Journal of King Saud University – Science (2015) **27**, 369–374

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Quasi-periodic non-stationary solutions of 3D Euler equations for incompressible flow

Sergey V. Ershkov

Institute for Time Nature Explorations, M.V. Lomonosov's Moscow State University, Leninskie Gory, 1-12, Moscow 119991, Russia

Received 5 February 2015; accepted 17 May 2015
Available online 27 May 2015

KEYWORDS

Euler equations;
Riccati equations;
Non-stationary solutions

Abstract A novel derivation of non-stationary solutions of 3D Euler equations for incompressible inviscid flow is considered here. Such a solution is the product of 2 separated parts: one consisting of the spatial component and the other being related to the time dependent part.

Spatial part of a solution could be determined if we substitute such a solution to the equations of motion (equation of momentum) with the requirement of *scale-similarity* in regard to the proper component of spatial velocity. So, the time-dependent part of equations of momentum should depend on the time-parameter only.

The main result, which should be outlined, is that the governing (time-dependent) ODE-system consists of 2 *Riccati*-type equations in regard to each other, which has no solution in general case. But we obtain conditions when each component of time-dependent part is proved to be determined by the proper *elliptical* integral in regard to the time-parameter t, which is a generalization of the class of inverse periodic functions.

1. Introduction: the Euler system of equations

In accordance with (Landau and Lifshitz, 1987; Ladyzhenskaya, 1969; Lighthill, 1986), the Euler system of equations for incompressible flow of inviscid fluid should be presented in the Cartesian coordinates as below (under the proper initial conditions):

$$\nabla \cdot \vec{u} = 0, \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \vec{F}, \tag{1.2}$$

where $\boldsymbol{u}$ is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, $\boldsymbol{F}$ represents external force (*per unit of mass in a volume*) acting on the fluid; besides, we assume external force $\boldsymbol{F}$ above to be the force, which has a potential ϕ represented by $\boldsymbol{F} = -\nabla\phi$.

2. The originating system of PDE for Euler equations

Using the identity $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$, we could present the Euler equations in the case of incompressible flow of inviscid fluid $\boldsymbol{u} = \{u_1, u_2, u_3\}$ as below (Saffman, 1995; Milne-Thomson, 1950):

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.05.005
1018-3647

$$\begin{aligned}&\nabla\cdot\vec{u}=0,\\&\frac{\partial\vec{u}}{\partial t}=\vec{u}\times\vec{\omega}-\left(\tfrac{1}{2}\nabla(\vec{u}^2)+\tfrac{\nabla p}{\rho}+\nabla\phi\right)\end{aligned}\tag{2.1}$$

here we denote *the curl field* $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$, a pseudovector field (*time-dependent*) Kamke, 1971:

$$\{\omega_x,\ \omega_y,\ \omega_z\}\equiv\left\{\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right),\quad\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right),\quad\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)\right\}\tag{2.2}$$

also we denote $\nabla\phi=\{f_x,f_y,f_z\}$ in (2.1); besides, let us choose $\rho=1$ for simplicity.

3. Conditions for the space-part of exact solution

Let us search for solutions $\{\boldsymbol{u},p\}$ of the system (2.1) in a form below:

$$\begin{aligned}&u_1=U(t)\cdot u(x,\ y,\ z),\ u_2=V(t)\cdot v(x,\ y,\ z),\\&u_3=W(t)\cdot w(x,\ y,\ z),\ p=P(t)\cdot p(x,\ y,\ z)\end{aligned}\tag{3.1}$$

then we should obtain from (2.1) and expression (2.2) the proper system of PDE:

$$\begin{cases}\frac{\partial u_1}{\partial t}=(u_2\cdot\omega_z-u_3\cdot\omega_y)-\frac{1}{2}\frac{\partial}{\partial x}(u_1^2+u_2^2+u_3^2)-\frac{\partial}{\partial x}p-f_x,\\[4pt]\frac{\partial u_2}{\partial t}=(u_3\cdot\omega_x-u_1\cdot\omega_z)-\frac{1}{2}\frac{\partial}{\partial y}(u_1^2+u_2^2+u_3^2)-\frac{\partial}{\partial y}p-f_y,\\[4pt]\frac{\partial u_3}{\partial t}=(u_1\cdot\omega_y-u_2\cdot\omega_x)-\frac{1}{2}\frac{\partial}{\partial z}(u_1^2+u_2^2+u_3^2)-\frac{\partial}{\partial z}p-f_z,\end{cases}\tag{3.2}$$

