Journal of King Saud University – Science (2016) **28**, 37–40

King Saud University

Journal of King Saud University – Science

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ORIGINAL ARTICLE

Generalized variational formulations for extended exponentially fractional integral

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Received 3 April 2015; accepted 22 May 2015
Available online 31 May 2015

KEYWORDS

Fractional calculus;
Generalized variational formulation;
Euler–Lagrange equation;
Extended exponentially fractional integral

Abstract Recently, the fractional variational principles as well as their applications yield a special attention. For a fractional variational problem based on different types of fractional integral and derivatives operators, corresponding fractional Lagrangian and Hamiltonian formulation and relevant Euler–Lagrange type equations are already presented by scholars. The formulations of fractional variational principles still can be developed more. We make an attempt to generalize the formulations for fractional variational principles. As a result we obtain generalized and complementary fractional variational formulations for extended exponentially fractional integral for example and corresponding Euler–Lagrange equations. Two illustrative examples are presented. It is observed that the formulations are in exact agreement with the Euler–Lagrange equations.

1. Introduction

Fractional calculus represents a generalization of ordinary differentiation and integration to arbitrary order. It is an area of current strong research with many different and important applications in different fields of sciences ranging from geophysical fluid dynamics to quantum field theory (Malinowska and Torres, 2012; Yang, 2012).

During the last few years a special attention was devoted to the fractional variational principles as well as their applications (Baleanu, 2008). The formulation of the fractional variational principles has an important role for elaboration of a consistent fractional quantization method for both discrete and continuous systems. The first attempt to find the fractional Lagrangian and Hamiltonian is due to Riewe (1996, 1997), who first applied fractional calculus to a non-conservative mechanics modeling, and formed the fractional Euler–Lagrange equations and the fractional Hamilton equations. The research made by Riewe opened the booming of the fractional variational principle. Since then, the fractional variational principles have been becoming one of the most popular researching areas. Important contributions were obtained by many scholars, for example, Klimek (2001, 2002), Agrawal (2002, 2006, 2007, 2010), Agrawal and Baleanu (2007), Baleanu and Muslih (2005a, 2005b), Muslih and Baleanu

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.05.006

(2005), Baleanu (2006), Baleanu et al. (2013), Rabei et al. (2007), Atanackovi'c (2008), Atanackovi'c and Pilipovi'c (2011), Atanackovic et al. (2012), He (2011, 2014), He et al. (2012), Malinowska and Torres (2010), Almeida and Torres (2011), Almeida (2012), Almeida and Malinowska (2014), El-Nabulsi (2011a, 2011b, 2014), Odzijewicz et al. (2012), Yang et al. (2013), Yang and Baleanu (2013), Bourdin et al. (2014) and Bahrami et al. (2015) and their collaborators and so on. These scholars from different angles put forward different kinds of fractional models and methods, and established corresponding fractional Lagrangian and Hamiltonian formulation and relevant Euler–Lagrange type equations. The formulations of fractional variational principles should still be more developed, continually.

In this paper, we will make an attempt to generalize the formulations for some fractional variational principles. The present paper is organized as follows: In Section 2, the extended exponentially fractional integral is reviewed briefly. In Section 3, the generalized variational formulations for the fractional variational principle based on extended exponentially fractional integral are proposed. In Section 4, two illustrative examples are given.

2. Extended fractional integral

Definition 1. Let f be a continuous function in the interval $[a,b]$. For $t \in [a,b]$, the left and right extended fractional integral of order $\alpha > 0$ are defined by:

$$K_{(z)}^{(-\alpha)}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(\cosh z - \cosh\zeta)^{\alpha-1}\mathrm{d}\zeta \tag{1}$$

where the multiplicity of $(\cosh z - \cosh\zeta)^{\alpha-1}$ is removed by requiring $\log(\cosh z - \cosh\zeta)$ to be real when $\cosh z - \cosh\zeta > 0$.

Eq. (1) is called an extended exponentially fractional integral (El-Nabulsi, 2011a).

3. Generalized variational formulation

Problem 1. Given the smooth generalized Lagrangian function

$$L(q, v, t) : \mathrm{R}^n \times \mathrm{R}^n \times [a,b] \to \mathrm{R}$$

assumed to be a C^2–function with respect to all its arguments. Find the stationary points of the extended exponentially fractional integral

$$S = \frac{1}{\Gamma(\alpha)}\int_a^t [L(q,v,\tau) + p(\dot{q} - v)]\cdot (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau, \tag{2}$$

under the initial condition

$$q(a) = q_a, \tag{3}$$

where q is the generalized coordinate, $\dot{q}$ can only be used as the derivative of q, v is the generalized velocity which is defined as

$$v = \frac{\mathrm{d}q}{\mathrm{d}\tau}, \tag{4}$$

p is the generalized momentum, τ is the intrinsic time, t is the observer time.

