Journal of King Saud University – Science (2016) **28**, 280–285

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

REVIEW ARTICLE

Multistage Bernstein polynomials for the solutions of the Fractional Order Stiff Systems

M.H.T. Alshbool [a,*], **I. Hashim** [b]

[a] *School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM, Bangi, Selangor, Malaysia*
[b] *School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Research Institute, Center for Modeling & Computer Simulation, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia*

Received 24 January 2015; accepted 3 June 2015
Available online 18 June 2015

KEYWORDS

Multistage Bernstein polynomials;
Fractional Order Stiff Systems

Abstract In this paper, a new modification of the Bernstein polynomials method called Multistage Bernstein polynomials (MB-polynomials) is applied to solve new topic, which is Fractional Order Stiff Systems. The MB-polynomials is a simple reliable modification based on adapting standard Bernstein polynomials method. The procedure of the method is explained briefly and supported with illustrative examples to demonstrate the validity of the method. The results of MB-polynomials are compared with the traditional Bernstein polynomials method and several other methods that solved stiff systems. The results attest to the efficiency of the proposed method.

Contents

1. Introduction

Fractional Order Stiff Systems have been employed to describe a variety of systems such as biology, physiology, medicine, hydraulics geology, and engineering. Fractional Order Stiff Systems are considered a new topic due to their potential applications especially in control processing. Stiff problems

* Corresponding author.
E-mail address: alshbool.mohammed@gmail.com (M.H.T. Alshbool).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.06.001
1018-3647

have been studied in many areas such as chemical engineering, non linear mechanics, biochemistry, and life sciences. Hence, the need for a reliable and efficient technique for the solution of stiff systems of differential equations is of high importance. Since 30 years, respected works have focused on the development of more advanced and efficient methods for stiff problems. BDF is a formula that is based on backward differentiation, and many modifications introduced by different authors are extended backward differentiation. First modification is (EBDF) formula introduced in Cash (1980). The MEBDF (modified EBDF) (Cash, 1983) and MF-MEBDF (matrix free MEBDF) (Hosseini and Hojjati, 1999), are modification methods of (BDF) formula used to solve stiff systems of ordinary differential equations. A-EBDF is also a modification of (BDF) applied to solve stiff systems of ordinary differential equations (Hojjati et al., 2004). Haar wavelets are used for linear and nonlinear stiff system of ordinary differential equations (Hsiao, 2004; Hsiao and Wang, 2001). Adomian decomposition method is applied on stiff problems (Saad Mahmood et al., 2005). Furthermore, modification of Homotopy perturbation methods which is called Rational Homotopy perturbation method (RHPM) is used to obtain an analytic approximation of stiff systems of ordinary differential equations (Biazar et al., 2015).

One of the important analytic methods for solving linear and nonlinear equations is Bernstein polynomials (B-polynomials). Bernstein operational matrix of differentiation proposed by Bhatti and Bracken (2007) used Bernstein polynomial basis to solve differential equation. Bernstein operational matrix for solving Lane–Emden type equations (Pandey and Kumar, 2012). operational matrices of Bernstein polynomials and their applications to solve Bessel differential equation (Yousefi and Behroozifar, 2010). Bernstein polynomials to solve fractional riccati type differential equations (Yuzbasi, 2013). Bernstein series solution of linear second-order partial differential equations with mixed conditions(Isik et al., 2012). Recently, Yiming et al. (2014) used Bernstein polynomials to find Numerical solution for the variable order linear cable equation. Approximate solutions of singular differential equations with estimation error by using Bernstein polynomials (Alshbool et al., 2015).

The interpolation polynomial used in (B-polynomials) method is a good approximation to the function $y(x)$, and for large n. Bernstein polynomials (B-polynomials) have many useful properties. The procedure takes advantage of the continuity and unity partition properties of the basis set of B-polynomials over an interval $[0,\mathrm{R}]$. This provides greater flexibility to impose boundary conditions at the end points of the interval. It also ensures that the sum at any point x of all B-polynomials is unity. For this reason we choose (B-polynomials) method.