Besides, there exists the proper restriction from continuity equation (1.1) as below ($\partial u/\partial x\neq 0$):

$$\begin{aligned}&U(t)\frac{\partial u}{\partial x}+V(t)\frac{\partial v}{\partial y}+W(t)\frac{\partial w}{\partial z}=0,\quad\Rightarrow\\&\frac{\partial v}{\partial y}=\chi\frac{\partial u}{\partial x},\quad\frac{\partial w}{\partial z}=\lambda\frac{\partial u}{\partial x},\quad\{\chi,\lambda\}=const\end{aligned}\tag{3.3}$$

The system of equations (3.2) should be transformed under conditions (3.1) as below:

$$\begin{cases}\frac{dU(t)}{dt}=\frac{V(t)\cdot v(x,y,z)\cdot\left(V(t)\cdot\frac{\partial v}{\partial x}-U(t)\cdot\frac{\partial u}{\partial y}\right)-W(t)\cdot w(x,y,z)\cdot\left(U(t)\cdot\frac{\partial u}{\partial z}-W(t)\cdot\frac{\partial w}{\partial x}\right)}{u(x,y,z)}\\[4pt]\quad-\frac{1}{2}\frac{\frac{\partial}{\partial x}\left(U(t)^2\cdot u^2(x,y,z)+V^2(t)\cdot v^2(x,y,z)+W^2(t)\cdot w^2(x,y,z)\right)}{u(x,y,z)}-\frac{\left(P(t)\cdot\frac{\partial}{\partial x}p(x,y,z)+f_x\right)}{u(x,y,z)},\\[4pt]\frac{dV(t)}{dt}=\frac{W(t)\cdot w(x,y,z)\cdot\left(W(t)\cdot\frac{\partial w}{\partial y}-V(t)\cdot\frac{\partial v}{\partial z}\right)-U(t)\cdot u(x,y,z)\cdot\left(V(t)\cdot\frac{\partial v}{\partial x}-U(t)\cdot\frac{\partial u}{\partial y}\right)}{v(x,y,z)}\\[4pt]\quad-\frac{1}{2}\frac{\frac{\partial}{\partial y}\left(U(t)^2\cdot u^2(x,y,z)+V^2(t)\cdot v^2(x,y,z)+W^2(t)\cdot w^2(x,y,z)\right)}{v(x,y,z)}-\frac{\left(P(t)\cdot\frac{\partial}{\partial y}p(x,y,z)+f_y\right)}{v(x,y,z)},\\[4pt]\frac{dW(t)}{dt}=\frac{U(t)\cdot u(x,y,z)\cdot\left(U(t)\cdot\frac{\partial u}{\partial z}-W(t)\cdot\frac{\partial w}{\partial x}\right)-V(t)\cdot v(x,y,z)\cdot\left(W(t)\cdot\frac{\partial w}{\partial y}-V(t)\cdot\frac{\partial v}{\partial z}\right)}{w(x,y,z)}\\[4pt]\quad-\frac{1}{2}\frac{\frac{\partial}{\partial z}\left(U(t)^2\cdot u^2(x,y,z)+V^2(t)\cdot v^2(x,y,z)+W^2(t)\cdot w^2(x,y,z)\right)}{w(x,y,z)}-\frac{\left(P(t)\cdot\frac{\partial}{\partial z}p(x,y,z)+f_z\right)}{w(x,y,z)},\end{cases}\tag{3.4}$$

thus, from the 1-st of Eq. (3.4) we should assume ($\{a_i\}=const$, $i=1,\ldots,9$):

$$\begin{aligned}&\frac{v(x,y,z)\cdot\frac{\partial v}{\partial x}}{u(x,y,z)}=a_1,\quad-\frac{v(x,y,z)\cdot\frac{\partial u}{\partial y}}{u(x,y,z)}=a_2,\quad-\frac{w(x,y,z)\cdot\frac{\partial u}{\partial z}}{u(x,y,z)}=a_3,\quad\frac{w(x,y,z)\cdot\frac{\partial w}{\partial x}}{u(x,y,z)}=a_4,\\&-\frac{1}{2}\frac{\frac{\partial}{\partial x}(u^2(x,y,z))}{u(x,y,z)}=a_5,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial x}(v^2(x,y,z))}{u(x,y,z)}=a_6,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial x}(w^2(x,y,z))}{u(x,y,z)}=a_7,\\&-\frac{\frac{\partial}{\partial x}p(x,y,z)}{u(x,y,z)}=a_8,\quad-\frac{f_x}{u(x,y,z)}=a_9,\end{aligned}\tag{3.5}$$

