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A Mulholland-type inequality in the whole plane with multi parameters

Bing He, Bicheng Yang *

Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 51003, PR China

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ABSTRACT

By introducing independent parameters, and applying the weight coefficients, we give a new Mulholland-type inequality in the whole plane with a best possible constant factor. Moreover, the equivalent forms, a few particular cases and the operator expressions are considered.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geqslant 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy-Hilbert's inequality (cf. Hardy et al., 1934):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$. The Mulholland's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ was provided as follows (cf. Theorem 343 of Hardy et al. (1934), replacing $\frac{a_m}{m}, \frac{b_n}{n}$ by a_m, b_n):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}}\right)^{1/p}\left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}}\right)^{1/q}. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. Hardy et al., 1934; Mitrinović et al., 1991).

In 2007, a Hilbert-type integral inequality in the whole plane was given as follows (cf. Wang and Yang, 2011):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\min\{1,|xy|\}f(x)g(y)}{\sqrt{1+xy+(xy)^2}}dxdy$$
$$< 2\ln(3+2\sqrt{3})\left(\int_{-\infty}^{\infty}|x|f^2(x)dx\int_{-\infty}^{\infty}|y|f^2(y)dy\right)^{\frac{1}{2}}, \tag{3}$$

where the constant factor $2\ln(3+2\sqrt{3})$ is the best possible. Some new results on inequalities (1)–(3) were obtained by Gao and Yang (1998), Yang et al. (2011), Krnić and Pečarić (2005), Perić and Vuković (2011), He (2015), Adiyasuren et al. (2016), Yang (2007), Li and He (2007), Krnić and Vukovic (2012), Huang (2015), Huang and Yang (2013), Huang et al. (2014a,b), Huang (2010). In 2016, Yang and Chen gave a more accurate extension of (1) in the whole plane as follows (cf. Zhong et al., 2017):

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}} < 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{4}$$

* Corresponding author.
E-mail addresses: hebing@gdei.edu.cn (B. He), bcyang@gdei.edu.cn (B. Yang).
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where the constant factor $2B(\lambda_1,\lambda_2)(0<\lambda_1,\lambda_2\leqslant 1,\lambda_1+\lambda_2=\lambda,\xi,\eta\in[0,\frac{1}{2}])$ is the best possible. Another result on this kind of inequalities was provided by Xin et al. (2016).

In this paper, by introducing independent parameters, applying the weight coefficients, we give a Mulholland-type inequality in the whole plane with a best possible constant factor similar to (4) and the main result of Xin et al. (2016). Moreover, the equivalent forms, a few particular cases and the operator expressions are considered.

2. Some lemmas

In the following, we agree that $p>1,\frac{1}{p}+\frac{1}{q}=1,\sigma\in\mathbf{R}=(-\infty,\infty)$, $-\sigma<\lambda_1,\lambda_2\leqslant 1-\sigma,\lambda_1+\lambda_2=\lambda,\rho\geqslant 1$,

$$\arccos\sqrt{1-\frac{1}{\rho}}\leqslant\gamma\leqslant\pi-\arccos\sqrt{1-\frac{1}{\rho}}(\gamma=\alpha,\beta),$$

$\zeta\in(-1,1)$, satisfying

$$\frac{1}{\rho(1-\cos\gamma)}-1\leqslant\zeta\leqslant 1-\frac{1}{\rho(1+\cos\gamma)}$$

$((\zeta,\gamma)=(\xi,\alpha)$ or $(\eta,\beta))$, and

$$h_\gamma(\lambda_1):=\frac{2(\lambda+2\sigma)\csc^2\gamma}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\in\mathbf{R}_+=(0,\infty)(\gamma=\alpha,\beta). \tag{5}$$

Remark 1. With regards to the above assumptions, it follows that $\rho(1\pm\zeta)(1\mp\cos\gamma)\geqslant 1$ $((\zeta,\gamma)=(\xi,\alpha)$ or $(\eta,\beta))$. In particular, for $\rho=1$, we have $\alpha=\beta=\frac{\pi}{2}$ and $\xi=\eta=0$.