In this paper, we present a new modification of Bernstein polynomials called Multistage Bernstein polynomials (MB-polynomials). The proposed method minimizes the error of the result of Fractional and ordinary Order Stiff Systems which is solved by standard Bernstein polynomials method. Moreover, the solution provided by MB-polynomials is valid in larger x than standard B-polynomials. The results of MB-polynomials are compared with B-polynomials, A-EBDFs methods, HPM and RHPM, and we show that MB-polynomials obtains more accurate result.

The rest of this paper is organized as follows: In Section 2, we present some definitions and properties of fractional calculus. In Section 3, some basic definitions of Fractional Order Stiff Systems are provided. In Section 4, we describe the standard B-polynomials and the MB-polynomials. Section 5, presents numerical comparisons with several methods which indicate that the MB-polynomials method is a simple, yet powerful method to give the approximate solutions for Fractional Order Stiff Systems. Finally we summarize our work in Section 6 and suggest some recommendations for future work.

2. Preliminaries and notations

In this section, we give some definitions and properties of fractional calculus according to Diethelm et al. (2005).

Definition 2.1. A real function $f(x),\ x>0$, is said to be in the space $C_\mu,\ \mu\in\Re$, if there exists a real number $p>\mu$, such that $f(x)=x^pf_1(x)$, where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu,\ n\in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha\geqslant 0$, of a function $f\in C_\mu,\ \mu\geqslant -1$, is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(x)ds \quad (\alpha>0),$$
$$J^0f(x)=f(x), \tag{1}$$

where $\Gamma(\alpha)$ is well-known gamma function. Some of the properties of the operator J^α, which we will need here, are as follows: For $f\in C_\mu,\ \mu\geqslant -1,\ \alpha,\beta\geqslant 0$ and $\gamma\geqslant -1$:

1. $J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x)$,
2. $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$.

Definition 2.3. The fractional derivative (D^α) of $f(t)$, in the Caputo sense is defined as

$$D^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (x-s)^{n-\alpha-1}f^{(n)}(x)ds \tag{2}$$

for $n-1<\alpha<n,\ n\in N,\ x>0,\ f\in C_{-1}^n$.

The following are two basic properties of the Caputo fractional derivative (Diethelm et al., 2005):

1. Let $f\in C_{-1}^n,\ n\in N$. Than $D^\alpha f,\ 0\leqslant\alpha\leqslant n$ is well defined and $D^\alpha f\in C_{-1}$.
2. Let $n-1\leqslant\alpha\leqslant n,\ n\in N$ and $f\in C_\mu^n,\ \mu\geqslant -1$. Then

$$(J^\alpha D^\alpha)f(x)=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{x^k}{k!}. \tag{3}$$

For the Caputo derivative we have

$$D_*^\alpha c=0 \quad (c \quad \text{constant}), \tag{4}$$

$$D_*^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \text{ and } \beta \geqslant \lceil \alpha \rceil \text{ or } \beta > \lfloor \alpha \rfloor. \end{cases} \quad (5)$$

We note that the approximate solutions will be found by using the Caputo fractional derivative and its properties in this study.

3. Fractional Order Stiff Systems

We consider a stiff system of FDEs:

$$D^\alpha y_j + f_j(x, y_1, y_2, \ldots, y_n) = g_j(x). \quad (6)$$

Subject to the initial conditions

$$y_j(x_0) = \beta_i, \quad (7)$$

where β_j are constants, $(j = 1, 2, \ldots n)$.

First, we write system (6) in the form

$$D^\alpha y_j + f_j(x, y_1, y_2, \ldots, y_n) - g_j(x) = 0 \quad (8)$$

Subject to the initial conditions (7). We will next present the solution approaches for (6) based on the standard B-polynomials and MB-polynomials separately.

4. Solution by Bernstein polynomials (B-polynomials)

The Bernstein polynomials of degree m are defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m,$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

There are $m+1$ nth-degree Bernstein polynomials. For mathematical convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > m$.