but the 2-nd of Eq. (3.4) yields as below ($\{b_i\}=const$, $i=1,\ldots,9$):

$$\begin{aligned}&\frac{w(x,y,z)\cdot\frac{\partial w}{\partial y}}{v(x,y,z)}=b_1,\quad-\frac{w(x,y,z)\cdot\frac{\partial v}{\partial z}}{v(x,y,z)}=b_2,\quad-\frac{u(x,y,z)\cdot\frac{\partial v}{\partial x}}{v(x,y,z)}=b_3,\quad\frac{u(x,y,z)\cdot\frac{\partial u}{\partial y}}{v(x,y,z)}=b_4,\\&-\frac{1}{2}\frac{\frac{\partial}{\partial y}(u^2(x,y,z))}{v(x,y,z)}=b_5,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial y}(v^2(x,y,z))}{v(x,y,z)}=b_6,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial y}(w^2(x,y,z))}{v(x,y,z)}=b_7,\\&-\frac{\frac{\partial}{\partial y}p(x,y,z)}{v(x,y,z)}=b_8,\quad-\frac{f_y}{v(x,y,z)}=b_9,\end{aligned}\tag{3.6}$$

besides, 3-rd of Eq. (3.4) yields ($\{c_i\}=const$, $i=1,\ldots,9$):

$$\begin{aligned}&\frac{u(x,y,z)\cdot\frac{\partial u}{\partial z}}{w(x,y,z)}=c_1,\quad-\frac{u(x,y,z)\cdot\frac{\partial w}{\partial x}}{w(x,y,z)}=c_2,\quad-\frac{v(x,y,z)\cdot\frac{\partial w}{\partial y}}{w(x,y,z)}=c_3,\quad\frac{v(x,y,z)\cdot\frac{\partial v}{\partial z}}{w(x,y,z)}=c_4,\\&-\frac{1}{2}\frac{\frac{\partial}{\partial z}(u^2(x,y,z))}{w(x,y,z)}=c_5,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial z}(v^2(x,y,z))}{w(x,y,z)}=c_6,\quad-\frac{1}{2}\frac{\frac{\partial}{\partial z}(w^2(x,y,z))}{w(x,y,z)}=c_7,\\&-\frac{\frac{\partial}{\partial z}p(x,y,z)}{w(x,y,z)}=c_8,\quad-\frac{f_z}{w(x,y,z)}=c_9.\end{aligned}\tag{3.7}$$

4. The space-part of exact solution

As for the structure of space part of exact solution (3.1), the system of equations (3.5)–(3.7) could be solved by the proper analytical way as below:

Eq. (3.5) yields:

$$\begin{aligned}&\frac{v(x,y,z)\cdot\frac{\partial v}{\partial x}}{u(x,y,z)}=a_1,\quad\frac{\partial u}{\partial y}=-\left(\frac{a_2}{a_1}\right)\cdot\frac{\partial v}{\partial x},\\&\frac{w(x,y,z)\cdot\frac{\partial u}{\partial z}}{u(x,y,z)}=-a_3,\quad\frac{\partial w}{\partial x}=-\left(\frac{a_4}{a_3}\right)\cdot\frac{\partial u}{\partial z},\\&\frac{\partial}{\partial x}(u(x,\ y,\ z))=-a_5\quad a_6=-a_1,\\&a_7=-a_4,\quad\frac{\partial}{\partial x}p(x,\ y,\ z)=a_8\cdot u(x,\ y,\ z),\\&f_x=-a_9\cdot u(x,\ y,\ z),\end{aligned}\tag{4.1}$$

