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An efficient semi-analytical method for solving the generalized regularized long wave equations with a new fractional derivative operator

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ABSTRACT

In this work, the newly developed optimal perturbation iteration technique with Laplace transform is applied to the generalized regularized long wave equations with a new fractional operator to obtain new approximate solutions. We transform the classical generalized regularized long wave equations to fractional differential form by using the Atangana-Baleanu fractional derivative which is defined with the Mittag-Leffler function. To show the efficiency of the proposed method, a numerical example is given for different values of physical parameters.

1. Introduction

Partial differential equations, especially nonlinear ones, have been used to model many scientific phenomena in applied mathematics and engineering. The importance of getting approximate solutions of them either numerically or analytically has always been emphasized. From the early 2000s onwards, many researchers have constructed a variety of techniques to analyze the solutions of nonlinear partial differential equations, such as the sine-Gordon expansion method (Baskonus et al., 2017), the extended sinh-Gordon equation expansion method (Cattani et al., 2018), Bernoulli sub-equation function method (Baskonus and Bulut, 2016), homotopy analysis method (Liao, 2004), modified simple equation method (Khan et al., 2016), homotopy perturbation method (HPM) (Dubey et al., 2016) and Adomian decomposition method (Deniz and Bildik, 2014). Due to the inability of these methods for many works, researchers have proposed new methods such as perturbation iteration method (Aksoy and Pakdemirli, 2010; Aksoy et al., 2012), optimal homotopy asymptotic method (Bildik and Deniz, 2020a; Marinca and Herisanu, 2008; Bildik and Deniz, 2018a; Iqbal et al., 2010) and optimal perturbation iteration method (OPIM) (Deniz and Bildik, 2017a,b, 2018; Deniz, 2017; Bildik and Deniz, 2017a,b, 2018b,c) to deeply analyze nonlinear models.

One of the most important nonlinear partial differential equations is the generalized regularized long wave (GRLW) equation which can be given as

$$u_t + u_x + \gamma(u^p)_x - \beta u_{xxt} = 0, \quad (x,t) \in (a,b) \times (0,T) \tag{1.1}$$

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where $\gamma,\ \beta$ are positive constants and p is a positive integer. The Eq. (1.1) was initially used by Peregrine for presenting small-amplitude long-waves on the surface of water in a channel (Peregrine, 1996). Besides that, GRLW equations are used for describing a variety of physical phenomena such as longitudinal dispersive waves in elastic rods, rotating flow down a tube and pressure waves in liquid–gas bubble mixtures. The nonlinear term $\gamma(u^p)_x$ in Eq. (1.1) causes steepening of the wave form. Also, the last term βu_{xxt} is named as dispersion effect term and this term is used to make the wave form spread. The solitons emerge by virtue of the balance between dispersion and nonlinearity (Mohammadi and Mokhtari, 2011). These solitons exist in many types of systems from sky to laboratory (Bhardwaj and Shankar, 2000). Many researchers have also demonstrated that GRLW equation is superior to the KdV equation and can be used instead of KdV equation in many nonlinear systems (Bona et al., 1983).

The Eq. (1.1) can be modified to the regularized long wave equation (RLW) for $p = 1$ or to the modified regularized long wave (MRLW) equation for $p = 2$. The development of an undular bore can be described by the RLW and MRLW equations. The constants $\gamma,\ \beta$ in the Eq. (1.1) characterise the behavior of an undular bore. In many fields of mathematics and engineering such as magneto-hydrodynamics waves in plasma, lossless propagation of shallow water waves, rotating flow down a tube, longitudinal dispersive waves in elastic rods, ion-acoustic waves in plasma,pressure waves in liquid–gas bubble mixture and thermally excited phonon packets in low temperature nonlinear crystals, RLW and MRLW equations play a pivotal role (Mirzaei and Dehghan, 2011; Dehghan and Salehi, 2011).