For $|t|>1,t=x,y,(\zeta,\gamma,t)=(\xi,\alpha,x)$ (or (η,β,y)), we set

$$A_{\zeta,\gamma}(t):=|t-\zeta|+(t-\zeta)\cos\gamma,$$

and the following function:

$$H(x,y):=\frac{[\min\{\ln\rho A_{\xi,\alpha}(x),\ln\rho A_{\eta,\beta}(y)\}]^\sigma}{[\max\{\ln\rho A_{\xi,\alpha}(x),\ln\rho A_{\eta,\beta}(y)\}]^{\lambda+\sigma}}.$$

Definition 1. Define two weight coefficients as follows:

$$\omega(\lambda_2,m):=\sum_{|n|=2}^{\infty}\frac{H(m,n)}{A_{\eta,\beta}(n)}\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)},|m|\in\mathbf{N}\setminus\{1\}, \tag{6}$$

$$\varpi(\lambda_1,n):=\sum_{|m|=2}^{\infty}\frac{H(m,n)}{A_{\xi,\alpha}(m)}\frac{\ln^{\lambda_2}\rho A_{\eta,\beta}(n)}{\ln^{1-\lambda_1}\rho A_{\xi,\alpha}(m)},|n|\in\mathbf{N}\setminus\{1\}, \tag{7}$$

where $\sum_{|j|}^{=2\infty}\cdots=\sum_{j=-2}^{-\infty}\cdots+\sum_{j=2}^{\infty}\cdots(j=m,n)$.

Lemma 1. *(cf. Xin et al., 2016) Suppose that $g(t)(>0)$ is strictly decreasing in $(1,\infty)$, satisfying $\int_1^\infty g(t)dt\in\mathbf{R}_+$. We have*

$$\int_2^\infty g(t)dt<\sum_{n=2}^{\infty}g(n)<\int_1^\infty g(t)dt. \tag{8}$$

If $(-1)^i g^{(i)}(t)>0$ $(i=0,1,2;t\in(\frac{3}{2},\infty))$, $\int_{\frac{3}{2}}^\infty g(t)dt\in\mathbf{R}_+$, then we have the following Hermite-Hadamard's inequality (cf. Chen and Yang, 2016):

$$\sum_{n=2}^{\infty}g(n)<\int_{\frac{3}{2}}^\infty g(t)dt. \tag{9}$$

Lemma 2. *The following inequalities are valid for $\lambda_1>-\sigma,-\sigma<\lambda_2\leqslant 1-\sigma$:*

$$h_\beta(\lambda_1)(1-\theta(\lambda_2,m))<\omega(\lambda_2,m)<h_\beta(\lambda_1),|m|\in\mathbf{N}\setminus\{1\}, \tag{10}$$

where

$$\theta(\lambda_2,m):=\frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{P\ln\rho A_{\xi,\alpha}(m)}}\frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du$$
$$=O\left(\frac{1}{\ln^{\lambda_2+\sigma}\rho A_{\xi,\alpha}(m)}\right)\in(0,1). \tag{11}$$

Proof. For $|m|\in\mathbf{N}\setminus\{1\}$, we put

$$H^{(1)}(m,y):=\frac{[\min\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y-\eta)(\cos\beta-1)\}]^\sigma}{[\max\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y-\eta)(\cos\beta-1)\}]^{\lambda+\sigma}},y<-1,$$
$$H^{(2)}(m,y):=\frac{[\min\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y-\eta)(\cos\beta+1)\}]^\sigma}{[\max\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y-\eta)(\cos\beta+1)\}]^{\lambda+\sigma}},y>1,$$

wherefrom

$$H^{(1)}(m,-y)=\frac{[\min\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y+\eta)(1-\cos\beta)\}]^\sigma}{[\max\{\ln\rho A_{\xi,\alpha}(m),\ln\rho(y+\eta)(1-\cos\beta)\}]^{\lambda+\sigma}},y>1.$$

We find

$$\omega(\lambda_2,m)=\sum_{n=-2}^{-\infty}\frac{H^{(1)}(m,n)}{(n-\eta)(\cos\beta-1)}\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}\rho(n-\eta)(\cos\beta-1)}$$
$$+\sum_{n=2}^{\infty}\frac{H^{(2)}(m,n)}{(n-\eta)(\cos\beta+1)}$$
$$\times\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}\rho(n-\eta)(\cos\beta+1)}$$
$$=\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1-\cos\beta}\sum_{n=2}^{\infty}\frac{H^{(1)}(m,-n)}{(n+\eta)\ln^{1-\lambda_2}\rho(n+\eta)(1-\cos\beta)}$$
$$+\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1+\cos\beta}\sum_{n=2}^{\infty}\frac{H^{(2)}(m,n)}{(n-\eta)\ln^{1-\lambda_2}\rho(n-\eta)(1+\cos\beta)}. \tag{12}$$