In general, we approximate any function $y_j(x)$ with the first $(m+1)$ Bernstein polynomials as

$$y_j(x) = \sum_{i=0}^{m} c_{j,i} B_{i,m}(x) = \mathrm{C}_j^{\mathrm{T}} \phi(x), \quad j = 1, 2 \ldots n, \quad (9)$$

where

$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^{\mathrm{T}}.$$

$$\phi(x) = AX, \quad (10)$$

where

$$A = \begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_n \\ \vdots & \vdots & \ldots & \vdots \\ k_0 & k_1 & \ldots & k_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{pmatrix}. \quad (11)$$

For $j = 1$

$$y_1(x) = \sum_{i=0}^{m} c_{1,i} B_{i,m}(x) = \mathrm{C}_1^{\mathrm{T}} \phi(x),$$

where

$$\mathrm{C}_1^{\mathrm{T}} = [\mathrm{c}_{1,0}, \mathrm{c}_{1,1}, \ldots, \mathrm{c}_{1,m}].$$

For $j = 2$

$$y_2(x) = \sum_{i=0}^{m} c_{2,i} B_{i,m}(x) = \mathrm{C}_2^{\mathrm{T}} \phi(x),$$

where

$$\mathrm{C}_2^{\mathrm{T}} = [\mathrm{c}_{2,0}, \mathrm{c}_{2,1}, \ldots, \mathrm{c}_{2,m}],$$

$\vdots$

For $j = n$

$$y_n(x) = \sum_{i=0}^{m} c_{m,i} B_{i,m}(x) = \mathrm{C}_n^{\mathrm{T}} \phi(x),$$

where

$$\mathrm{C}_n^{\mathrm{T}} = [\mathrm{c}_{n,0}, \mathrm{c}_{n,1}, \ldots, \mathrm{c}_{n,m}].$$

The fractional derivatives of the vector $\phi(x)$ can be expressed as

$$\frac{\mathrm{d}^\alpha \phi(x)}{\mathrm{d}x} = D^\alpha \phi(x), \quad (12)$$

where D^α is the $(m+1) \times (m+1)$ operational matrix of derivative given as

$$D^\alpha = A \Omega \Psi A^{-1}, \quad (13)$$

where

$$A = \begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_n \\ \vdots & \vdots & \ldots & \vdots \\ k_0 & k_1 & \ldots & k_n \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \end{pmatrix}. \quad (14)$$

By means of the operational matrix of derivative (Yousefi and Behroozifar, 2010), we approximate $g(x)$ as

$$g_j(x) \simeq \mathrm{G}_j^{\mathrm{T}} \phi(x), \quad j = 1, \ldots, n, \quad (15)$$

where the vector $\mathrm{G}_j^{\mathrm{T}} = [g_{j,0}(x), \ldots, g_{j,m}(x)]^{\mathrm{T}}$, by applying (9), (12) and (15) on the system (8), we have the residual $\Re(x)$ as

$$\mathrm{C}_1^{\mathrm{T}} D^\alpha \phi(x) + f_1(x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)) - \mathrm{G}_1^{\mathrm{T}} \phi(x) = 0 \quad (16)$$

$$\mathrm{C}_2^{\mathrm{T}} D^\alpha \phi(x) + f_2(x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)) - \mathrm{G}_2^{\mathrm{T}} \phi(x) = 0$$

$\vdots$

$$\mathrm{C}_n^{\mathrm{T}} D^\alpha \phi(x) + f_n(x, \mathrm{C}_1^{\mathrm{T}} \phi(x), \mathrm{C}_2^{\mathrm{T}} \phi(x), \ldots, \mathrm{C}_n^{\mathrm{T}} \phi(x)) - \mathrm{G}_n^{\mathrm{T}} \phi(x) = 0$$

with the initial conditions

$$\mathrm{C}_1^{\mathrm{T}} \phi(x_0) = \beta_1, \quad \mathrm{C}_2^{\mathrm{T}} \phi(x_0) = \beta_2, \ldots, \quad \mathrm{C}_n^{\mathrm{T}} \phi(x_0) = \beta_n. \quad (17)$$

To solve the system (16), we have to find collocation points $x_0, x_1, \ldots, x_m$ which will be substituted in (16), then we will have $(m-1)$ equations, with initial condition in (17). Now we have (m) equations where the unknowns are c_i, which can be solved by using Newton's iterative method. To find the collocation points $x_0, x_1, \ldots, x_m$, follow as

$$\begin{pmatrix} B_{0,m}(x) - B_{0,m-1}(x) \\ B_{1,m}(x) - B_{1,m-1}(x) \\ \vdots \\ B_{m-1,m}(x) - B_{m-1,m-1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{18}$$

The points presented from (18) are called the collocation points $(x_0, x_1, \ldots, x_m)$.

