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On Skolem odd and even difference mean graphs

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ABSTRACT

Let $G=(V,E)$ be a simple, finite and undirected (p,q)-graph with p vertices and q edges. A graph G is Skolem odd difference mean if there exists an injection $f:V(G)\rightarrow\{0,1,2,\ldots,p+3q-3\}$ and an induced bijection $f^*:E(G)\rightarrow\{1,3,5,\ldots,2q-1\}$ such that each edge uv (with $f(u)>f(v)$) is labeled with $f^*(uv)=\left\lceil\frac{f(u)-f(v)}{2}\right\rceil$. We say G is Skolem even difference mean if there exists an injection $f:V(G)\rightarrow\{0,1,2,\ldots,p+3q-1\}$ and an induced bijection $f^*:E(G)\rightarrow\{2,4,6,\ldots,2q\}$ such that each edge uv (with $f(u)>f(v)$) is labeled with $f^*(uv)=\left\lceil\frac{f(u)-f(v)}{2}\right\rceil$. A graph that admits a Skolem odd (or even) difference mean labeling is called a Skolem odd (or even) difference mean graph. In this paper, first, we construct some new Skolem odd difference mean graphs and then investigate the Skolem even difference meanness of some standard graphs.

1. Background

Let $G=(V,E)$ be a simple, finite and undirected (p,q)-graph of order $|V|=p$ and size $|E|=q$. A graph labeling is an assignment of integers to the vertices or edges (or both) of a graph subject to certain conditions. Many types of labeling have been introduced over the last few decades. An excellent survey of graph labeling is available in Gallian (2016). Terms and notations not defined here are used in the sense of Harary (1972). The concept of mean graph was introduced in Somasundaram and Ponraj (2003). A graph G is called a mean graph if there is an injection $f:V(G)\rightarrow\{0,1,2,\ldots,q\}$ such that each edge uv is labeled with $\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$ and the resulting edge labels are distinct. In 2006, Manickam and Marudai studied the odd mean labeling of graphs. A graph G is said to be odd mean if there exists an injection $f:V(G)\rightarrow\{0,1,2,\ldots,2q-1\}$ and an induced bijection $f^*:E(G)\rightarrow\{1,3,5,\ldots,2q-1\}$ such that edge uv is labeled with $\left\lceil\frac{f(u)+f(v)}{2}\right\rceil$. The notion of Skolem difference mean graph was due to Murugan and Subramanian (2011) and further studied by Ramya et al. (2013) and Ramya and Selvi (2014). A graph is difference mean if there exists an injection $f:V(G)\rightarrow\{1,2,\ldots,p+q\}$ and an induced bijection $f^*:E(G)\rightarrow\{1,2,3,\ldots,q\}$ such that each edge uv (with $f(u)>f(v)$) is labeled with $\left\lceil\frac{f(u)-f(v)}{2}\right\rceil$. Ramya et al. (2014) defined the concept of Skolem odd difference mean graph and further studied in Jeyanthi et al. (2016). A graph G is Skolem odd difference mean if there exists an injection $f:V(G)\rightarrow\{0,1,2,\ldots,p+3q-3\}$ and an induced bijection $f^*:E(G)\rightarrow\{1,3,5,\ldots,2q-1\}$ such that each edge uv (with $f(u)>f(v)$) is labeled with $\left\lceil\frac{f(u)-f(v)}{2}\right\rceil$. Moreover, G is a Skolem even vertex odd difference mean graph if for each vertex $v, f(v)$ is even. Motivated by the results in Ramya et al. (2014) and Jeyanthi et al. (2016), in this paper, we introduce the dual version of Skolem odd difference mean labeling. A graph G is Skolem even difference mean if there exists an injection $f:V(G)\rightarrow\{0,1,2,\ldots,p+3q-1\}$ and an induced bijection $f^*:E(G)\rightarrow\{2,4,6,\ldots,2q\}$ such that each edge uv (with $f(u)>f(v)$) is labeled with $\left\lceil\frac{f(u)-f(v)}{2}\right\rceil$. Moreover, G is a

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Skolem even vertex even difference mean graph if for each vertex $v, f(v)$ is even. In the present study, we also use the following definitions.

Let $K_{m,n}$ be the complete bipartite graph with partite sets of size m and n respectively. A caterpillar $S(n_1, n_2, \ldots, n_m)$ is obtained from a path $P_m : v_1v_2v_3 \ldots v_m$ by adding n_i pendant edges to vertex v_i $(1 \leqslant i \leqslant m)$. The coconut tree $T(n, m)$ is obtained by identifying the central vertex of the star $K_{1,n}$ with a pendant vertex of a path P_m. The graph $P_m@P_n$ is obtained from P_m and m copies of P_n by identifying one pendant vertex of the i-th copy of P_n with the i-th vertex of P_m. The graph mP_n is the disjoint union of m copies of P_n. Let $B(m, n)$ be the bistar obtained from a $K_{1,m}$ and a $K_{1,n}$ by joining their central vertex u and v with an edge. The graph $B(m, n{:}\, P_w), w \geqslant 2$, is obtained from the bistar $B(m, n)$ by replacing the edge uv with P_w. A graph obtained from a path by attaching exactly two pendant edges to each internal vertex of a path $P_m, m \geqslant 3$, is called a twig and is denoted by $Tg(m)$.