Eq. (3.6) yields:

$$\begin{aligned}&\frac{w(x,y,z)\cdot\frac{\partial w}{\partial y}}{v(x,y,z)}=b_1,\quad\frac{\partial w}{\partial y}=-\left(\frac{b_1}{b_2}\right)\cdot\frac{\partial v}{\partial z},\\&\left(\frac{u(x,y,z)\cdot\frac{\partial v}{\partial x}}{v(x,y,z)}\right)\cdot\left(\frac{v(x,y,z)\cdot\frac{\partial v}{\partial x}}{u(x,y,z)}\right)=-b_3\cdot a_1,\Rightarrow\frac{\partial v}{\partial x}=\sqrt{(-b_3\cdot a_1)},\\&\Rightarrow u(x,y,z)=-\left(\frac{b_3}{\sqrt{(-b_3\cdot a_1)}}\right)\cdot v(x,y,z),\\&\Rightarrow b_3=a_5,\frac{\partial u}{\partial y}=-\left(\frac{b_4}{b_3}\right)\cdot\frac{\partial v}{\partial x},\Rightarrow\left(\frac{a_2}{a_1}\right)=\left(\frac{b_4}{b_3}\right),\quad b_5\\&=-b_4,\quad\frac{\partial}{\partial y}(v(x,y,z))=-b_6,\quad b_7=-b_1,\frac{\partial}{\partial y}p(x,y,z)\\&=-b_8\cdot v(x,y,z),\quad\frac{\partial}{\partial x}p(x,y,z)\\&=-a_8\cdot\left(\frac{b_3}{\sqrt{(-b_3\cdot a_1)}}\right)\cdot v(x,y,z),\quad f_y\\&=-b_9\cdot v(x,y,z),\end{aligned}\tag{4.2}$$

and Eq. (3.7) yields:

$$\left(\frac{u(x,\ y,\ z)\cdot\frac{\partial u}{\partial z}}{w(x,\ y,\ z)}\right)\cdot\left(\frac{w(x,\ y,\ z)\cdot\frac{\partial u}{\partial z}}{u(x,\ y,\ z)}\right) = -a_3\cdot c_1,$$

$$\Rightarrow \frac{\partial u}{\partial z} = \sqrt{(-a_3\cdot c_1)}, \quad \Rightarrow \frac{\partial v}{\partial z} = -\frac{\sqrt{(a_3\cdot c_1\cdot b_3\cdot a_1)}}{b_3}, w(x,\ y,\ z)$$

$$= -\left(\frac{a_3}{\sqrt{(-a_3\cdot c_1)}}\right)\cdot u(x,\ y,\ z)$$

$$= \left(\frac{a_3}{\sqrt{(-a_3\cdot c_1)}}\right)\cdot\left(\frac{b_3}{\sqrt{(-b_3\cdot a_1)}}\right)\cdot v(x,\ y,\ z), \quad \frac{\partial w}{\partial x} = -\left(\frac{c_2}{c_1}\right)\cdot\frac{\partial u}{\partial z},$$

$$\Rightarrow \left(\frac{a_4}{a_3}\right) = \left(\frac{c_2}{c_1}\right), -\frac{v(x,\ y,\ z)\cdot\frac{\partial w}{\partial y}}{w(x,\ y,\ z)} = c_3,$$

$$\Rightarrow \frac{\partial w}{\partial y} = \sqrt{(-b_1\cdot c_3)}, \frac{\partial w}{\partial y} = -\left(\frac{c_3}{c_4}\right)\cdot\frac{\partial v}{\partial z},$$

$$\Rightarrow \left(\frac{c_3}{c_4}\right) = \left(\frac{b_1}{b_2}\right) c_5 = -c_1, c_6 = -c_4, \frac{\partial}{\partial z}(w(x,\ y,\ z))$$

$$= -c_7, \frac{\partial}{\partial z}p(x,\ y,\ z)$$

$$= -c_8\cdot w(x,\ y,\ z) = -c_8\cdot\left(\sqrt{\frac{a_3\cdot b_3}{a_1\cdot c_1}}\right)\cdot v(x,\ y,\ z), \quad f_z$$

$$= -c_9\cdot w(x,\ y,\ z). \tag{4.3}$$

So, the space part of the solution should be presented as below:

$$u = -\left(\frac{b_3}{\sqrt{(-b_3\cdot a_1)}}\right)\cdot v(x,\ y,\ z) = -b_3\cdot x + b_3\cdot b_6\cdot y$$

$$+\sqrt{(-a_3\cdot c_1)}\cdot \mathrm{z},\ \mathrm{v} = x - b_6\cdot y - \frac{\sqrt{(-a_3\cdot c_1)}}{b_3}\cdot z,$$

$$w = \left(\sqrt{\frac{a_3\cdot b_3}{a_1\cdot c_1}}\right)\cdot v(x,\ y,\ z) = \sqrt{\left(-\frac{a_3}{c_1}\right)}\cdot|b_3|\cdot x$$

