q\left( \zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, 1 \right) = \frac{1}{2} \sum_{j=0}^{i-1} h_j (\zeta(z_j) + \zeta(z_{j+1}))$$

and

$$\begin{aligned} &\left| R\left( \zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, 1 \right) \right| \\ &\qquad \leq \frac{1}{2\sqrt{3}} \sum_{j=0}^{i-1} h_j \left[ h_j \|\zeta'\|_2^2 - (\zeta(z_{j+1}) - \zeta(z_j))^2 \right]^{\frac{1}{2}} \leq \frac{1}{4\sqrt{3}} (\Gamma - \gamma) \sum_{j=0}^{i-1} h_j^2. \end{aligned}$$

**Special Case II:** Put $\lambda = 0$ in (5.2) and (5.3), we have

$$Q\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}\right) = \sum_{j=0}^{i-1} h_j \zeta(\frac{z_j + z_{j+1}}{2})$$

and

$$\left| R\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}\right) \right| \leq \frac{1}{2\sqrt{3}} \sum_{j=0}^{i-1} h_j \left[h_j \|\zeta'\|_2^2 - (\zeta(z_{j+1}) - \zeta(z_j))^2\right]^{\frac{1}{2}} \leq \frac{1}{4\sqrt{3}} (\Gamma - \gamma) \sum_{j=0}^{i-1} h_j^2.$$

**Special Case III:** Put $\lambda = \frac{1}{2}$ in (5.2) and (5.3), we have

$$Q\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, \frac{1}{2}\right) = \frac{1}{4} \sum_{j=0}^{i-1} h_j \left(\zeta(z_j) + 2\zeta(\frac{z_j + z_{j+1}}{2}) + \zeta(z_{j+1})\right)$$

and

$$\left| R\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, \frac{1}{2}\right) \right| \leq \frac{1}{4\sqrt{3}} \sum_{j=0}^{i-1} h_j \left[h_j \|\zeta'\|_2^2 - (\zeta(z_{j+1}) - \zeta(z_j))^2\right]^{\frac{1}{2}} \leq \frac{1}{8\sqrt{3}} (\Gamma - \gamma) \sum_{j=0}^{i-1} h_j^2.$$

**Special Case IV:** Put $\lambda = \frac{1}{3}$ in (5.2) and (5.3), we have

$$Q\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, \frac{1}{3}\right) = \frac{1}{6} \sum_{j=0}^{i-1} h_j \left(\zeta(z_j) + 4\zeta(\frac{z_j + z_{j+1}}{2}) + \zeta(z_{j+1})\right)$$

and

$$\left| R\left(\zeta, \zeta', I_j, \frac{z_j + z_{j+1}}{2}, \frac{1}{3}\right) \right| \leq \frac{1}{6} \sum_{j=0}^{i-1} h_j \left[h_j \|\zeta'\|_2^2 - (\zeta(z_{j+1}) - \zeta(z_j))^2\right]^{\frac{1}{2}} \leq \frac{1}{12} (\Gamma - \gamma) \sum_{j=0}^{i-1} h_j^2.$$

## 6. Conclusion

Using three step kernel, we have obtained new generalized Ostrowski−Grüss type inequalities (2.1) which is a variant of (1.4) which was obtained in article [2]. By fixing $z = \frac{m+n}{2}$ and by choosing different values of parameter $\lambda$ we captured many results stated in [2] and [14]. Furthermore, we also got different important results as our main results' special cases including trapezoidal inequality, mid-point inequality, averaged mid−point and trapezoidal inequality and Simpson's inequality. Moreover, applications are deduced for probability theory, special means and numerical integration.

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