It is widely known that any dynamical system defined with the help of fractional order differential operators has a memory effect. In other words, the future state of a physical system depends on the present as well as the past states (Kumar et al., 2017, 2018; Singh et al., 2017; Singh and Srivastava, 2020; Srivastava et al., 2019a,b, 2020a). Therefore, it is reasonable to transform any differential equations to fractional ones to deeply analyze the solutions. Many problems have been reconsidered via fractional derivatives and newly developed techniques. Numerical solution of Caputo-Fabrizio time fractional distributed order reaction–diffusion equation has been obtained via quasi wavelet based numerical method (Kumar and Gómez-Aguilar, 2020). Shifted Chebyshev collocation of the fourth kind with convergence analysis has been used for solving the space–time fractional advection–diffusion equation (Safdari et al., 2020). Modified fractional derivatives with non-singular kernel have been analyzed via Laplace variational- iteration method (Yépez-Martínez and Gómez-Aguilar, 2020). Numerical solutions of the fractional Fisher's type equations have been obtained by using spectral collocation methods (Saad et al., 2019a). Laplace homotopy analysis method has been applied for solving linear partial differential equations using a fractional derivative (Morales-Delgado et al., 2016). Many other papers can also be seen in Pandey et al. (2020), Bhangale et al. (2020), Dwivedi et al. (2020), Bonyah et al. (2021) and Deniz (2020a)).

In 2016, Baleanu and Atangana came up with new operators, namely AB operators, with fractional order based upon the well-known Mittag–Leffler function to come through the kernel problems of the Caputo-Fabrizio and Caputo-Riemann–Liouville derivatives (Atangana and Baleanu, 2016). AB operators have all the utilities of those of past counterpasts apart from all these; the kernel of the operator is nonsingular and nonlocal. Additionally, AB fractional integral is the fractional average of the Riemann–Liouville (RL) fractional integral of the function. These new prospective ideas on fractional operators have drawn attention of many researchers. A lot of manuscripts have been written in only 4–5 years. One can see the most effective ones in Saad (2018), Algahtani (2016), Gómez-Aguilar (2018), Bildik and Deniz (2019), Saad et al. (2018, 2019b), Jajarmi and Baleanu (2018), Kilbas et al. (2006), Srivastava (2020a,b), Srivastava et al. (2019c, 2020b) and references therein.

In this research paper, we aim to extend the GRLW equations by interchanging the derivative with a newly constructed AB – derivative to get

$$ {}_0^{ABC}D_t^{\alpha}u + u_x + \gamma(u^p)_x - \beta u_{xxt} = 0, (x,t) \in (a,b)\times(0,T), 0 < \alpha \leqslant 1 \tag{1.2} $$

and additionally to exhibit the approximate solutions of the modified fractional GRLW equations with the help of optimal perturbation iteration technique and Laplace technique.

2. Preliminaries and some definitions

In this current part, we present some important preliminaries and results of AB derivative. One can see Atangana and Baleanu (2016) for much more information.

Definition 1. The Atangana-Baleanu (AB) time-fractional derivative in the Caputo sense is given as

$$ {}_a^{ABC}D_t^{\alpha}f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f\prime(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau \tag{2.1} $$

where $B(\alpha)$ holds the property that $B(0) = B(1) = 1,\ f \in L_1(a,b)$, $\alpha \in [0,1]$ and $E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$ is Mittag-Leffler function.

Definition 2. The Atangana-Baleanu (AB) time-fractional derivative in the RL sense is given as (Atangana and Baleanu, 2016)

$$ {}_a^{ABR}D_t^{\alpha}f(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(\tau)E_{\alpha}\left[-\frac{\alpha(t-\tau)^{\alpha}}{1-\alpha}\right]d\tau \tag{2.2} $$

where $\alpha \in [0,1],\ f \in L_1(a,b)$ and not differentiable.