In virtue of $\lambda_1>\sigma,-\sigma<\lambda_2\leqslant 1-\sigma$, and $\lambda_1+\lambda_2=\lambda$, we find that for $y>1,i=1,2$,

$$\frac{H^{(i)}(m,(-1)^i y)}{[y-(-1)^i\eta]\ln^{1-\lambda_2}\rho\left[y-(-1)^i\eta\right][1+(-1)^i\cos\beta]}$$
$$=\begin{cases}\frac{1}{[y-(-1)^i\eta]\left(\ln\rho A_{\xi,\alpha}(m)\right)^{\lambda+\sigma}\ln^{1-\lambda_2-\sigma}\rho\left[y-(-1)^i\eta\right][1+(-1)^i\cos\beta]}, \\ \frac{1}{\rho}<[y-(-1)^i\eta][1+(-1)^i\cos\beta]\leqslant A_{\xi,\alpha}(m) \\ \frac{\left(\ln\rho A_{\xi,\alpha}(m)\right)^\sigma}{[y-(-1)^i\eta]\ln^{1+\lambda_1+\sigma}\rho\left[y-(-1)^i\eta\right][1+(-1)^i\cos\beta]}, \\ \left[y-(-1)^i\eta\right][1+(-1)^i\cos\beta]>A_{\xi,\alpha}(m)\end{cases}$$

are strictly decreasing in $(1,\infty)$. By (12) and (8), we find

$$\omega(\lambda_2,m)<\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1-\cos\beta}\int_1^\infty\frac{H^{(1)}(m,-y)dy}{(y+\eta)\ln^{1-\lambda_2}\rho(y+\eta)(1-\cos\beta)}$$
$$+\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1+\cos\beta}\int_1^\infty\frac{H^{(2)}(m,y)dy}{(y-\eta)\ln^{1-\lambda_2}\rho(y-\eta)(1+\cos\beta)}.$$

Setting $u=\frac{\ln\rho(y+\eta)(1-\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}$ $(u=\frac{\ln\rho(y-\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)})$ in the above first (second) integral, in view of Remark 1, by simplifications, we obtain

$$\omega(\lambda_2, m) < \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_0^\infty \frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du$$
$$= 2\csc^2\beta\left(\int_0^1 u^{\lambda_2+\sigma-1}du + \int_1^\infty \frac{u^{\lambda_2-1}}{u^{\lambda+\sigma}}du\right) = h_\beta(\lambda_1).$$

By (12) and (8), in the same way, we still have

$$\omega(\lambda_2, m) > \frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1-\cos\beta}\int_2^\infty \frac{H^{(1)}(m,-y)dy}{(y+\eta)\ln^{1-\lambda_2}\rho(y+\eta)(1-\cos\beta)}$$
$$+ \frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1+\cos\beta}\int_2^\infty \frac{H^{(2)}(m,y)dy}{(y-\eta)\ln^{1-\lambda_2}\rho(y-\eta)(1+\cos\beta)}$$
$$\geqslant 2\csc^2\beta\int_{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}}^\infty \frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du$$
$$= h_\beta(\lambda_1) - 2\csc^2\beta\int_0^{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}} \frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du$$
$$= h_\beta(\lambda_1)(1-\theta(\lambda_2,m)) > 0,$$

where $\theta(\lambda_2, m)(<1)$ is indicated by (11). It follows that for $A_{\xi,\alpha}(m) \geqslant (2+\eta)(1+\cos\beta)$,

$$0 < \theta(\lambda_2, m) = \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}} u^{\lambda_2+\sigma-1}du$$
$$= \frac{\lambda_1+\sigma}{\lambda+2\sigma}\left[\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}\right]^{\lambda_2+\sigma}.$$

Hence, (10) and (11) are valid. □

In the same way, we still have.