Therefore, according to B-polynomials, the m-term approximations for the solutions of (6) can be expressed as

$$y_1(x) = C_1^T\phi(x), \quad y_2(x) = C_2^T\phi(x), \ldots, y_n(x) = C_n^T\phi(x), \tag{19}$$

4.1. Solution by Multistage Bernstein polynomials (MB-polynomials)

The approximate solution (19) is generally, as will be shown in the numerical of the paper, not valid for large x. For this reason, we present this method (MB-polynomials) which is a simple way of ensuring validity of the approximation for large x to treat (19) as an algorithm for approximating the solutions of (6) in a sequence of intervals.

Choosing the initial approximation as

$$\begin{aligned} y_{1,0}(x) &= y_1(x^*) = \beta_1^* \\ y_{2,0}(x) &= y_2(x^*) = \beta_2^* \\ &\vdots \\ y_{n,0}(x) &= y_n(x^*) = \beta_n^*, \end{aligned} \tag{20}$$

where x^* is the left end point of each interval.

Now, we solve (6) for the unknowns $y_{j,k}(x)$, $(j = 1, 2, \ldots, n,$ $k = 1, 2, \ldots)$, by applying the collocation method in (18). In order to carry out the iteration in every subinterval of equal length Δx, $[0, x_1)$, $[x_1, x_2)$, $[x_2, x_3) \ldots [x_{k-1}, x_k)$, we need to know the values of the following:

$$y_{1,0}^*(x) = y_1(x^*), \quad y_{2,0}^*(x) = y_2(x^*), \ldots, y_{n,0}^*(x) = y_n(x^*). \tag{21}$$

A simple way for obtaining the necessary value (21) could be conducted by means of the previous m-term approximation of the preceding subinterval given by (19), that is

$$y_1^*(x) = C_1^T\phi(x^*), \quad y_2^*(x) = C_2^T\phi(x^*), \ldots, y_n^*(x) = C_n^T\phi(x^*). \tag{22}$$

To get more accurate and efficient result, we have to pay attention on collocation points for each subinterval; so that, we subdivide the interval and arrange the stages of the method as

- Stage 1. Apply the method on the intervals $[0, x_1)$, $[x_1, x_2)$, $[x_2, x_3) \ldots [x_{k-1}, 1)$, the collocation points $(x_0, x_1, \ldots, x_m)$ which will be used are the points presented from (18).
- Stage 2. Apply the method on the intervals $[1, x_1)$, $[x_1, x_2)$, $[x_2, x_3) \ldots [x_{k-1}, 2)$, but the collocation points which will be used are $(x_0 + 1, x_1 + 1, \ldots, x_m + 1)$.⋮
- Stage r. Apply the method on the intervals $[r-1, x_1)$, $[x_1, x_2)$, $[x_2, x_3) \ldots$ $[x_{k-1}, r)$, where $r = 1, 2 \ldots$, and the collocation points which will be used are $(x_0 + (r-1),$ $x_1 + (r-1), \ldots, x_m + (r-1))$.

5. Numerical experiments

In this section, some numerical examples are given to illustrate the properties and effectiveness of the method. We also compare the approximate solution with some other numerical solutions.

Example 1. Consider the stiff system of FDE.

$$\begin{cases} D^\alpha y_1(x) = -y_1(x) + 95y_2(x), \\ D^\alpha y_2(x) = -y_1(x) - 97y_2(x) \end{cases} \tag{23}$$

with the initial conditions $y_1(0) = 1,\ y_2(0) = 1,\ 0 < \alpha \leqslant 1$. The exact solution when $\alpha = 1$ is

$$\begin{cases} y_1(x) = \frac{1}{47}(95e^{-2x} - 48e^{-96x}) \\ y_2(x) = \frac{1}{47}(48e^{-96x} - e^{-2x}). \end{cases} \tag{24}$$

According to the B-polynomials method in Section 4, after that, we apply our modification MB-polynomials on this example, with $m = 9,\ j = 1, 2,\ k = 6,\ r = 5$ and $0 < \alpha \le 1$.