Definition 3. The Laplace transform of the AB fractional derivative in the Caputo sense ${}_a^{ABC}D_t^{\alpha}f(t)$ has the form

$$ \mathscr{L}\left\{{}_a^{ABC}D_t^{\alpha}f(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha}\,\frac{s^{\alpha}Lf(t)(s) - s^{\alpha-1}f(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.3} $$

Definition 4. The Laplace transform of the AB fractional derivative in the RL sense ${}_a^{ABR}D_t^{\alpha}f(t)$ has the form

$$ \mathscr{L}\left\{{}_a^{ABR}D_t^{\alpha}f(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha}\,\frac{s^{\alpha}Lf(t)(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.4} $$

3. Analysis of fractional GRLW equations via OPIM

Optimal perturbation iteration method (OPIM) is first constructed by Bildik and Deniz by using the ideas of perturbation iteration (Aksoy and Pakdemirli, 2010; Aksoy et al., 2012) and optimal homotopy asymptotic methods (Marinca and Herisanu, 2008; Bildik and Deniz, 2018a; Iqbal et al., 2010). Many different types of problems have been solved by using this techniques (Deniz, 2020b, c; Agarwal et al., 2020; Deniz, 2020d; Bildik et al., 2020; Deniz et al., 2020; Bildik and Deniz, 2020b,c). In this section, we use OPIM and Laplace transform to obtain approximate solutions of the extended fractional GRLW equations.

Let us consider the Eq. (1.2) with the initial condition $u(x,0) = A$. Then, by applying the Laplace transform to the Eq. (1.2) and with the help of the definitions in the previous section, one can get

$$\mathscr{L}[u] - \frac{1}{s}A + \frac{\alpha(-1+s^{-\alpha})+1}{B(\alpha)}\mathscr{L}[F(u_{xxt}, u_x, u)] = 0 \tag{3.1}$$

where

$$F = F(u_{xxt}, u_x, u, \varepsilon) = u_x + \gamma(u^p)_x - \beta u_{xxt} \tag{3.2}$$

is a nonlinear term. Now, we use OPIM to decompose the nonlinear term. The following formulation can be used to summarize the technique:

(a) The perturbation parameter can be artificially embedded into (3.2) as

$$F(u_{xxt}, u_x, u, \varepsilon) = 0 \tag{3.3}$$

to handle with nonlinear terms. For instance, in our case, $\varepsilon = 1$ can be inserted into the Eq. (3.2) as:

$$F = u_x + \varepsilon\gamma(u^p)_x - \beta u_{xxt} = 0. \tag{3.4}$$

(b) One can take an approximate solution with one correction term in the perturbation expansion as follows:

$$u_{n+1} = u_n + \varepsilon(u_c)_n \tag{3.5}$$

where $n \in \mathbb{N}$. Upon substitution of (3.5) into (3.4), expanding in a Taylor series with first derivatives only gives the following algorithm:

$$F + F_u(u_c)_n\varepsilon + F_{u_x}((u_c)_n)_x\varepsilon + F_{u_{xxt}}((u_c)_n)_{xxt}\varepsilon + F_\varepsilon\varepsilon = 0 \tag{3.6}$$

where

$$F_u = \frac{\partial F}{\partial u}, \; F_{u_x} = \frac{\partial F}{\partial u_x}, \; F_{u_{xxt}} = \frac{\partial F}{\partial u_{xxt}}, \; F_\varepsilon = \frac{\partial F}{\partial \varepsilon}.$$

Using the Eq. (3.1) and computing all derivatives, functions at $\varepsilon = 0$ gives

$$\mathscr{L}[u_n] - \frac{1}{s}A + \frac{\alpha(-1+s^{-\alpha})+1}{B(\alpha)}\mathscr{L}\left[((u_c)_n)_x + \beta(u_n)_{xxt} - (u_n)_x - ((u_c)_n)_{xxt}\right] = 0. \tag{3.7}$$

(3.7) is an iteration procedure for OPIM algorithms of fractional GRLW Eqs. (1.2). One can begin to iterate by picking a first trial function u_0 which should satisfy the prescribed conditions. By doing that, first correction term $(u_c)_0$ can be obtained from the algorithm (3.7) by using u_0 and condition(s).

c) In order to enhance the accuracy of the results and effectiveness of the method, we offer to use the following formula

$$u_{n+1} = u_n + P_n(u_c)_n \tag{3.8}$$

where $P_0, P_1, P_2, \ldots$ are convergence control parameters which alters us to adjust the convergence.