$$-b_6\cdot\left(\sqrt{\frac{a_3\cdot b_3}{a_1\cdot c_1}}\right)\cdot y - a_3\cdot z,$$

$$(a_1\cdot b_3) = -1, \quad a_2 = b_6, \quad |a_3|\cdot|b_3| = a_3\cdot b_3,$$

$$a_3 = c_7, \quad a_5 = b_3, \quad a_8\cdot b_3\cdot b_6 = b_8,$$

$$a_8 = \frac{c_8}{|a_1|\cdot|c_1|}, p(x,\ y,\ z) = a_8\cdot(-b_3\cdot b_6\cdot x\cdot y$$

$$-\sqrt{(-a_3\cdot c_1)}\cdot x\cdot z + b_6\sqrt{(-a_3\cdot c_1)}\cdot y\cdot z + \frac{b_3}{2}x^2$$

$$+\left(\frac{b_3\cdot(b_6)^2}{2}\right)\cdot y^2 + \frac{|c_1|\cdot|a_3|}{2b_3}\cdot z^2) \tag{4.4}$$

Thus, if we choose for simplicity the proper constants as:

$$c_1 = -1, \quad a_8 = 1 \quad (\Rightarrow b_8 = b_3\cdot b_6, c_8 = 1/|b_3|) \tag{4.5}$$

the space part of the solution should be presented as below ($|a_3|\cdot|b_3| = a_3\cdot b_3$):

$$u = -b_3\cdot x + b_3\cdot b_6\cdot y + \sqrt{a_3}\cdot z$$

$$v = x - b_6\cdot y - \frac{\sqrt{a_3}}{b_3}\cdot z,$$

$$w = \sqrt{a_3}\cdot|b_3|\cdot x - b_6\cdot|b_3|\cdot\sqrt{a_3}\cdot y - a_3\cdot z, \tag{4.6}$$

$$p(x,\ y,\ z) = -b_3\cdot b_6\cdot x\cdot y - \sqrt{a_3}\cdot x\cdot z$$

$$+b_6\sqrt{a_3}\cdot y\cdot z + \frac{b_3}{2}x^2 + \left(\frac{b_3\cdot(b_6)^2}{2}\right)\cdot y^2 + \frac{|a_3|}{2b_3}\cdot z^2$$

5. Time-dependent part of exact solution

As for the structure of time-dependent part of exact solution (3.1) with space part (4.6), it could be obtained from system of Eq. (3.4) which should be transformed as below:

$$\begin{cases} \dfrac{dU(t)}{dt} = V(t)\cdot U(t)\cdot a_2 + W(t)\cdot U(t)\cdot a_3 + U(t)^2\cdot a_5 + P(t)\cdot a_8 + a_9, \\ \dfrac{dV(t)}{dt} = W(t)\cdot V(t)\cdot b_2 + U(t)\cdot V(t)\cdot b_3 + V^2(t)\cdot b_6 + P(t)\cdot b_8 + b_9, \\ \dfrac{dW(t)}{dt} = U(t)\cdot W(t)\cdot c_2 + V(t)\cdot W(t)\cdot c_3 + W^2(t)\cdot c_7 + P(t)\cdot c_8 + c_9, \end{cases} \tag{5.1}$$

where $a_2 = b_6$, $|a_3|\cdot|b_3| = a_3\cdot b_3$, $a_5 = b_3$, $a_8 = 1$, $b_8 = b_3\cdot b_6$, $c_7 = a_3$, $c_8 = 1/|b_3|$; besides, it should be accomplished along with the continuity equation (3.3):

$$U(t)\frac{\partial u}{\partial x} + V(t)\frac{\partial v}{\partial y} + W(t)\frac{\partial w}{\partial z} = 0,$$

$$\Rightarrow U(t)\cdot b_3 + V(t)\cdot b_6 + W(t)\cdot a_3 = 0,$$

so, we have 4 equations for the obtaining of 4 functions $U(t)$, $V(t)$, $W(t)$, $P(t)$.