We can approximate solution of the system as

$$y_1(x) \simeq \sum_{i=0}^{14} c_{1,i} B_{i,14}(x) = C_1^T\phi(x), \tag{25}$$

$$y_2(x) \simeq \sum_{i=0}^{14} c_{2,i} B_{i,14}(x) = C_2^T\phi(x) \tag{26}$$

with the initial condition

$$C_1^T\phi(0) = 1, \quad C_2^T\phi(0) = 1. \tag{27}$$

By (12), we define the derivative of (25) as

$$D^\alpha y_1(x) = C_1^T D^\alpha\phi(x), \quad D^\alpha y_2(x) = C_2^T D^\alpha\phi(x). \tag{28}$$

By substituting (25) and (28) in (23), we obtain

$$\begin{cases} C_1^T D^\alpha\phi(x) = -C_1^T\phi(x) + 95C_2^T\phi(x), \\ C_2^T D^\alpha\phi(x) = -C_1^T\phi(x) - 97C_2^T\phi(x), \end{cases} \tag{29}$$

the residual $\Re(x)$ for the system of Eq. (23) can be written as

$$\begin{cases} \Re_1(x) \simeq C_1^T D^\alpha\phi(x) + C_1^T\phi(x) - 95C_2^T\phi(x), \\ \Re_2(x) \simeq C_2^T D^\alpha\phi(x) + C_1^T\phi(x) + 97C_2^T\phi(x). \end{cases} \tag{30}$$

To find the unknowns c_i, we have to find the collocation points by (18) and substitute it in (30), we report the collocation points value as

$$x_0 = 1, \quad x_1 = 0.1 \ldots x_8 = 0.8 \tag{31}$$

with initial conditions in (27), then we have m equations where the unknowns are c_i, which can be solved by using Newton's iterative method and maple programming.

Hence, the solution to the stiff system (23) is:

Table 1 Comparisons between B-polynomials solutions and MB-polynomials solutions for system (23), with $m = 9$ and $\alpha = 0.90$.

x	y_i	B-polynomials	MB-polynomials
0.0	$\|y_1\|$	1	1
	$\|y_2\|$	1	1
1.0	$\|y_1\|$	0.32031	0.32965
	$\|y_2\|$	0.00673	0.00262
2.0	$\|y_1\|$	8935.894	0.11470
	$\|y_2\|$	8935.794	0.00082
3.0	$\|y_1\|$	7.882e+05	0.04028
	$\|y_2\|$	7.882e+05	0.00023
4.0	$\|y_1\|$	1.394e+07	0.02554
	$\|y_2\|$	1.394e+07	0.00013
5.0	$\|y_1\|$	1.163e+08	0.01308
	$\|y_2\|$	1.163e+08	0.00008

$$y_1(x) = C_1^T \phi(x) \tag{32}$$
$$y_2(x) = C_2^T \phi(x). \tag{33}$$

To carry out the iterations on every subinterval of equal length Δx, we need to know the values of the following initial conditions:

$$y_1(x^*) = \beta_1^*, \quad y_2(x^*) = \beta_2^*, \tag{34}$$

where x^* is the left end point of each interval, we can obtain these values by following the MB-polynomials method as given in Section 4.1. The collocation points which will be used, as we present it in Section 4.1, can be arranged in stages as follows

- Stage 1. On the intervals $[0, x_1),\ [x_1, x_2),\ [x_2, x_3) \ldots [x_{k-1}, 1)$, the collocation points $(x_0 = 1, \quad x_1 = 0.1 \ldots x_8 = 0.8)$ which were obtained from (18).
- Stage 2. On the intervals $[1, x_1),\ [x_1, x_2),\ [x_2, x_3) \ldots [x_{k-1}, 2)$, but the collocation points which will be used are $(x_0 = 2,\ x_1 = 1.1 \ldots x_{13} = 1.8)$.⋮
- Stage 5. On the intervals $[5, x_1),\ [x_1, x_2),\ [x_2, x_3) \ldots\ [x_{k-1}, 6)$. The collocation points which will be used are $(x_0 = 5, x_1 = 5.1 \ldots x_{13} = 5.8)$.