Performing the calculations for $n = 0, 1, \ldots$, one can get more approximate solutions as follows:

$$\begin{aligned} &u_1 = u(x,t;P_0) = u_0 + P_0(u_c)_0 \\ &u_2(x,t;P_0,P_1) = u_1 + P_1(u_c)_1 \\ &\quad\vdots \\ &u_m(x,t;P_0,\ldots,P_{m-1}) = u_{m-1} + P_{m-1}(u_c)_{m-1} \end{aligned} \tag{3.9}$$

d) Substituting the approximate solution u_m into the Eq. (1.2), the general problem is transformed to the following residual:

$$Re(x,t;P_0,\ldots,P_{m-1}) = F((u_m)_{xxt}, (u_m)_x, (u_m)_t, (u_m)) \tag{3.10}$$

Undoubtedly, if $Re(x,t;P_0,\ldots,P_{m-1}) = 0$ then the approximation $u_m(x,t;P_0,\ldots,P_{m-1})$ is the exact solution. However, this case doesn't usually arise in nonlinear differential equations, but the functional can be minimized as:

$$J(P_0,\ldots,P_{m-1}) = \int_0^T \int_a^b Re^2(x,t;P_0,\ldots,P_{m-1})dxdt \tag{3.11}$$

where a, b and T are taken from the domain of the problem. Optimal values of $P_0, P_1, \ldots$ can be received from the conditions

$$\frac{\partial J}{\partial P_0} = \frac{\partial J}{\partial P_1} = \ldots = \frac{\partial J}{\partial P_{m-1}} = 0. \tag{3.12}$$

If the Eq. (3.12) may be very difficult to solve or it can take too much CPU time. In that case, the constants $P_0, P_1, \ldots$may be obtained from

$$Re(x_0,t_0;P_i) = Re(x_1,t_1;P_i) = \cdots = Re(x_{m-1},t_{m-1};P_i) = 0,\quad i = 0,1,\ldots,m-1 \tag{3.13}$$

where $x_i, t_i \in (a,b)\times(0,T)$. As it is known, this method is called collocation technique. There is no general theorem for selecting collocation points. For more information about obtaining these constants, readers may refer to Marinca and Herisanu (2008), Iqbal et al. (2010) and Deniz and Sezer (2020).

4. Convergence analysis of the proposed technique

In this section, convergence analysis of the optimal perturbation iteration technique is investigated by using Banach fixed-point theorem. To achieve that, we first reorganize the approximate solutions with different indexes such as:

$$\begin{aligned} &u_0 = \Upsilon_0, \\ &P_n(u_c)_n = \Upsilon_{n+1} \end{aligned} \tag{4.1}$$

and correspondingly one can get

$$\begin{aligned} &u_0 = \Upsilon_0 \\ &u_1 = u(x,t;P_0) = u_0 + P_0(u_c)_0 = \Upsilon_0 + \Upsilon_1 \\ &u_2 = u(x,t;P_0,P_1) = u_1 + P_1(u_c)_1 = \Upsilon_0 + \Upsilon_1 + \Upsilon_2 \\ &\quad\vdots \\ &u_n = u(x,t;P_0,\ldots,P_{n-1}) = \Upsilon_0 + \Upsilon_1 + \ldots + \Upsilon_n \end{aligned} \tag{4.2}$$

Thereby, the n-th order OPIM solution may be represnted as:

$$u_n(x,t;P_0,\ldots,P_{n-1}) = \Upsilon_0(x,t) + \sum_{j=1}^{n}\Upsilon_j(x,t;P_0,\ldots P_{j-1}). \tag{4.3}$$