This study is organized into three sections. In Section 2, we show that there exist Skolem odd difference mean graphs with non-cycle and non-tree component(s). In Section 3, we investigate the Skolem even difference meanness of some standard graphs.

2. Construction of Skolem odd difference mean graphs

By definition, it is a fact that if G is a Skolem odd difference mean (p, q)-graph, then $p \geqslant q$ (also see Theorem 2.6 in Selvi et al. (2015) for a proof by contrapositive). If G is connected, then $q = p$ or $q = p - 1$. Hence, G is a graph with one cycle or a tree.

Lemma 2.1. *If G is a connected Skolem odd difference mean (p, p)-graph, then $f(u)$ is odd for some $u \in V(G)$.*

Proof. By definition, $p + 3q - 3 = 4p - 3$ which is odd. Hence the largest edge label $2p - 1$ must be obtained by labeling two adjacent vertices with 0 and $4p - 3$ giving us an odd vertex label. □

Theorem 2.2. *The disjoint union of paths of length at least 2 is a Skolem even vertex odd difference mean graph.*

Proof. Suppose the paths are $\bigcup P_{n_i}, 1 \leqslant i \leqslant t, t \geq 2$ and $n_i \leqslant n_j$ for $i < j$. Let the vertices of the i-th path P_{n_i} be $u_{i,1}$ to u_{i,n_i}. Now we have $\sum n_i$ vertices and $\sum n_i - t$ edges. Hence, we need to label the vertices by integers in $[0, 4\sum n_i - 3t - 3]$.

1. Label $u_{1,j}$ by $0, 4, 8, \ldots$ for $j = 1, 3, 5, \ldots, n_1$ (or $n_1 - 1$ if n_1 is even).
2. For $i = 1, 2, \ldots, t - 1$, let a_i be the largest used label for P_{n_i}. Label $u_{i+1,j}$ of $P_{n_{i+1}}$ by $a_i + 2, a_i + 6, a_i + 10, \ldots$ for $j = 1, 3, 5, \ldots, n_{i+1}$ (or $n_{i+1} - 1$ if n_{i+1} is even).
3. Label $u_{t,j}$ by $a_t + 2, a_t + 6, a_t + 10, \ldots$ for $j = n_t, n_t - 2, n_t - 4, \ldots, 2$ (or $n_t - 1, n_t - 3, n_t - 5, \ldots, 2$ if n_t is odd).
4. For $i = t, t - 1, \ldots, 2$, let b_i be the largest used label for P_{n_i}. Label $u_{i-1,j}$ of $P_{n_{i-1}}$ by $b_i + 2, b_i + 6, b_i + 10, \ldots$ for $j = n_t, n_t - 2, n_t - 4, \ldots, 2$ (or $n_t - 1, n_t - 3, n_t - 5, \ldots, 2$ if n_t is odd).

It is easy to verify that the largest vertex label used is $4\sum n_i - 4t - 2$. Moreover, all the vertex labels are even such that the induced edge labels are odd integers from 1 to $2\sum n_i - 2t - 1$. □

For example, we can label the vertices of $2P_3 \cup P_4 \cup P_6 \cup P_7$ by $0, 70, 4; 6, 68, 10; 12, 66, 16, 62; 18, 60, 22, 56, 26, 52; 28, 50, 32, 46, 36, 42, 40$ according to the labeling function as defined above.

Theorem 2.3. *The graph $C_m \cup P_n$ is a Skolem odd difference mean graph for $m = 4, 6$ and $n \geqslant 2$.*

Proof. Consider $C_4 \cup P_n$. Let $C_4 = u_1u_2u_3u_4u_1$. Define a function $f : V \to \{0, 1, \ldots, 4n + 10\}$ by $f(u_1) = 0, f(u_2) = 4n + 10, f(u_3) = 8, f(u_4) = 4n + 6, f(v_{2i-1}) = 4n + 2 - 4(i - 1)$ for $1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil$, and $f(v_{2i}) = 5 + 4i$ for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.

The induced edge label function f^* is defined as $f^*(u_1u_2) = 2n + 5, f^*(u_2u_3) = 2n + 1, f^*(u_3u_4) = 2n - 1, f^*(u_4u_1) = 2n + 3$, and $f^*(v_iv_{i+1}) = 2n - 3 - 2(i - 1)$ for $1 \leqslant i \leqslant n - 1$.