Lemma 3. *The following inequalities are valid for* $\lambda_2 > -\sigma, -\sigma < \lambda_1 \leqslant 1-\sigma$:

$$h_\alpha(\lambda_1)(1-\widetilde{\theta}(\lambda_1, n)) < \varpi(\lambda_1, n) < h_\alpha(\lambda_1), |n| \in \mathbf{N}\setminus\{1\}, \quad (13)$$

where

$$\widetilde{\theta}(\lambda_1, n) := \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\xi)(1+\cos\alpha)}{\ln\rho A_{\eta,\beta}(n)}} \frac{(\min\{1,u\})^\sigma u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\sigma}}du$$
$$= O\left(\frac{1}{\ln^{\lambda_1+\sigma}\rho A_{\eta,\beta}(n)}\right) \in (0,1). \quad (14)$$

Lemma 4. *If* $(\zeta,\gamma) = (\xi,\alpha)$ *(or* (η,β)*), then for* $\varepsilon > 0$*, we have*

$$H_\varepsilon(\zeta,\gamma) := \sum_{|k|=2}^\infty \frac{\ln^{-1-\varepsilon}\rho[|k-\zeta| + (k-\zeta)\cos\gamma]}{|k-\zeta|(k-\zeta)\cos\gamma}$$
$$= \frac{1}{\varepsilon}(2\csc^2\gamma + o(1)) \qquad (\varepsilon \to 0^+). \quad (15)$$

Proof. By (9), we find

$$H_\varepsilon(\zeta,\gamma) = \sum_{k=-2}^{-\infty}\frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma-1)}{(k-\zeta)(\cos\gamma-1)} + \sum_{k=2}^\infty\frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma+1)}{(k-\zeta)(\cos\gamma+1)}$$
$$= \sum_{k=2}^\infty\left[\frac{\ln^{-1-\varepsilon}\rho(k+\zeta)(1-\cos\gamma)}{(k+\zeta)(1-\cos\gamma)} + \frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma+1)}{(k-\zeta)(\cos\gamma+1)}\right]$$
$$< \int_{\frac{3}{2}}^\infty\left[\frac{\ln^{-1-\varepsilon}\rho(y+\zeta)(1-\cos\gamma)}{(y+\zeta)(1-\cos\gamma)} + \frac{\ln^{-1-\varepsilon}\rho(y-\zeta)(\cos\gamma+1)}{(y-\zeta)(\cos\gamma+1)}\right]dy$$
$$= \frac{1}{\varepsilon}\left[\frac{\ln^{-\varepsilon}\rho(\frac{3}{2}+\zeta)(1-\cos\gamma)}{1-\cos\gamma} + \frac{\ln^{-\varepsilon}\rho(\frac{3}{2}-\zeta)(1+\cos\gamma)}{1+\cos\gamma}\right]$$
$$= \frac{1}{\varepsilon}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_1(1)\right)(\varepsilon\to 0^+).$$

By (8), we still can find that

$$H_\varepsilon(\zeta,\gamma) = \sum_{k=2}^\infty\left[\frac{\ln^{-1-\varepsilon}\rho(k+\zeta)(1-\cos\gamma)}{(k+\zeta)(1-\cos\gamma)} + \frac{\ln^{-1-\varepsilon}\rho(n-\zeta)(\cos\gamma+1)}{(n-\zeta)(\cos\gamma+1)}\right]$$
$$> \int_2^\infty\left[\frac{\ln^{-1-\varepsilon}\rho(y+\zeta)(1-\cos\gamma)}{(y+\zeta)(1-\cos\gamma)} + \frac{\ln^{-1-\varepsilon}\rho(y-\zeta)(\cos\gamma+1)}{(y-\zeta)(\cos\gamma+1)}\right]dy$$
$$= \frac{1}{\varepsilon}\left[\frac{\ln^{-\varepsilon}\rho(2+\zeta)(1-\cos\gamma)}{1-\cos\gamma} + \frac{\ln^{-\varepsilon}\rho(2-\zeta)(1+\cos\gamma)}{1+\cos\gamma}\right]$$
$$= \frac{1}{\varepsilon}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_2(1)\right)(\varepsilon\to 0^+).$$

Hence, we prove that (15) is valid. □

3. Main results and a few particular cases

We set

$$k_{\alpha,\beta}(\lambda_1) := h_\beta^{1/p}(\lambda_1)h_\alpha^{1/q}(\lambda_1) = \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^{2/p}\beta\csc^{2/q}\alpha. \quad (16)$$