Theorem 1. *Let $\mathscr{B}$be a Banach space denoted by a norm $\|.\|$over which the series (4.3) is given. In addition to that, we assume that the initial function $u_0 = \Upsilon_0$falls into the ball of the desired solution. (4.3) converges if there is a β such that*

$$\|\Upsilon_{n+1}\| \leqslant \beta\|\Upsilon_n\|. \tag{4.4}$$

Proof. At first, a sequence may be constructed as:

$$\begin{aligned} &R_0 = \Upsilon_0 \\ &R_1 = \Upsilon_0 + \Upsilon_1 \\ &R_2 = \Upsilon_0 + \Upsilon_1 + \Upsilon_2 \\ &\quad\vdots \\ &R_n = \Upsilon_0 + \Upsilon_1 + \Upsilon_2 + \cdots + \Upsilon_n. \end{aligned} \tag{4.5}$$

Table 1
Absolute residual errors of the third and fourth order OPIM solutions at $x = 30$.

t	Third Order, $\alpha = 0.5$	Third Order, $\alpha = 0.8$	Fourth Order, $\alpha = 0.5$	Fourth Order, $\alpha = 0.8$
0.1	2.0850×10^{-12}	5.1034×10^{-12}	6.0147×10^{-15}	8.0125×10^{-16}
0.2	4.0293×10^{-12}	2.2231×10^{-12}	5.2066×10^{-15}	8.8804×10^{-16}
0.3	6.4350×10^{-12}	5.7742×10^{-13}	3.0156×10^{-15}	8.9034×10^{-16}
0.4	2.9716×10^{-12}	8.1069×10^{-12}	9.088×10^{-14}	8.111×10^{-17}
0.5	5.1784×10^{-12}	2.7072×10^{-12}	5.0124×10^{-14}	5.3307×10^{-16}
0.6	7.7258×10^{-12}	8.0142×10^{-12}	3.052×10^{-14}	7.0125×10^{-16}
0.7	7.4391×10^{-12}	5.138×10^{-12}	6.077×10^{-14}	2.1055×10^{-16}
0.8	9.5418×10^{-12}	1.0054×10^{-13}	1.0452×10^{-14}	7.1104×10^{-16}
0.9	1.2037×10^{-12}	8.7924×10^{-12}	3.7434×10^{-14}	6.022×10^{-16}
1.	3.3024×10^{-12}	9.9602×10^{-12}	6.9985×10^{-14}	1.4457×10^{-16}

Table 2
Absolute residual errors of the third and fourth order OPIM solutions at $x = 100$.

t	Third Order, $\alpha = 0.5$	Third Order, $\alpha = 0.8$	Fourth Order, $\alpha = 0.5$	Fourth Order, $\alpha = 0.8$
0.1	1.0124×10^{-10}	2.0564×10^{-14}	8.1051×10^{-14}	9.5099×10^{-15}
0.2	2.1112×10^{-10}	7.1051×10^{-10}	9.8222×10^{-14}	9.0441×10^{-15}
0.3	5.3224×10^{-9}	6.9901×10^{-11}	1.3655×10^{-13}	7.4301×10^{-15}
0.4	7.7758×10^{-9}	7.0201×10^{-11}	2.0215×10^{-12}	1.7893×10^{-16}
0.5	6.0447×10^{-10}	8.3069×10^{-10}	5.2217×10^{-14}	1.4520×10^{-17}
0.6	6.1044×10^{-9}	7.0132×10^{-12}	9.6321×10^{-15}	3.5714×10^{-17}
0.7	2.0057×10^{-9}	5.0433×10^{-11}	1.0635×10^{-14}	1.4223×10^{-18}
0.8	9.0291×10^{-10}	8.7852×10^{-11}	5.5524×10^{-14}	7.0211×10^{-17}
0.9	1.2552×10^{-8}	7.1717×10^{-11}	2.1105×10^{-13}	6.3336×10^{-17}
1.	7.1046×10^{-9}	1.0012×10^{-10}	9.0307×10^{-14}	4.5503×10^{-17}