Along with the invariant from the continuity equation (besides, $a_9\cdot b_3 + b_9\cdot b_6 + c_9\cdot a_3 = 0$):

$$\frac{d(U(t)\cdot b_3 + V(t)\cdot b_6 + W(t)\cdot a_3)}{dt} = 0,$$

a system of Eq. (5.1) immediately yields the invariant for function $P(t)$ as below:

$$P(t)\cdot(a_8\cdot b_3 + b_8\cdot b_6 + c_8\cdot a_3)$$

$$= -[V(t)\cdot U(t)\cdot a_2 + W(t)\cdot U(t)\cdot a_3 + U(t)^2\cdot a_5]\cdot b_3$$

$$-[W(t)\cdot V(t)\cdot b_2 + U(t)\cdot V(t)\cdot b_3 + V^2(t)\cdot b_6]\cdot b_6$$

$$-[U(t)\cdot W(t)\cdot c_2 + V(t)\cdot W(t)\cdot c_3 + W^2(t)\cdot c_7]\cdot a_3, \tag{5.2}$$

thus, we should exclude expression (5.2) for function $P(t)$ from the analysis of equations of system (5.1) and also we should exclude the continuity equation, which means that one of 3 functions $U(t)$, $V(t)$, $W(t)$ is the linear combination of two others:

$$W(t) = -U(t)\cdot\frac{b_3}{a_3} - V(t)\cdot\frac{b_6}{a_3} \tag{5.3}$$

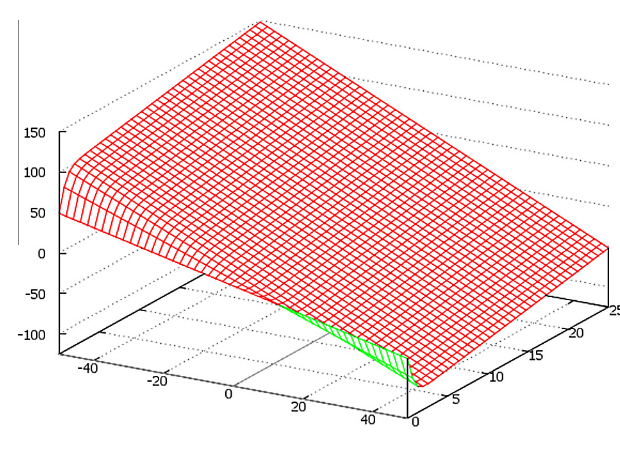

Figure 1 A *schematic* plot of the function $\sim (x - y + 1) * \{\tanh(-t) - 1\}$, here we designate: $x \in (-50,\ 50)$, $t = y \in (0,\ 25)$.

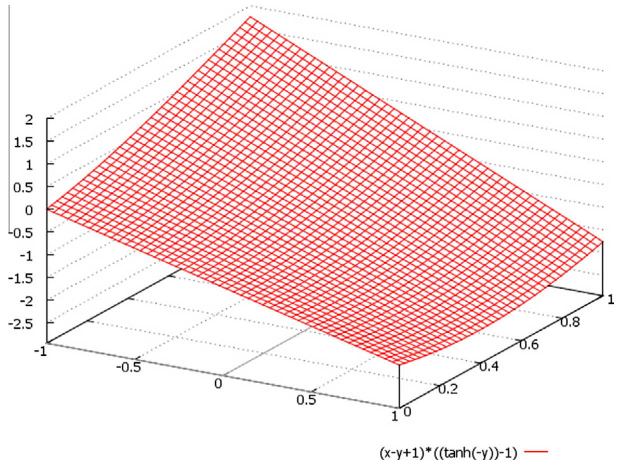

Figure 2 A *schematic* plot of the function $\sim (x - y + 1) * \{\tanh(-t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

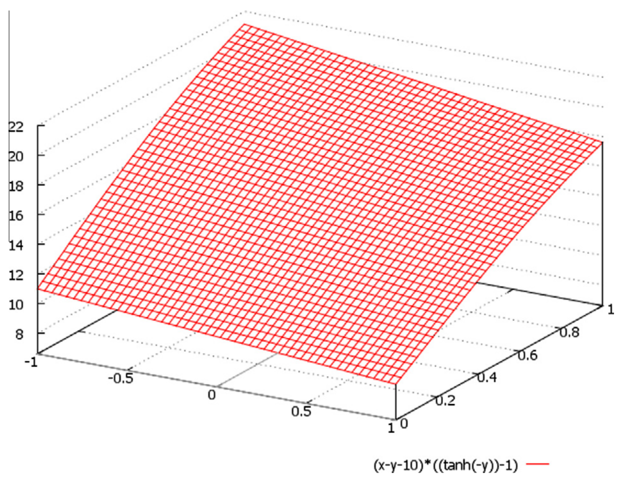

Figure 3 A *schematic* plot of the function $\sim (x - y - 10) * \{\tanh(-t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

So, analyzing the system (5.1), we finally should obtain the system of 2 ordinary differential equations of the 1-st order for any 2 of 3 functions $U(t)$, $V(t)$, $W(t)$ (*the last 3-rd function could be obtained by expressing it from the continuity equation above*).