Theorem 1. *Suppose that* $a_m, b_n \geqslant 0$ $(|m|,|n| \in \mathbf{N}\setminus\{1\})$, *satisfy*

$$0 < \sum_{|m|=2}^\infty \frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p < \infty, < \sum_{|n|=2}^\infty \frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q < \infty.$$

We have the following equivalent inequalities:

$$I := \sum_{|n|=2}^\infty\sum_{|m|=2}^\infty H(m,n)a_m b_n$$
$$< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^\infty \frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=2}^\infty \frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}, \quad (17)$$

$$J := \left[\sum_{|n|=2}^\infty \frac{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\left(\sum_{|m|=2}^\infty H(m,n)a_m\right)^p\right]^{\frac{1}{p}}$$
$$< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^\infty \frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{1/p}. \quad (18)$$

In particular, (i) for $\alpha = \beta = \frac{\pi}{2}, \frac{1}{p} - 1 \leqslant \xi, \eta \leqslant 1 - \frac{1}{p}$, we have the following equivalent inequalities similar to (4):

$$\sum_{|n|=2}^\infty\sum_{|m|=2}^\infty \frac{(\min\{\ln\rho|m-\xi|, \ln\rho|n-\eta|\})^\sigma a_m b_n}{(\max\{\ln\rho|m-\xi|, \ln\rho|n-\eta|\})^{\lambda+\sigma}}$$
$$< \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}$$
$$\times\left[\sum_{|m|=2}^\infty \frac{\ln^{p(1-\lambda_1)-1}\rho|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^\infty \frac{\ln^{q(1-\lambda_2)-1}\rho|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}, \quad (19)$$

$$\left\{\sum_{|n|=2}^\infty \frac{\ln^{p\lambda_2-1}\rho|n-\eta|}{|n-\eta|}\left[\sum_{|m|=2}^\infty \frac{(\min\{\ln\rho|m-\xi|, \ln\rho|n-\eta|\})^\sigma a_m}{(\max\{\ln\rho|m-\xi|, \ln\rho|n-\eta|\})^{\lambda+\sigma}}\right]^p\right\}^{\frac{1}{p}}$$
$$< \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\left[\sum_{|m|=2}^\infty \frac{\ln^{p(1-\lambda_1)-1}\rho|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{1/p}. \quad (20)$$

(ii) For $\xi=\eta=0$, $\arccos\sqrt{1-\frac{1}{\rho}}\leqslant\alpha,\beta\leqslant\pi-\arccos\sqrt{1-\frac{1}{\rho}}$, we have the following equivalent inequalities:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{[\min\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+n\cos\beta)\}]^{\sigma}a_m b_n}{[\max\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+\cos\beta)\}]^{\lambda+\sigma}}$$
$$<k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}\rho(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{1/p}$$
$$\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho(|n|+\cos\beta)}{(|n|+\cos\beta)^{1-q}}b_n^q\right]^{1/q}, \quad (21)$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}\rho(|n|+\cos\beta)}{|n|+\cos\beta}\right.$$
$$\left.\times\left[\sum_{|m|=2}^{\infty}\frac{[\min\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+n\cos\beta)\}]^{\sigma}a_m}{[\max\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+\cos\beta)\}]^{\lambda+\sigma}}\right]^p\right\}^{\frac{1}{p}}$$
$$<k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}\rho(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}}. \quad (22)$$

Proof. By Hölder's inequality with weight (cf. Kuang, 2010) and (7), we find

$$\left(\sum_{|m|=2}^{\infty}H(m,n)a_m\right)^p=\left\{\sum_{|m|=2}^{\infty}H(m,n)\left[\frac{(A_{\xi,\alpha}(m))^{1/q}\ln^{(1-\lambda_1)/q}\rho A_{\xi,\alpha}(m)}{\ln^{(1-\lambda_2)/p}\rho A_{\eta,\beta}(n)}a_m\right]\right.$$
$$\left.\times\left[\frac{\ln^{(1-\lambda_2)/p}\rho A_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{1/q}\ln^{(1-\lambda_1)/q}\rho A_{\xi,\alpha}(m)}\right]\right\}^p$$
$$\leqslant\sum_{|m|=2}^{\infty}H(m,n)\frac{(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p$$
$$\times\left[\sum_{|m|=2}^{\infty}H(m,n)\frac{\ln^{(1-\lambda_2)q/p}\rho A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1}\rho A_{\xi,\alpha}(m)}\right]^{p-1}$$
$$=\frac{(\varpi(\lambda_1,n))^{p-1}A_{\eta,\beta}(n)}{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty}H(m,n)\frac{(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p.$$