Consider $C_6 \cup P_n$. Let $C_6 = u_1u_2u_3u_4u_5u_6u_1$. Define a function $f : V \to \{0, 1, 2, \ldots, 4n + 18\}$ by $f(u_1) = 0, f(u_2) = 4n + 14, f(u_3) = 4, f(u_4) = 4n + 6, f(u_5) = 12, f(u_6) = 4n + 18, f(v_1) = 4n + 2, f(v_{2i-1}) = 2(2n - 2i + 1)$ for $2 \leqslant i \leqslant \lceil \frac{n}{2} \rceil$, and $f(v_{2i}) = 1 + 4i$ for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.

The induced edge label function f^* is defined as $f^*(u_1u_2) = 2n + 7, f^*(u_2u_3) = 2n + 5, f^*(u_3u_4) = 2n + 1, f^*(u_4u_5) = 2n - 3, f^*(u_5u_6) = 2n + 3, f^*(u_6u_1) = 2n + 9, f^*(v_1v_2) = 2n - 1, f^*(v_iv_{i+1}) = 2n - 2i - 1$ for $2 \leqslant i \leqslant n - 1$. Thus, f is a Skolem odd difference mean labeling of $C_m \cup P_n$ for $m = 4, 6$ and $n \geqslant 2$. □

Theorem 2.4. *The graph $K_{2,n} \cup (n - 1)K_2$ is a Skolem even vertex odd difference mean graph for all $n \geqslant 2$.*

Proof. Let the vertices of $K_{2,n}$ be u_1, u_2 and $v_i, 1 \leqslant i \leqslant n$ whereas the vertices of $(n - 1)K_2$ be $x_j, y_j, 1 \leqslant j \leqslant n - 1$ so that the edges are u_1v_i, u_2v_i and x_jy_j. Define a function $f : V \to \{0, 1, 2, \ldots, 12n - 6\}$ as $f(u_1) = 0, f(u_2) = 4n, f(v_i) = 12n - 2 - 4i, 1 \leqslant i \leqslant n$ and $f(x_j) = 2j, f(y_j) = 4n - 2 - 2j, 1 \leqslant j \leqslant n - 1$ (see Fig. 1).

The induced edge label function f^* is defined as $f^*(u_1v_i) = 6n - 1 - 2i, f^*(u_2v_i) = 4n - 1 - 2i, 1 \leqslant i \leqslant n$ and $f^*(x_jy_j) = 2(n - j) - 1$ for $1 \leqslant j \leqslant n - 1$. Thus f is a Skolem even vertex odd difference mean labeling of $K_{2,n} \cup (n - 1)K_2$. □

Corollary 2.5. *The graph nK_2 is Skolem odd difference mean for $n \geqslant 1$ with all vertex labels of same parity.*

Theorem 2.6. *If G is a Skolem even vertex odd difference mean $(q + 1, q)$-graph, then $G \cup nK_2$ is a Skolem odd difference mean graph for all $n \geqslant 1$.*

Proof. Let G be a Skolem even vertex odd difference mean $(q + 1, q)$-graph with a labeling $f : V(G) \to \{0, 1, 2, \ldots, 4q - 2\}$ and its induced edge labeling f^*. The edge set labels are $1, 3, 5, \ldots, 2q - 1$. Define $G\prime = G \cup nK_2$ with $V(nK_2) = \{u_i, v_i : 1 \leqslant i \leqslant n\}$ and $E(nK_2) = \{u_iv_i : 1 \leqslant i \leqslant n\}$ so that $|V(G\prime)| = q + 1 + 2n$ and $|E(G\prime)| = q + n$. Define an injective function $g : V(G\prime) \to \{0, 1, 2, \ldots, 4q - 2 + 5n\}$ such that $g(v) = f(v)$ for each $v \in V(G)$. Similarly to the vertex labeling of $(n - 1)K_2$ for Theorem 2.4, also define $g(u_i) = 2i - 1, g(v_i) = 4q + 4n - 2i + 1$ for $1 \leqslant i \leqslant n$. The induced edge label function g^* is defined as $g^*(e) = f^*(e)$ and $g^*(u_iv_i) = (4q + 4n - 4i + 2)/2 = 2(q + n - i) + 1$ for $1 \leqslant i \leqslant n$. Thus, $g(u_i), g(v_i)$ are odd and $\{g^*(u_iv_i)\} = \{2q + 1, 2q + 3, \ldots, 2(q + n) - 1\}$. Hence g is a Skolem odd difference mean labeling of $G \cup nK_2$. □

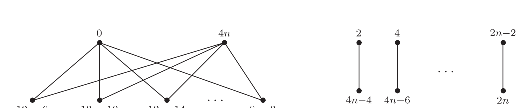

Fig. 1. Vertex labeling of $K_{2,n} \cup (n - 1)K_2$.