Table 3 Comparisons between B-polynomials solutions and MB-polynomials solutions for system (35), with $m = 9$ and $\alpha = 0.90$.

x	y_i	B-polynomials	MB-polynomials
1.0	$\|y_1\|$	0.3070	0.3070
	$\|y_2\|$	0.2918	0.2918
	$\|y_3\|$	0.3115	0.3115
2.0	$\|y_1\|$	406.229	0.2024
	$\|y_2\|$	4112.052	0.1950
	$\|y_3\|$	4111.652	0.2047
3.0	$\|y_1\|$	2.13e+4	0.1401
	$\|y_2\|$	3.745e+5	0.1352
	$\|y_3\|$	3.745e+5	0.1416
4.0	$\|y_1\|$	2.57e+5	0.0993
	$\|y_2\|$	6.71e+6	0.0959
	$\|y_3\|$	6.71e+6	0.1004
5.0	$\|y_1\|$	1.55e+6	0.0669
	$\|y_2\|$	5.63e+7	0.0655
	$\|y_3\|$	5.63e+7	0.0673

In Table 1 we present the numerical solutions which are applied by the 9-term B-polynomials and the 9-term MB-polynomials, with $\alpha = 0.90$ and the step-length is $h = 0.2$. Evidently, the classical B-polynomials solutions are not valid for a long time. The MB-polynomials overcomes this lack of the B-polynomials. In Table 2 we listed the error of the computed solution obtained by the MB-polynomials and compared it with that given by HPM, RHPM and standard Bernstein polynomials method B-polynomials. To evaluate the approximation values of the solution at a given x, with $\alpha = 1,\ m = 14$ and step-length is $h = 0.2$.

Example 2. Consider another stiff system of FDE.

$$\begin{cases} D^\alpha y_1 = -20y_1 - 0.25y_2 - 19.75y_3, \\ D^\alpha y_2 = 20y_1 - 20.25y_2 + 0.25y_3, \\ D^\alpha y_3 = 20y_1 - 19.75y_2 - 0.25y_3 \end{cases} \tag{35}$$

with the initial conditions $y_1(0) = 1,\ y_2(0) = 0,\ y_3(0) = -1$.

The theoretical solution when $\alpha = 1$ is

Table 2 Absolute error on $[0, 5]$ for system (23), with $m = 14$ and $\alpha = 1$.

x	y_i	HPM (Biazar et al., 2015)	RHPM (Biazar et al., 2015)	B-polynomials	MB-polynomials
0.0	y_1	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
	y_2	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
1.0	y_1	3.7309e−05	8.7089e−07	1.5045e−02	2.3163e−09
	y_2	1.5358e−03	3.9449e−05	1.5045e−02	2.3331e−09
2.0	y_1	4.5307e−04	8.1898e−07	1.2735e+09	5.4795e−08
	y_2	1.4548e−02	3.1085e−07	1.2735e+09	5.4614e−08
3.0	y_1	1.1801e−03	7.7735e−07	2.2043e+12	1.8224e−07
	y_2	2.7451e−02	1.3643e−05	2.2043e+12	2.0729e−09
4.0	y_1	1.6779e−03	1.3902e−06	2.6717e+14	9.1267e−06
	y_2	2.4065e−02	1.8229e−05	2.6717e+14	6.9711e−07
5.0	y_1	1.7258e−03	1.9343e−06	9.3587e+15	3.4942e−04
	y_2	9.1965e−03	1.0232e−05	9.3587e+15	2.0905e−05

Table 4 Absolute error on $[0, 10]$ for system (35), with $m = 14$ and $\alpha = 1$.

x	y_i	A-EBDF (Hojjati et al., 2004)	MB-polynomials
1.0	y_1	0.38e−07	0.38e−7
	y_2	0.39e−07	0.35e−7
	y_3	0.38e−07	0.35e−7
5.0	y_1	0.14e−08	0.18e−13
	y_2	0.14e−08	0.18e−13
	y_3	0.14e−08	0.18e−13
10.0	y_1	0.22e−09	0.15e−14
	y_2	0.22e−09	0.15e−14
	y_3	0.22e−09	0.15e−14

$$y_1 = \frac{1}{2}\left(e^{-0.5} + e^{-20x}(\cos(20x) + \sin(20x))\right),$$
$$y_2 = \frac{1}{2}\left(e^{-0.5} - e^{-20x}(\cos(20x) - \sin(20x))\right),$$
$$y_3 = -\frac{1}{2}\left(e^{-0.5} + e^{-20x}(\cos(20x) - \sin(20x))\right).$$