By (13), it follows that

$$J<h_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{H(m,n)(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}}$$
$$=h_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}\frac{H(m,n)(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}}$$
$$=h_{\alpha}^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\omega(\lambda_2,m)\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}. \quad (23)$$

By (10) and (16), we have (18).

Using Hölder's inequality again, we have

$$I=\sum_{|n|=2}^{\infty}\left[\frac{\ln^{\lambda_2-(1/p)}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1/p}}\sum_{|m|=2}^{\infty}H(m,n)a_m\right]\left[\frac{\ln^{(1/p)-\lambda_2}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{-1/p}}b_n\right]$$
$$\leqslant J\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}, \quad (24)$$

and then by (18), we have (17).

On the other hand, assuming that (17) is valid, we set

$$b_n:=\frac{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty}H(m,n)a_m\right)^{p-1},\quad |n|\in\mathbf{N}\setminus\{1\},$$

and find

$$J=\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{1/p}.$$

By (23), it follows that $J<\infty$. If $J=0$, then (18) is trivially valid; if $J>0$, then we have

$$0<\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2-1)}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q=J^p=I<k_{\alpha,\beta}(\lambda_1)$$
$$\times\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1-1)}\rho A_{\xi,\beta}(m)}{(A_{\xi,\alpha}(m))^{1-q}}\alpha_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2-1)}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}},$$
$$=\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2-1)}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{p}}$$
$$<k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1-1)}\rho A_{\xi,\beta}(m)}{(A_{\xi,\alpha}(m))^{1-q}}\alpha_m^p\right]^{\frac{1}{p}}.$$

Hence, (18) is valid, which is equivalent to (17). □

Theorem 2. *With regards to the assumptions of Theorem 1, the constant factor$k_{\alpha,\beta}(\lambda_1)$ in (17) and (18) is the best possible.*

Proof. For $0<\varepsilon<q(\lambda_2+\sigma)$, we set $\widetilde{\lambda}_1=\lambda_1+\frac{\varepsilon}{q}(>-\sigma)$, $\widetilde{\lambda}_2=\lambda_2-\frac{\varepsilon}{q}$ $(\in(-\sigma,1-\sigma))$, and

$$\widetilde{a}_m:=\frac{\ln^{\lambda_1-(\varepsilon/p)-1}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}=\frac{\ln^{\widetilde{\lambda}_1-\varepsilon-1}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}\ (|m|\in\mathbf{N}\setminus\{1\}),$$

$$\widetilde{b}_n:=\frac{\ln^{\lambda_2-(\varepsilon/q)-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}=\frac{\ln^{\widetilde{\lambda}_2-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\ (|n|\in\mathbf{N}\setminus\{1\}).$$

By (15) and (13), we find

$$\widetilde{I}_1:=\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}\widetilde{b}_n^q\right]^{\frac{1}{q}}$$
$$=\left[\sum_{|m|=2}^{\infty}\frac{\ln^{-1-\varepsilon}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{-1-\varepsilon}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\right]^{\frac{1}{q}}$$
$$=\frac{1}{\varepsilon}(2\csc^2\alpha+o(1))^{1/p}(2\csc^2\beta+\widetilde{o}(1))^{1/q}(\varepsilon\to0^+),$$