Corollary 2.7. *The graph $G \cup nK_2$ is a Skolem odd difference mean graph if G is a Skolem even vertex odd difference mean graph as in Theorems 2.2 and 2.3 or in Somasundaram and Ponraj (2003).*

Theorem 2.8. *If G is a Skolem even vertex odd difference mean $(q+1, q)$-graph, then $G \cup P_n$ is a Skolem odd difference mean graph for all $n \geqslant 2$.*

Proof. Let G be a Skolem even vertex odd difference mean $(q+1, q)$-graph with a labeling $f: V(G) \to \{0, 1, 2, \ldots, 4q-2\}$ and its induced edge labeling f^*. The edge set labels are $1, 3, 5, \ldots, 2q-1$. Define $G\prime = G \cup P_n$ with $V(P_n) = \{u_i, 1 \leqslant i \leqslant n\}$ and $E(P_n) = \{u_iu_{i+1}, 1 \leqslant i \leqslant n-1\}$ so that $|V(G\prime)| = q+n+1$ and $|E(G\prime)| = q+n-1$. Define an injective function $g: V(G\prime) \to \{0, 1, 2, \ldots, 4q+4n-5\}$ such that $g(v) = f(v)$ for each $v \in V(G), g(u_{2i-1}) = 4i-3$ for $1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil$ and $g(u_{2i}) = 4q+4(n-i)-1$ for $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. The induced edge label function g^* is defined as $g^*(e) = f^*(e)$ and $g^*(u_iu_{i+1}) = 2(q+n-i)-1$ for $1 \leqslant i \leqslant n-1$. It can be verified that $g(u_i)$ is odd for $1 \leqslant i \leqslant n$, and $\{g^*(u_iu_{i+1})\} = \{2q+1, 2q+3, \ldots, 2(q+n)-1\}$. Hence g is a Skolem odd difference mean labeling of $G \cup P_n$ for all $n \geqslant 2$. □

Corollary 2.9. *The graph $G \cup P_n$ is a Skolem odd difference mean graph if G is a Skolem even vertex odd difference mean graph as in Theorems 2.2 and 2.3 or in Somasundaram and Ponraj (2003).*

Theorem 2.10. *If G is a Skolem even vertex odd difference mean $(q+1, q)$-graph, then $G \cup K_{1,n}$ is a Skolem odd difference mean graph for all $n \geqslant 1$.*

Proof. Let G be a Skolem even vertex odd difference mean $(q+1, q)$-graph with a labeling $f: V(G) \to \{0, 1, 2, \ldots, 4q-2\}$ and its induced edge labeling f^*. The edge set labels are $1, 3, 5, \ldots, 2q-1$. Define $G\prime = G \cup K_{1,n}$ with $V(K_{1,n}) = \{u, u_i, 1 \leqslant i \leqslant n\}$ and $E(K_{1,n}) = \{uu_i, 1 \leqslant i \leqslant n\}$ so that $|V(G\prime)| = q+n+2$ and $|E(G\prime)| = q+n$. Define an injective function $g: V(G\prime) \to \{0, 1, 2, \ldots, 4q+4n-1\}$ such that $g(v) = f(v)$ for each $v \in V(G)$ and $g(u) = 1, g(u_i) = 4q+4i-1$ for $1 \leqslant i \leqslant n$ (see Fig. 2).

The induced edge label function g^* is defined as $g^*(e) = f^*(e)$ and $g^*(uu_i) = 2(q+i)-1$ for $1 \leqslant i \leqslant n$. It can be verified that $g(u_i)$ is odd for $1 \leqslant i \leqslant n$, and $\{g^*(u_iu_{i+1})\} = \{2q+1, 2q+3, \ldots, 2(q+n)-1\}$. Hence g is a Skolem odd difference mean labeling of $G \cup K_{1,n}$ for all $n \geqslant 1$. □

Corollary 2.11. *The graphs $G \cup K_{1,n}$ are Skolem odd difference mean graph if G is a Skolem even vertex odd difference mean graph as in Theorems 2.2 and 2.3 or in Somasundaram and Ponraj (2003).*

3. Skolem even difference mean graphs

As a natural extension, we introduce in this section the Skolem even difference mean labeling of graphs.

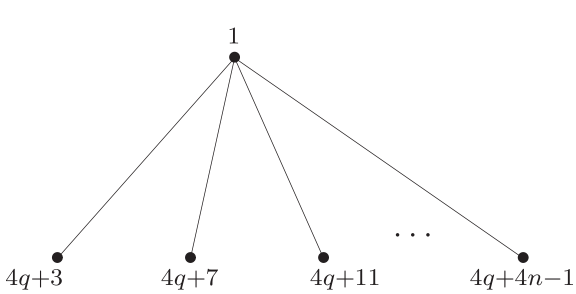

Fig. 2. Vertex labeling of $K_{1,n}$ in $G \cup K_{1,n}$.

Observation 3.1. *If G is a Skolem even difference mean (p, q)-graph, then $p \geqslant q$* (similar to Theorem 2.6 in Selvi et al. (2015)).