In Table 3 we present the numerical solutions which are applied by the 9-term B-polynomials and the 9-term MB-polynomials, with $\alpha = 0.90$ and the step-length is $h = 0.2$. Evidently, the classical B-polynomials solutions are not valid for a long time. MB-polynomials method solves this problem and solutions are valid for large x. In Table 4 we listed the error of the computed solution obtained by the MB-polynomials and compared it with B-polynomials and A-EBDF (Hojjati et al., 2004). To evaluate the approximation values of the solution at a given x, the step-length is $h = 0.2$, with $m = 14,\ j = 1, 2, 3,\ k = 6$ and $r = 10$.

6. Conclusions

In this paper, the MB-polynomials is considered a simple modification of the standard B-polynomials. We applied MB-polynomials to solve Fractional Order Stiff Systems. Comparison between MB-polynomials and other several methods as B-polynomials, A-EBDF, HPM and RHBM indicates that MB-polynomials can solve stiff problems more accurately with less iterations, also MB-polynomials is considered valid in large x than standard B-polynomials. The subjects of our future works can be exemplified by applying MB-polynomials for solving different systems, like Chaotic Fractional Order Systems and Lorenz system.

References

Alshbool, M.H.T., Bataineh, A.S., Hashim, I., Isik, O., 2015. Approximate solutions of singular differential equations with estimation error by using Bernstein polynomials. Int. J. Pure Appl. Math. 100, 109–125.

Bhatti, M.I., Bracken, P., 2007. Solutions of differential equations in a Bernstein polynomial basis. Comput. Appl. Math. 205, 272–280.

Biazar, J., Ali, M., Salehi, F., 2015. Rational Homotopy perturbation method for solving stiff systems of ordinary differential equations. Appl. Math. Model. 39 (3–4), 1291–1299.

Cash, J.R., 1980. On the integration of stiff system of ODEs using extended backward differentiation formula. Numer. Math. 34, 235–246.

Cash, J.R., 1983. The integration of stiff initial value problems in ODEs using modified extended backward differentiation formula. Comput. Math. Appl. 9, 645–657.

Diethelm, K., Ford, N.J., Freed, A.D., 2005. Algorithms for the fractional calculus: a selection of numerical methods. Comput. Methods Appl. Mech. Eng. 194, 743–773.

Hojjati, G., Rahimi, M.Y., Hosseini, S.M., 2004. A-EBDF: an od for numerical solution of stiff system of ODEs. Math. Comput. Simul. 66, 33–41.

Hosseini, S.M., Hojjati, G., 1999. Matrix-free MEBDF method for numerical solution of system of ODEs. Math. Comput. Model. 29, 67–77.

Hsiao, C.H., 2004. Haar Wavelet approach to linear stiff system. Math. Comput. Simul. 64, 561–567.

Hsiao, C.H., Wang, W.J., 2001. Haar Wavelet approach to nonlinear stiff systems. Math. Comput. Simul. 57, 347–353.

Isik, O., Sezer, M., Guney, Z., 2012. Bernstein series solution of linear second-order partial differential equations with mixed conditions. Math. Method Appl. Sci. http://dx.doi.org/10.1002/mma.2817.

Pandey, K., Kumar, N., 2012. Solution of Lane–Emden type equations using Bernstein operational matrix of differentiation. New Astron. 17, 303–308.

Saad Mahmood, A., Casasus, L., AL-Hayani, W., 2005. The decomposition method for stiff systems of ordinary differential equations. Appl. Math. Comput. 167, 964–975.

Yiming, C., Liqing, L., Baofeng, L., Yannan, S., 2014. Numerical solution for the variable order linear cable equation with Bernstein polynomials. Appl. Math. Comput. 238, 329–341.

Yousefi, S.A., Behroozifar, M., 2010. Operational matrices of Bernstein polynomials and their applications. Int. J. Syst. Sci. 41, 709–716.

Yousefi, S.A., Behroozifar, M., 2010. Operational matrices of Bernstein polynomials and their applications. Int. J. Syst. Sci. 41, 709–716.

Yuzbasi, S., 2013. Numerical solution of fractional Riccati type differential equations by means of the Bernstein polynomials. Appl. Math. Comput. 219, 6328–6343.