$$\widetilde{I}:=\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}H(m,n)\widetilde{a}_m\widetilde{b}_n$$
$$=\sum_{|m|}^{=2\infty}\sum_{|n|=2}^{\infty}H(m,n)\frac{\ln^{\widetilde{\lambda}_1-\varepsilon-1}\rho A_{\xi,\alpha}(m)\ln^{\widetilde{\lambda}_2-1}\rho A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)A_{\eta,\beta}(n)}$$
$$=\sum_{|m|=2}^{\infty}\omega(\widetilde{\lambda}_2,m)\frac{\ln^{-1-\varepsilon}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}>h_{\beta}(\widetilde{\lambda}_1)\sum_{|m|=2}^{\infty}(1-\theta(\widetilde{\lambda}_2,m))\frac{\ln^{-1-\varepsilon}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}$$
$$=h_{\beta}(\widetilde{\lambda}_1)\left[\sum_{|m|=2}^{\infty}\frac{\ln^{-1-\varepsilon}\rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)}-\sum_{|m|=2}^{\infty}\frac{O\left(\ln^{-1-(\frac{\varepsilon}{p}+\lambda_2+\sigma)}\rho A_{\xi,\alpha}(m)\right)}{A_{\xi,\alpha}(m)}\right]$$
$$=\frac{1}{\varepsilon}h_{\beta}\left(\lambda_1+\frac{\varepsilon}{q}\right)(2\csc^2\alpha+o(1)-\varepsilon O(1)).$$

If there exists a positive number $K\leqslant k_{\alpha,\beta}(\lambda_1)$, such that (17) is still valid when replacing $k_{\alpha,\beta}(\lambda_1)$ by K, then in particular, we have

$$\varepsilon\widetilde{I} = \varepsilon\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}H(m,n)\widetilde{a}_m\widetilde{b}_n < \varepsilon K\widetilde{I}_1.$$

Hence, in view of the above results, it follows that

$$h_\beta\left(\lambda_1+\frac{\varepsilon}{q}\right)(2\csc^2\alpha + o(1) - \varepsilon O(1)) < K\cdot\left(2\csc^2\alpha + o(1)\right)^{1/p}\left(2\csc^2\beta + \widetilde{o}(1)\right)^{1/q},$$

and then

$$\frac{4(\lambda+2\sigma)\csc^2\beta}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^2\alpha \leqslant 2K\csc^{2/p}\alpha\csc^{2/q}\beta(\varepsilon\to 0^+),$$

namely,

$$k_{\alpha,\beta}(\lambda_1) = \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^{2/p}\beta\csc^{2/q}\alpha \leqslant K.$$

Hence, $K = k_{\alpha,\beta}(\lambda_1)$ is the best possible constant factor in (17).

The constant factor $k_{\alpha,\beta}(\lambda_1)$ in (18) is still the best possible. Otherwise we would reach a contradiction by (24)' that the constant factor in (17) is not the best possible. □

4. Operator expressions and a remark

Setting $\varphi(m) := \frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}$ $(|m|\in\mathbf{N}\setminus\{1\})$, $\psi(n) := \frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}$, wherefrom, $\psi^{1-p}(n) = \frac{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}$ $(|n|\in\mathbf{N}\setminus\{1\})$, we define the following real weighted normed function spaces:

$$l_{p,\varphi} := \left\{a = \{a_m\}_{|m|}^{=2\infty}; |a|_{p,\varphi} = \left(\sum_{|m|=2}^{\infty}\varphi(m)|a_m|^p\right)^{1/p} < \infty\right\},$$

$$l_{q,\psi} := \left\{b = \{b_n\}_{|n|}^{=2\infty}; |b|_{q,\psi} = \left(\sum_{|n|=2}^{\infty}\psi(n)|b_n|^q\right)^{1/q} < \infty\right\},$$

$$l_{p,\psi^{1-p}} := \left\{c = \{c_n\}_{|n|}^{=2\infty}; |c|_{p,\psi^{1-p}} = \left(\sum_{|n|=2}^{\infty}\psi^{1-p}(n)|c_n|^p\right)^{1/p} < \infty\right\}.$$

For $a = \{a_m\}_{|m|}^{=2\infty} \in l_{p,\varphi}$, putting $c_n = \sum_{|m|}^{=2\infty}H(m,n)a_m$ and $c = \{c_n\}_{|n|}^{=2\infty}$, it follows by (18) that $|c|_{p,\psi^{1-p}} < k_{\alpha,\beta}(\lambda_1)|a|_{p,\varphi}$, namely $c\in l_{p,\psi^{1-p}}$.