Observation 3.2. *If $p = q+1$, then any Skolem even difference mean labeling of G must admit an even vertex labeling.*

Theorem 3.1. *The caterpillar $S(n_1, n_2, \ldots, n_m)$ is a Skolem even vertex even difference mean graph.*

Proof. Let $V(S(n_1, n_2, \ldots, n_m)) = \{v_j, u_i^j : 1 \leqslant i \leqslant n_j, 1 \leqslant j \leqslant m\}$ and $E(S(n_1, n_2, \ldots, n_m)) = \{v_jv_{j+1} : 1 \leqslant j \leqslant m-1\} \cup \{v_ju_i^j : 1 \leqslant i \leqslant n_j, 1 \leqslant j \leqslant m\}$. Define a function $f: V(S(n_1, n_2, \ldots, n_m)) \to \{0, 1, 2, 3, 4, \ldots, p+3q-1 = 4(m+n_1+n_2+\ldots+n_m-1)\}$ such that

1. $f(v_1) = 0, f(v_{2j-1}) = 4(n_2+n_4+\ldots+n_{2j-2})+4(j-1)$ for $2 \leqslant j \leqslant \lceil \frac{m}{2} \rceil$,
2. $f(v_{2j}) = 4(m+n_1+n_2+\ldots+n_m) - 4(n_1+n_3+\ldots+n_{2j-1}+j)$ for $1 \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor$,
3. $f(u_i^1) = 4(m+n_1+n_2+\ldots+n_m) - 4i$ for $1 \leqslant i \leqslant n_1$, and $f(u_i^{2j-1}) = 4(m+n_1+n_2+\ldots+n_m) - 4(n_1+n_3+\ldots+n_{2j-3}+ +i+j-1)$ for $2 \leqslant j \leqslant \lceil \frac{m}{2} \rceil, 1 \leqslant i \leqslant n_{2j-1}$,
4. $f(u_i^2) = 4i$ for $1 \leqslant i \leqslant n_2$, and $f(u_i^{2j}) = 4(n_2+n_4+\ldots +n_{2j-2}+i+j-1)$ for $2 \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor, 1 \leqslant i \leqslant n_{2j}$.

Let $e_j = v_jv_{j+1}$ for $1 \leqslant j \leqslant m-1$ and $e_i^j = v_ju_i^j$ for $1 \leqslant i \leqslant n_j, 1 \leqslant j \leqslant m$. For each vertex label f the induced edge label f^* is defined as follows:$f^*(e_i^j) = 2(m+n_j+n_{j+1}+\ldots+n_m) - 2(i+j-1)$ for $1 \leqslant j \leqslant m, 1 \leqslant i \leqslant n_j f^*(e_j) = 2(m+n_{j+1}+n_{j+2} +\ldots+n_m) - 2j$ for $1 \leqslant j \leqslant m-1$.

Thus f is a Skolem even difference mean labeling of $S(n_1, n_2, \ldots, n_m)$. Hence $S(n_1, n_2, \ldots, n_m)$ is a Skolem even difference mean graph. □

For example, the Skolem even difference mean labeling of $S(4, 2, 3, 2)$ is shown in Fig. 3.

Corollary 3.2. *The graphs (i) $T(n, m)$, (ii) $B(r, s\colon P_w)$ and (iii) $Tg(m)$ are Skolem even difference mean.*

Proof. It follows from Theorem 3.1 such that for (i), we take $n_1 = n, n_2 = n_3 = \cdots = n_m = 0$; for (ii), we take $n_1 = r$, $n_2 = n_3 = \cdots = n_{w-1} = 0, n_w = s$; for (iii), we take $n_1 = n_m = 0$ and $n_2 = n_3 = \cdots = n_{m-1} = 2$. □

The following examples illustrate the three cases in the corollary.

- **Case (i).** The Skolem even difference mean labeling of $T(5, 6)$ is shown in Fig. 4.
- **Case (ii).** The Skolem even difference mean labeling of $B(4, 3\colon P_5)$ is shown in Fig. 5.
- **Case (iii).** The Skolem even difference mean labeling of $Tg(4)$ is shown in Fig. 6.

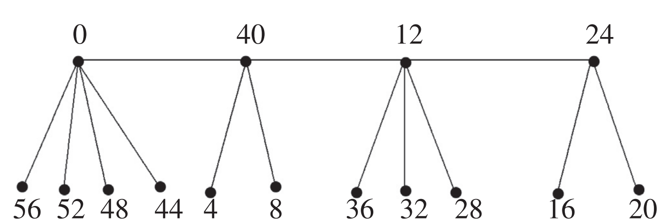

Fig. 3. $S(4, 2, 3, 2)$ is a Skolem even difference mean graph.

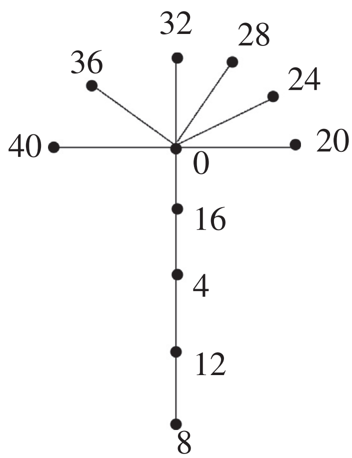

Fig. 4. $T(5,6)$ is a Skolem even difference mean graph.