These governing ODE-equations form together a system of 2 *Riccati*-type equations in regard to each other, which is the system of 2 ordinary differential equations of the 1-st kind with the right parts, consisting of polynomials of the 2-nd extent in regard to the functions $U(t)$, $V(t)$, $W(t)$.

Riccati type of equations has no analytical solution in general case (Kamke, 1971). We should note also that modern methods exist for obtaining the solution of *Riccati* equations with a good approximation (Bender and Orszag, 1999; Rosu et al., 2012; Christianto and Smarandache, 2008). But if we choose proper constants for the system (5.1):

$$\begin{aligned} &a_9 = b_9 = c_9 = 0, \quad (a_8 \cdot b_3 + b_8 \cdot b_6 + c_8 \cdot a_3) = 1, \Rightarrow \\ &(b_3 + b_3 \cdot (b_6)^2 + a_3/|b_3|) = 1, \Rightarrow \quad a_3 = |b_3| - b_3 \cdot |b_3| - b_3 \cdot |b_3| \cdot (b_6)^2, \end{aligned} \tag{5.4}$$

the 1-st and 2-nd equations of system (5.1) could be transformed as presented below:

$$\begin{cases} \frac{dU(t)}{dt} = V(t) \cdot U(t) \cdot \left(b_3 \cdot \left(\frac{b_2 \cdot b_6}{a_3} + c_3 - 2b_6\right) + b_6 \cdot (c_2 - b_3)\right) \\ \qquad + U(t)^2 \cdot b_3 \cdot (c_2 - b_3) + V^2(t) \cdot b_6 \cdot \left(\frac{b_2 \cdot b_6}{a_3} + c_3 - 2b_6\right), \\ \frac{dV(t)}{dt} = U(t) \cdot V(t) \cdot \left(b_3 \cdot \left(1 - \frac{b_2}{a_3}\right) + b_6 \cdot b_8 \cdot (c_2 - b_3) + b_3 \cdot b_8 \cdot \left(\frac{b_2 \cdot b_6}{a_3} + c_3 - 2b_6\right)\right) \\ \qquad + V^2(t) \cdot \left(b_6 \cdot \left(1 - \frac{b_2}{a_3}\right) + b_6 \cdot b_8 \cdot \left(\frac{b_2 \cdot b_6}{a_3} + c_3 - 2b_6\right)\right) + U^2(t) \cdot b_3 \cdot b_8 \cdot (c_2 - b_3), \end{cases} \tag{5.5}$$

where $a_2 = b_6$, see (4.4). Besides, if we additionally choose $b_2 = a_3$, system (5.5) above could be reduced to the simplified regular form below:

$$\begin{cases} \frac{dU(t)}{dt} = (b_8 \cdot U(t) + C) \cdot U(t) \cdot (b_3 \cdot (c_3 - b_6) + b_6 \cdot (c_2 - b_3)) \\ \qquad + U(t)^2 \cdot b_3 \cdot (c_2 - b_3) + (b_8 \cdot U(t) + C)^2 \cdot b_6 \cdot (c_3 - b_6), \\ \frac{dV(t)}{dt} = b_8 \cdot \frac{dU(t)}{dt} \quad \Rightarrow \quad V(t) = b_8 \cdot U(t) + C, \quad C = const \end{cases} \tag{5.6}$$

where the 1-st equation of system (5.6) has a proper solution below ($b_8 = b_3 \cdot b_6$):

$$\frac{dU(t)}{(A \cdot U^2(t) + B \cdot U(t) + D)} = dt, \tag{5.7}$$

$$\begin{aligned} &A = (b_8 \cdot b_3 \cdot (c_3 - b_6) + b_8 \cdot b_6 \cdot (c_2 - b_3) + b_3 \cdot (c_2 - b_3) + (b_8)^2 \cdot b_6 \cdot (c_3 - b_6)) \\ &B = C \cdot (b_3 \cdot (c_3 - b_6) + b_6 \cdot (c_2 - b_3) + 2b_8 \cdot b_6 \cdot (c_3 - b_6)), \quad D = C^2 \cdot b_6 \cdot (c_3 - b_6) \end{aligned}$$