Definition 2. Define a Mulholland-type operator $T{:}l_{p,\varphi}\to l_{p,\psi^{1-p}}$ as follows: For $a_m \geqslant 0, a = \{a_m\}_{|m|}^{=2\infty}\in l_{p,\varphi}$, there exists a unique representation $Ta = c\in l_{p,\psi^{1-p}}$. We also define the following formal inner product of Ta and $b = \{b_n\}_{|n|}^{=2\infty}\in l_{q,\psi}$ $(b_n \geqslant 0)$ as follows:

$$(Ta,b) := \sum_{|n|=2}^{\infty}\left(\sum_{|m|=2}^{\infty}H(m,n)a_m\right)b_n. \tag{25}$$

Hence, we may rewrite (17) and (18) in the following operator expressions:

$$(Ta,b) < k_{\alpha,\beta}(\lambda_1)|a|_{p,\varphi}|b|_{q,\psi}, \tag{26}$$

$$|Ta|_{p,\psi^{1-p}} < k_{\alpha,\beta}(\lambda_1)|a|_{p,\varphi}. \tag{27}$$

It follows that the operator T is bounded with

$$|T| := \sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{|Ta|_{p,\psi^{1-p}}}{|a|_{p,\varphi}} \leqslant k_{\alpha,\beta}(\lambda_1). \tag{28}$$

Since the constant factor $k_{\alpha,\beta}(\lambda_1)$ in (18) is the best possible, we have

$$|T| = k_{\alpha,\beta}(\lambda_1) = \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^{2/p}\beta\csc^{2/q}\alpha. \tag{29}$$

Remark 2. (i) For $\rho = 1, \xi = \eta = 0$ in (19), we have the following inequality:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{(\min\{\ln|m|,\ln|n|\})^\sigma a_m b_n}{(\max\{\ln|m|,\ln|n|\})^{\lambda+\sigma}} < \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)} \times\left[\sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}|m|}{|m|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n|}{|n|^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{30}$$

It follows that (19) is an extension of (30).

(ii) If $a_{-m} = a_m$ and $b_{-n} = b_n$ $(m,n\in\mathbf{N}\setminus\{1\})$, then (19) reduces to

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}&\left\{\frac{[\min\{\ln\rho(m-\xi),\ln\rho(n-\eta)\}]^\sigma}{[\max\{\ln\rho(m-\xi),\ln\rho(n-\eta)\}]^{\lambda+\sigma}}\right.\\&+\frac{[\min\{\ln\rho(m-\xi),\ln\rho(n+\eta)\}]^\sigma}{[\max\{\ln\rho(m-\xi),\ln\rho(n+\eta)\}]^{\lambda+\sigma}}+\frac{[\min\{\ln\rho(m+\xi),\ln\rho(n-\eta)\}]^\sigma}{[\max\{\ln\rho(m+\xi),\ln\rho(n-\eta)\}]^{\lambda+\sigma}}\\&\left.+\frac{[\min\{\ln\rho(m+\xi),\ln\rho(n+\eta)\}]^\sigma}{[\max\{\ln\rho(m+\xi),\ln\rho(n+\eta)\}]^{\lambda+\sigma}}\right\}a_m b_n < \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\\&\times\left\{\sum_{m=2}^{\infty}\left[\frac{\ln^{p(1-\lambda_1)-1}\rho(m-\xi)}{(m-\xi)^{1-p}}+\frac{\ln^{p(1-\lambda_1)-1}\rho(m+\xi)}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}\\&\times\left\{\sum_{n=2}^{\infty}\left[\frac{\ln^{q(1-\lambda_2)-1}\rho(n-\eta)}{(n-\eta)^{1-q}}+\frac{\ln^{q(1-\lambda_2)-1}\rho(n+\eta)}{(n+\eta)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{31}$$

In particular, for $\rho = 1, \xi = \eta = 0$, we have the following new Mulholland-type inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln m,\ln n\})^\sigma}{(\max\{\ln m,\ln n\})^{\lambda+\sigma}}a_m b_n < \frac{\lambda+2\sigma}{(\lambda_1+\sigma)(\lambda_2+\sigma)} \times\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{32}$$

5. Conclusions

In this paper, by introducing independent parameters, applying the weight coefficients, we give a new Mulholland-type inequality in the whole plane with a best possible constant factor in Theorems 1 and 2. Moreover, the equivalent forms, a few particular cases and the operator expressions are considered. The lemmas and theorems provide an extensive account of this type of inequalities.

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