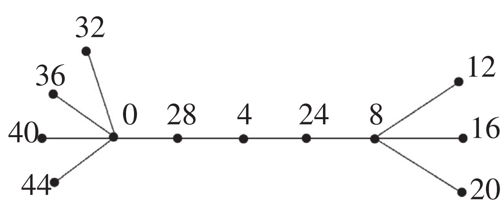

Fig. 5. $B(4,3\colon P_5)$ is a Skolem even difference mean graph.

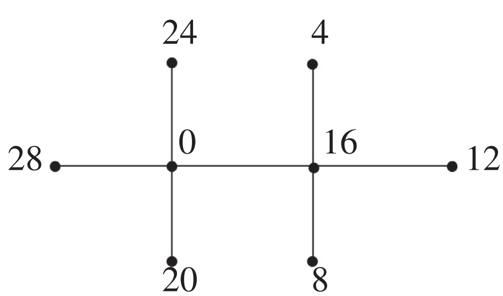

Fig. 6. $Tg(4)$ is a Skolem even difference mean graph.

Theorem 3.3. *The graph $P_m@P_n$ is a Skolem even difference mean graph.*

Proof. Let $V(P_m@P_n) = \{u_i^j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ and $E(P_m@\,P_n) = \{u_n^j u_n^{j+1} : 1 \leqslant j \leqslant m-1\} \cup \{u_i^j u_{i+1}^j : 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m\}$. Define a function $f : V(P_m@P_n) \to \{0,1,2,3,\ldots p+3q-1 = \ 4(mn-1)\}$ such that

$$f(u_i^j) = 2n(j-1) + 2(i-1) \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \text{ and } i \text{ is odd}, j \text{ is odd},$$
$$f(u_i^j) = 2n(2m-j+1) - 2i \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \text{ and } i \text{ is even}, j \text{ is odd},$$
$$f(u_i^j) = 2n(2m-j) + 2(1-i) \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \text{ and } i \text{ is odd}, j \text{ is even},$$
$$f(u_i^j) = 2nj - 2i \quad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \text{ and } i \text{ is even}, j \text{ is even}.$$

Let $e_j = u_n^j u_n^{j+1}$ for $1 \leqslant j \leqslant m-1$ and $e_i^j = u_i^j u_{i+1}^j$ for $1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m$. For each vertex label of f, the induced edge label function f^* is defined as follows:

$$f^*(e_j) = 2n(m-j) \quad \text{for } 1 \leqslant j \leqslant m-1,$$
$$f^*(e_i^j) = 2n(m-j+1) - 2i \quad \text{for } 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m \text{ and } j \text{ is odd},$$
$$f^*(e_i^j) = 2n(m-j) + 2i \quad \text{for } 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m \text{ and } j \text{ is even}.$$

Thus f is a Skolem even difference mean labeling of $P_m@P_n$. □

The Skolem even difference mean labeling of $P_4@P_4$ is shown in Fig. 7.

Theorem 3.4. *The graph mP_n is a Skolem even difference mean graph.*

Proof. Let $V(mP_n) = \{u_i^j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ and $E(mP_n) = \{u_i^j u_{i+1}^j : 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m\}$. Define a function $f : V(mP_n) \to \{0,1,2,\ldots,p+3q-1 = 4mn-3m-1\}$ as follows:

If n is odd, then

$$f(u_{2i-1}^j) = 2n(2m-j+1) - 4(m+i-1) \quad \text{for } 1 \leqslant i \leqslant \left\lceil \frac{n}{2} \right\rceil, 1 \leqslant j \leqslant m,$$
$$f(u_{2i}^j) = 2(n-2)(j-1) + 4(i-1) \quad \text{for } 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor, 1 \leqslant j \leqslant m.$$

If n is even, then

$$f(u_{2i-1}^j) = 4m(n-1) + 2n(1-j) + 2(1+j-2i) \quad \text{for } 1 \leqslant i \leqslant \left\lceil \frac{n}{2} \right\rceil, 1 \leqslant j \leqslant m,$$
$$f(u_{2i}^j) = 2(n-1)(j-1) + 4(i-1) \quad \text{for } 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor, 1 \leqslant j \leqslant m.$$

For each vertex labeling of f, the induced edge label function f^* is defined as $f^*(u_i^j u_{i+1}^j) = 2(n-1)(m-j+1) - 2(i-1)$ for $1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant m$. Thus f is a Skolem even difference mean labeling of mP_n.

The Skolem even difference mean labeling of $3P_4$ is shown in Fig. 8.