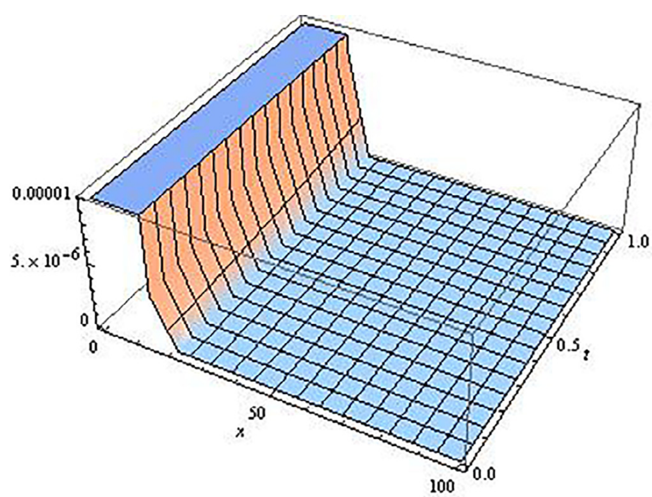

Fig. 1. Third order OPIM approximate solution of the Eq. (5.1) for $\alpha = 0.5$.

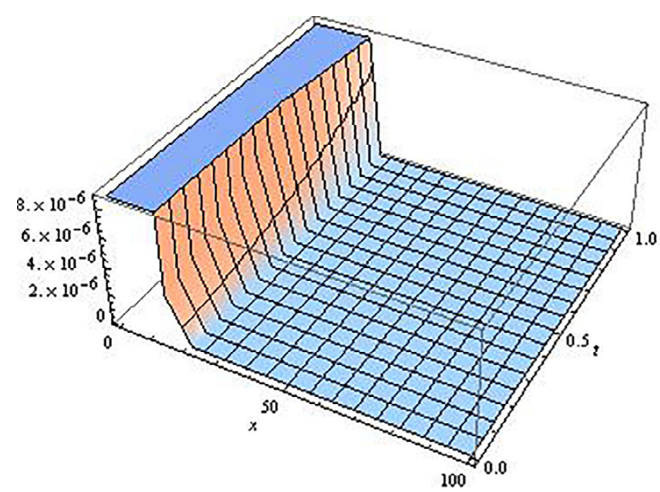

Fig. 2. Third order OPIM approximate solution of the Eq. (5.1) for $\alpha = 0.8$.

Secondly, it is required to show that $\{A_n\}_{n=0}^{\infty}$is a Cauchy sequence in $\mathscr{B}$. To do that, let us consider

$$\|R_{n+1} - R_n\| = \|\Upsilon_{n+1}\| \leqslant \beta\|\Upsilon_n\| \leqslant \beta^2\|\Upsilon_{n-1}\| \leqslant \cdots \leqslant \beta^{n+1}\|\Upsilon_0\|. \tag{4.6}$$

For every $n, k \in \mathbb{N}, n \geqslant k$, one can get

$$\begin{aligned} \|R_n - R_k\| &= \|(R_n - R_{n-1}) + (R_{n-1} - R_{n-2}) + \cdots + (R_{k+1} - R_k)\| \\ &\leqslant \|R_n - R_{n-1}\| + \|R_{n-1} - R_{n-2}\| + \cdots + \|R_{k+1} - R_k\| \\ &\leqslant \beta^n\|\Upsilon_0\| + \beta^{n-1}\|\Upsilon_0\| + \cdots + \beta^{k+1}\|\Upsilon_0\| = \tfrac{1-\beta^{n-k}}{1-\beta}\beta^{k+1}\|\Upsilon_0\| \end{aligned} \tag{4.7}$$

Since, it is given that $0 < \beta < 1$, one may obtain from (4.7) that

$$\lim_{n,k\to\infty} \|R_n - R_k\| = 0. \tag{4.8}$$

Consequently, $\{R_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathscr{B}$ and this implies that approximate solution (4.3) is convergent.