Theorem 1. *If q, v, and p are solutions to the previous problem, i.e., q, v, and p are critical points of the functional S, then q, v, and p satisfy the following Euler–Lagrange equations:*

$$v = \dot{q}, \tag{5}$$

$$p = \frac{\partial L}{\partial v}, \tag{6}$$

$$\dot{p} - \frac{\partial L}{\partial q} = \frac{(\alpha-1)\sinh\tau}{\cosh t - \cosh\tau}p. \tag{7}$$

Proof. The variation of the functional S reads

$$\delta S = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial v}\delta v + (\dot{q} - v)\delta p + p(\delta\dot{q} - \delta v)\right] (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau, \tag{8}$$

where all of q, v, and p are the independent variables.

Using the following formula of integration by part,

$$\int_a^t pg\delta\dot{q}\mathrm{d}\tau = -\int_a^t \frac{\mathrm{d}(pg)}{\mathrm{d}\tau}\delta q\mathrm{d}\tau, \tag{9}$$

where

$$g = (\cosh t - \cosh\tau)^{\alpha-1}, \tag{10}$$

$$\dot{g} = (1-\alpha)(\cosh t - \cosh\tau)^{\alpha-2}\sinh\tau, \tag{11}$$

we obtain the variation of the functional S, which takes the form

$$\delta S = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\left(\frac{\partial L}{\partial q} - \dot{p} - \frac{\dot{g}}{g}p\right)g\delta q + \left(\frac{\partial L}{\partial v} - p\right)g\delta v + (\dot{q} - v)g\delta p\right]\mathrm{d}\tau, \tag{12}$$

and we obtain the required result (5)–(7). □

4. Complementary variational formulation

Problem 2. Find the stationary points of the complementary extended exponentially fractional integral

$$S^c = \frac{1}{\Gamma(\alpha)}\int_a^t \left[L(q,v,\tau) - pq\frac{(1-\alpha)\sinh\tau}{\cosh t - \cosh\tau} - \dot{p}q - pv\right]\cdot (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau. \tag{13}$$

Theorem 2. *If q, v, and p are critical points of the complement functional S^c, then q, v, and p satisfy the generalized Euler–Lagrange Eqs. (5)–(7).*

Proof. The variation of the functional S^c reads

$$\delta S^c = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\left(\frac{\partial L}{\partial q} - \dot{p} - p\frac{\dot{g}}{g}\right)\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - q\delta\dot{p} - \left(q\frac{\dot{g}}{g} + v\right)\delta p\right]g\mathrm{d}\tau, \tag{14}$$

where all of q, v, and p are the independent variables.

Using the following formula of integration by part,

$$-\int_a^t qg\delta\dot{p}\mathrm{d}\tau = \int_a^t \frac{\mathrm{d}(qg)}{\mathrm{d}\tau}\delta p\mathrm{d}\tau, \tag{15}$$

we obtain the variation of the functional, which takes the form

$$\delta S = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\left(\frac{\partial L}{\partial q} - \dot{p} - \frac{\dot{g}}{g}p\right)g\delta q + \left(\frac{\partial L}{\partial v} - p\right)g\delta v + (\dot{q} - v)g\delta p\right]\mathrm{d}\tau, \tag{16}$$

and we obtain the required result (5)–(7). □

5. Examples

Example 1. We discuss the case of generalized Caldirola-Kanai Lagrangian

$$L(v, q, t) = m(\tau)\left(\frac{v^2}{2} - \omega^2\frac{q^2}{2}\right). \tag{17}$$

The extended exponentially fractional action takes the form

$$S_1 = \frac{1}{\Gamma(\alpha)}\int_a^t \left[m(\tau)\left(\frac{v^2}{2} - \omega^2\frac{q^2}{2}\right) + p(\dot{q} - v)\right] \cdot (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau, \tag{18}$$

where ω is the frequency, and

$$m(\tau) = m_0 e^{-\gamma(t-\tau)} = \bar{m}_0 e^{\gamma\tau}, \tag{19}$$

where $\bar{m}_0 = m_0 e^{-\gamma t}$ is an effective parameter which depends only on t and consequently, as the derivative is performed with respect to τ, it may be considered as an effective constant.