Theorem 3.5. *If G is a Skolem even vertex even difference mean $(q+1,q)$-graph, then $G \cup nK_2$ is a Skolem even difference mean graph for all $n \geqslant 1$.*

Proof. Let G be a Skolem even vertex even difference mean $(q+1,q)$-graph with labeling $f : V(G) \to \{0,1,2,\ldots,4q\}$ and its induced edge labeling function f^*. Clearly, $f(v)$ is even for each $v \in V(G)$ and $\{f^*(e)\} = \{2,4,6,\ldots,2q\}$. Define $G\prime = G \cup K_2$ with $V(nK_2) = \{u_i, v_i, 1 \leqslant i \leqslant n\}$ and $E(nK_2) = \{u_i v_i, 1 \leqslant i \leqslant n\}$ so that $|V(G\prime)| = q+1+2n$ and $|E(G\prime)| = q+n$. Define an injective function $g : V(G\prime) \to \{0,1,2,\ldots,4q+5n\}$ such that $g(v) = f(v)$ for each $v \in V(G)$ and $g(u_i) = 2i-1, g(v_i) = 4q+4n-2i+3$ for $1 \leqslant i \leqslant n$. The induced edge label function g^* is defined as $g^*(e) = f^*(e)$ and $g^*(u_i v_i) = (4q+4n-4i+4)/2 = 2(q+n-i+1)$ for $1 \leqslant i \leqslant n$. Thus, $g(u_i), g(v_i)$ are even and $\{g^*(u_i v_i)\} = \{2q+2, 2q+4,\ldots,2(q+n)\}$. Hence, g is a Skolem even difference mean labeling of $G \cup K_2$. □

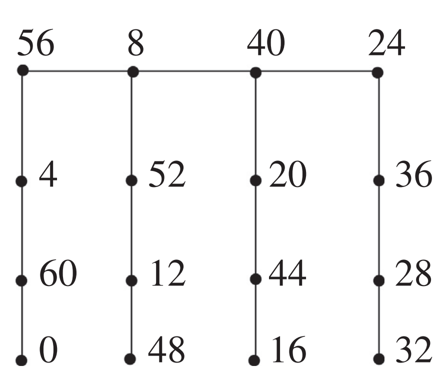

Fig. 7. $P_4@P_4$ is a Skolem even difference mean graph.

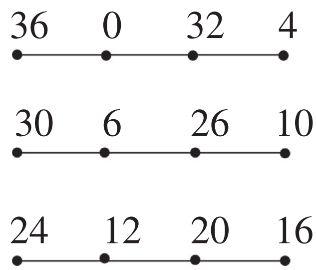

Fig. 8. $3P_4$ is a Skolem even difference mean graph.

By an argument similar to that in Theorems 2.8 and 2.10, we have the following corollary.

Corollary 3.6. *If G is a Skolem even vertex even difference mean $(q+1, q)$-graph, then $G \cup P_n$ $(n \geq 2)$ and $G \cup K_{1,n}$ $(n \geq 1)$ are Skolem even difference mean graph.*

Theorem 3.7. *The graph $K_{m,n} \cup (m-1)(n-1)K_2$ is a Skolem even difference mean graph for all $m, n \geqslant 2$.*

Proof. Let $V(K_{m,n} \cup (m-1)(n-1)K_2) = \{u_i, v_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\} \cup \{x_i, y_i : 1 \leqslant i \leqslant (m-1)(n-1)\}$ and $E(K_{m,n} \cup (m-1)(n-1)\ K_2) = \{u_i v_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, x_i y_i : 1 \leqslant i \leqslant (m-1)(n-1)\}$. Define a function $f : V(K_{m,n} \cup (m-1)(n-1)K_2) \rightarrow \{0, 1, 2, 3, 4, \ldots, p+\ 3q-1 = m+n+3mn+5(m-1)(n-1)-1\}$ such that

1. $f(u_i) = 4n(i-1)$ for $1 \leqslant i \leqslant m$,
2. $f(v_j) = m+n+3mn+5(m-1)(n-1)-4j+3$ for $1 \leqslant j \leqslant n$,
3. $f(x_i) = 1+2(i-1)$ for $1 \leqslant i \leqslant (m-1)(n-1)$,
4. $f(y_i) = 4(m-1)(n-1)-2(i-1)+1$ for $1 \leqslant i \leqslant (m-1)(n-1)$.

The induced edge label function f^* is defined as follows: $f^*(u_i v_j) = (m+n+3mn+5(m-1)(n-1)-1)/2 - 2n(i-1)\ -2(j-1)$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, and $f^*(x_i y_i) = 2(m-1)\ (n-1)-2(i-1)$ for $1 \leqslant i \leqslant (m-1)(n-1)$. Thus, f is a Skolem even difference mean labeling of $K_{m,n} \cup (m-1)(n-1)K_2$. □

The Skolem even difference mean labeling of $K(2,3) \cup 2K_2$ is shown in Fig. 9.