Theorem 2. *Let us suppose that the starting function $u_0 = \Upsilon_0$ falls into the ball of the solution $u(x,t)$. Then, $R_n = \sum_{i=0}^{n}\Upsilon_i$ also stays inside the ball of the solution.*

Proof. Let us assume that

$$\Upsilon_0 \in B_r(u) \tag{4.9}$$

where

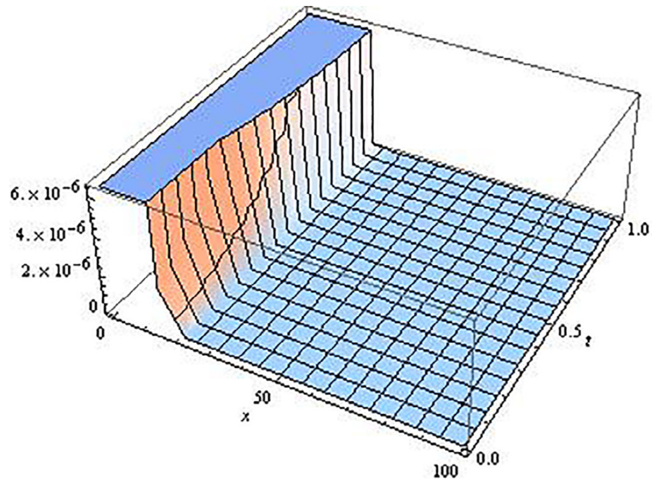

Fig. 3. Fourth order OPIM approximate solution of the Eq. (5.1) for $\alpha = 0.5$.

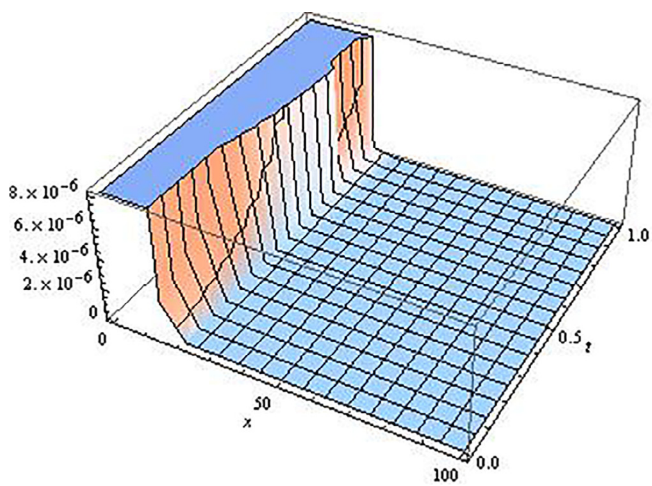

Fig. 4. Fourth order OPIM approximate solution of the Eq. (5.1) for $\alpha = 0.8$.

$$B_r(u) = \{\Upsilon \in R | \|u - \Upsilon\| < r\} \tag{4.10}$$

denoted the ball of the solution $u(x,t)$. From the hypothesis $u = \lim_{n\to\infty} R_n = \sum_{i=0}^{\infty}\Upsilon_i$and from Theorem 1, we have

$$\|u - R_n\| \leqslant \beta^{n+1}\|\Upsilon_0\| < \|\Upsilon_0\| < r \tag{4.11}$$

where $n \in \mathbb{N}$ and $\beta \in (0,1)$.

It should be emphasized here that the selection of the starting function directly affects convergence of the approximations. However, there is no general theorem about the election of the initial function.