The complementary fractional action takes the form

$$S_1^c = \frac{1}{\Gamma(\alpha)}\int_a^t \left[m(\tau)\left(\frac{v^2}{2} - \omega^2\frac{q^2}{2}\right) - \dot{p}q - pv - pq\frac{(1-\alpha)\sinh\tau}{\cosh t - \cosh\tau}\right] \cdot (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau. \tag{20}$$

The Euler–Lagrange equations are

$$v = \dot{q}, \tag{21}$$

$$p = mv, \tag{22}$$

$$\dot{p} + m\omega^2 q = \frac{(\alpha-1)\sinh\tau}{\cosh t - \cosh\tau}p. \tag{23}$$

For very large time, $\tau \to +\infty$, (23) is reduced to

$$\dot{p} + m\omega^2 q = (1-\alpha)p, \tag{24}$$

here,

$$\lim_{\tau\to+\infty}\frac{\sinh\tau}{\cosh t - \cosh\tau} = -1, \tag{25}$$

while for very early time, $\tau \to +0$, it is reduced to

$$\dot{p} + m\omega^2 q = 0. \tag{26}$$

here,

$$\dot{p} = m\dot{v} + \dot{m}v. \tag{27}$$

Inserting (21) into (22), we obtain

$$p = m\dot{q}. \tag{28}$$

Inserting (28) into (23), we obtain

$$\ddot{q} + \omega^2 q = -\left(\frac{(1-\alpha)\sinh\tau}{\cosh t - \cosh\tau} + \gamma\right)\dot{q}. \tag{29}$$

Hence, the generalized variational principles can propose the extended weak dissipations.

Example 2. We discuss the following special case of generalized Caldirola-Kanai Lagrangian

$$L = \mathrm{e}^{\gamma\tau}\frac{mv^2}{2} - \mathrm{e}^{-\gamma\tau}\frac{m\omega^2 q^2}{2}. \tag{30}$$

The extended exponentially fractional action takes the form

$$S_2 = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\mathrm{e}^{\gamma\tau}\frac{mv^2}{2} - \mathrm{e}^{-\gamma\tau}\frac{m\omega^2 q^2}{2} + p(\dot{q} - v)\right] (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau. \tag{31}$$

The complement fractional action takes the form

$$S_2^c = \frac{1}{\Gamma(\alpha)}\int_a^t \left[\mathrm{e}^{\gamma\tau}\frac{mv^2}{2} - \mathrm{e}^{-\gamma\tau}\frac{m\omega^2 q^2}{2} - \dot{p}q - pv - pq\frac{(1-\alpha)\sinh\tau}{\cosh t - \cosh\tau}\right] \cdot (\cosh t - \cosh\tau)^{\alpha-1}\mathrm{d}\tau. \tag{32}$$

The Euler–Lagrange equations are

$$v = \dot{q}, \tag{33}$$

$$p = \mathrm{e}^{\gamma\tau}mv, \tag{34}$$

$$\dot{p} + \mathrm{e}^{-\gamma\tau}m\omega^2 q = \frac{(\alpha-1)\sinh\tau}{\cosh t - \cosh\tau}p. \tag{35}$$

For very large time, $\tau \to +\infty$, (35) is reduced to

$$\dot{p} = (1-\alpha)p, \tag{36}$$

which gives

$$p = p_0\mathrm{e}^{(1-\alpha)\tau}, \tag{37}$$

while for very early time, $\tau \to +0$, it is reduced to

$$\dot{p} + m\omega^2 q = 0. \tag{38}$$

Inserting (33) into (34), we obtain

$$p = \mathrm{e}^{\gamma\tau}m\dot{q}. \tag{39}$$

Inserting (39) into (35), we obtain

$$\ddot{q} + \mathrm{e}^{-2\gamma\tau}\omega^2 q = -\left[\frac{(1-\alpha)\sinh\tau}{\cosh t - \cosh\tau} + \left(\gamma + \frac{\dot{m}}{m}\right)\right]\dot{q}. \tag{40}$$

Hence, the generalized variational principles can also propose the extended weak dissipations.

6. Conclusion

In this paper, we obtain the generalized and complementary fractional variational formulations and corresponding Euler–Lagrange equations based on extended exponentially fractional integral. In the new actions, the parameters, q, v, and p, are all chosen as variable functions. Therefore, the Euler–Lagrange equations are reduced from second-order to first-order, which consist of velocity-displacement relations, momentum-velocity relations and equations of motion.

The result can be further extended to the fractional variational principles based on different types of fractional integral and derivatives operators, e.g., Riemann-Liouville, Caputo, Riesz, Caputo-Riesz, Erdélyi-Kober, Grünwald-Letnikov, Weyl and Marchaud etc. In addition, we hope this work will bring new opportunities in studying the fractional variational principles as well as their applications.

Acknowledgement

This work is supported by the Natural Science Foundation of Hebei Province, China (No. E2012203192).

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