Theorem 3.8. *The graph $K_{1,n} \cup nK_2$ is a a Skolem even difference mean graph graph for all $n \geqslant 1$.*

Proof. Let $V(K_{1,n} \cup nK_2) = \{v_0, v_j : 1 \leqslant j \leqslant n\} \cup \{x_i, y_i : 1 \leqslant i \leqslant n\}$ and $E(K_{1,n} \cup nK_2) = \{v_0 v_i, x_i y_i : 1 \leqslant i \leqslant n\}$. Define a function $f : V(K_{1,n} \cup nK_2) \rightarrow \{0, 1, 2, 3, \ldots, p+3q-1 = 9n\}$ such that $f(v_0) = 0$, $f(v_i) = 8n - 4(i-1)$ for $1 \leqslant i \leqslant n, f(x_i) = 2i-1$ for $1 \leqslant i \leqslant n$, and $f(y_i) = 4n - 2(i-1)$ for $1 \leqslant i \leqslant n$. The induced edge label function f^* is defined as $f^*(v_0 v_i) = 4n - 2(i-1)$ for $1 \leqslant i \leqslant n$, and $f^*(x_i y_i) = 2n - 2(i-1)$ for $1 \leqslant i \leqslant n$. Thus, f is a Skolem even difference mean labeling of $K_{1,n} \cup nK_2$. □

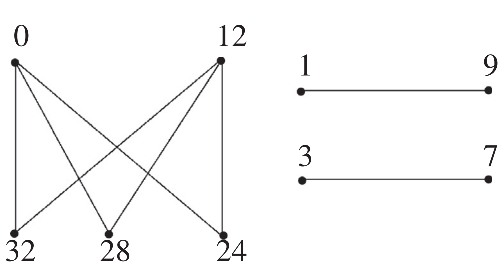

Fig. 9. $K(2,3) \cup 2K_2$ is a Skolem even difference mean graph.

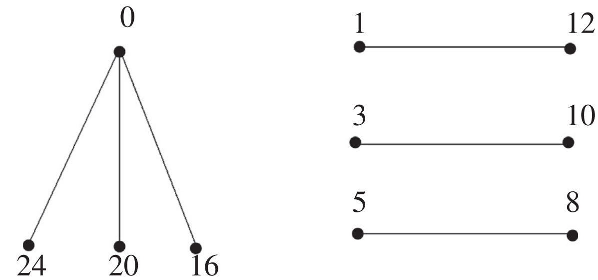

Fig. 10. $K_{1,3} \cup 3K_2$ is a Skolem even difference mean graph.

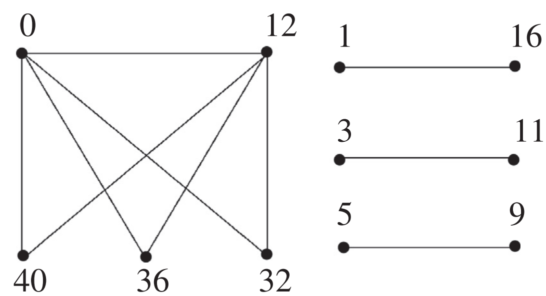

Fig. 11. $K(1,1,3) \cup 3K_2$ is a Skolem even difference mean graph.

The Skolem even difference mean labeling of $K_{1,3} \cup 3K_2$ is shown in Fig. 10.

Theorem 3.9. *The graph $K_{1,1,n} \cup nK_2$ is a Skolem even difference mean graph for all $n \geqslant 1$.*

Proof. Let $V(K_{1,1,n} \cup nK_2) = \{u, w, u_i, x_i, y_i : 1 \leqslant i \leqslant n\}$ and $E(K_{1,1,n} \cup nK_2) = \{uw, uu_i, wu_i, x_i y_i : 1 \leqslant i \leqslant n\}$. Define a function $f : V(K_{1,1,n} \cup nK_2) \rightarrow \{0, 1, 2, 3, \ldots, p+3q-1 = 4(3n+1)\}$ such that $f(u) = 0, f(w) = 4n, f(u_i) = 4(3n-i+2)$ for $1 \leqslant i \leqslant n$, $f(x_i) = 2i-1$ for $1 \leqslant i \leqslant n, f(y_1) = 4(n+1), f(y_i) = 4n-2i+3$ for $2 \leqslant i \leqslant n$. The induced edge label function f^* is defined as follows:

1. $f^*(uw) = 2n$,
2. $f^*(uu_i) = 2(3n-i+2)$ for $1 \leqslant i \leqslant n$,
3. $f^*(wu_i) = 2(2n-i+2)$ for $1 \leqslant i \leqslant n$,
4. $f^*(x_1 y_1) = 2(n+1)$,
5. $f^*(x_i y_i) = 2(n-i+1)$ for $2 \leqslant i \leqslant n$.

Thus f is a Skolem even difference mean labeling of $K_{1,1,n} \cup nK_2$. □

The Skolem even difference mean labeling of $K(1,1,3) \cup 3K_2$ is shown in Fig. 11.

4. Conclusion

In this paper first we show that there exist Skolem odd difference mean labeling for graphs with non-cycle and non-tree components. Further, we introduce the concept of Skolem even difference mean labeling. We conclude the paper with the following open problem.

Problem 4.1. Establish the Skolem even difference mean labeling of $G \cup nK_2$ where G is a (complete) multipartite graph and $n \geqslant 1$.

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