5. Numerical example

In the current part, we solve the following fractional GRLW equation with $p = 8,\ \gamma = \beta = 1$ as follows:

$${}_0^{ABC}D_t^{\alpha}u + u_x + (u^8)_x - u_{xxt} = 0 \tag{5.1}$$

with the initial condition

$$u(x,0) = \sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right). \tag{5.2}$$

One can initiate the procedures by taking the Eq. (5.2) as an initial function u_0. Then by applying Laplace transform, we have

$$\mathscr{L}[u] - \frac{\sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{s} + \frac{\alpha(-1+s^{-\alpha})+1}{B(\alpha)}\mathscr{L}[F(u_{xxt}, u_x, u)] = 0 \tag{5.3}$$

and correspondingly

$$\mathscr{L}[u] - \frac{\sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{s} + \frac{\alpha(-1+s^{-\alpha})+1}{B(\alpha)}\mathscr{L}[u_x + (u^8)_x - u_{xxt}] = 0 \tag{5.4}$$

Now, by using the iterative formula, one can obtain

$$\begin{aligned}(u_1)_{OPIM} &= \sqrt[7]{18}\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)\\ &+ P_0 t\left[\begin{array}{l}\frac{2\sqrt{23}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{9}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}} + \\ \frac{288\sqrt{23}\sinh\left(\frac{7(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{23}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)}{\sqrt{5}}\end{array}\right]\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right)\end{aligned} \tag{5.5}$$

$$\begin{aligned}(u_2)_{OPIM} &= -\frac{7}{5}\sqrt{23}P_0 t^2\,\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)\\ &- \frac{1008}{5}\sqrt{23}P_1 t^2\,\mathrm{sech}^{\frac{16}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)\\ &+ P_0P_1\left[\begin{array}{l}-12694\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right) + 4313\cosh\left(\frac{42(x+1)}{\sqrt{5}}\right) + \\ -28850\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right) + 8331\cosh\left(\frac{42(x+1)}{\sqrt{5}}\right)\\ -4316\sqrt{5}\cosh\left(\frac{28(x+1)}{\sqrt{5}}\right)\mathrm{sech}^{\frac{2}{7}}\left(\frac{7(x+1)}{\sqrt{5}}\right)\\ +2051\sqrt{7}\cosh\left(\frac{52(x+1)}{\sqrt{17}}\right)\mathrm{sech}^{\frac{5}{7}}\left(\frac{13(x+1)}{\sqrt{19}}\right)\\ +18051\sqrt{17}\cosh\left(\frac{29(x+1)}{\sqrt{22}}\right)\mathrm{sech}^{\frac{15}{7}}\left(\frac{21(x+1)}{\sqrt{101}}\right) + \\ \ldots\end{array}\right]\\ &\times\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right)\end{aligned} \tag{5.6}$$

and so on.

It can be easily deduced that as the number of iterations increase, the approximate solution becomes more tortuous and the use of the symbolic computer program becomes indispensable. Mathematica 9.0 is used to handle the complex calculations in this paper.

One can prefer to use the collocation method to get the parameters $P_0, P_1, \ldots$. For third order OPIM solutions, we reach the values $P_0 = 0.80635,\ P_1 = 0.40536,\ P_2 = 0.10589$ for $\alpha = 0.5$ and $P_0 = 0.96302,\ P_1 = 0.09965,\ P_2 = 0.85012$ for $\alpha = 0.8$. Tables 1 and 2 display the absolute residual error for approximate OPIM solutions of fractional GRLW equation for different α's at some constants x. Additionally, Figs. 1–4 demonstrate the different behaviour of the OPIM solutions for different α's.

6. Conclusion

In this research, we first aim to reconstruct the generalized regularized long wave equations with a new fractional operator. Then, optimal perturbation iteration technique has been implemented to get the approximate solutions of the extended version of the GRLW. Laplace transform is also used to form the OPIM algorithms. We can say that the most important portion of the present study is the usage of the AB fractional derivative instead of integer order derivative in generalized regularized long wave equations to examine the nature of displacement of ion acoustic plasma waves and shallow water waves.Numerical results also show that the suggested scheme is highly methodical and can be used to investigate nonlinear partial fractional mathematical models modeling natural phenomena. It can be also deduced that the use of fractional derivative brings the new paradigms in the area of mathematics or engineering. Future studies can be carried out by using another fractional operators or integral equations with the proposed technique.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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