The left side of expression (5.7) could be transformed to the proper *elliptical* integral (Lawden, 1989) in regard to function $U(t)$:

$$\int \frac{dU(t)}{(A \cdot U^2(t) + B \cdot U(t) + D)} = \begin{cases} \frac{2}{\sqrt{\Delta}} \arctan\left(\frac{2A \cdot U(t) + B}{\sqrt{\Delta}}\right), & \Delta > 0 \\ -\frac{2}{\sqrt{-\Delta}} Arth\left(\frac{2A \cdot U(t) + B}{\sqrt{-\Delta}}\right), & \Delta < 0 \end{cases} \quad \Delta = (4A \cdot D - B^2) \tag{5.8}$$

6. Discussion

In fluid mechanics, a lot of authors have been executing their researches to obtain the analytical solutions of Euler and Navier–Stokes equations (Drazin and Riley, 2006), even for 3D case of *compressible* gas flow (Ershkov and Schennikov, 2001). But there is an essential deficiency of non-stationary solutions indeed.

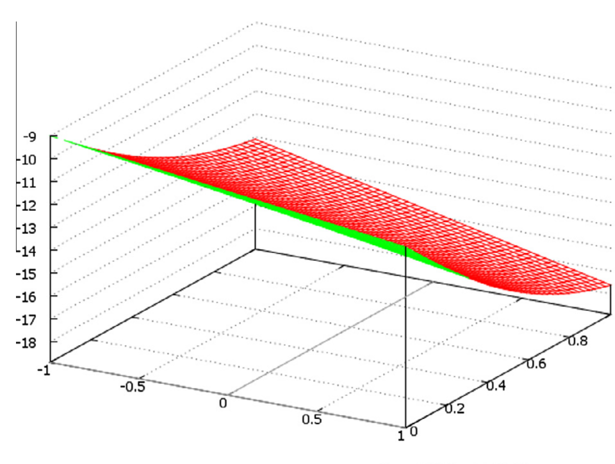

Figure 4 A *schematic* plot of the function $\sim (x - y + 10) * \{\tanh(-t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

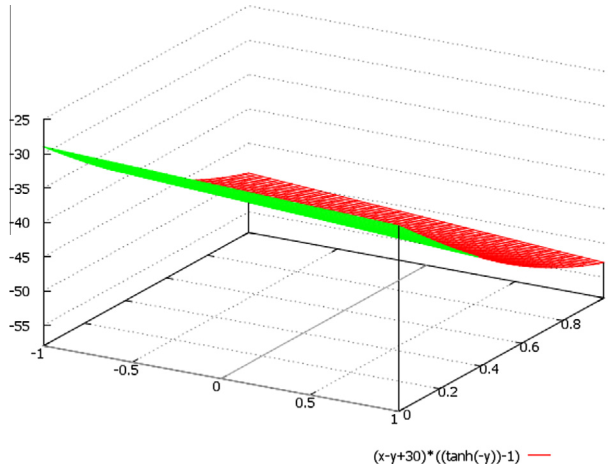

Figure 5 A *schematic* plot of the function $\sim (x - y + 30) * \{\tanh(-t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

Our presentation (3.1) of the non-stationary solutions of 3D Euler equations (1.1) and (1.2) for incompressible flow is considered here. The spatial part of such a solution is determined by equalities (4.4), under the given initial conditions; but the time-dependent part is determined by Eqs. (5.1)–(5.3).

Besides, the real example of exact solution is obtained. The spatial part of such a solution is presented by the equalities (4.5) and (4.6), but the time-dependent part is presented by equalities (5.2)–(5.4) and (5.8).

Also, we should especially note that the components of flow velocity (4.6) of the solution (3.1) will be uniformly increasing when $(x, y, z) \to \infty$. So, such a solution should be defined within the limited domain of the meanings of variables (x, y, z), it should be given by the initial conditions.

The explicit solutions for $U(t)$ can easily be obtained from Eqs. (5.7) and (5.6), therefore, the explicit form of the special solutions (3.1) should be provided:

$$u_1 = U(t) \cdot u(x, y, z),\; u_2 = V(t) \cdot v(x, y, z),\; u_3 = W(t) \cdot w(x, y, z),\; p = P(t) \cdot p(x, y, z),$$

where:

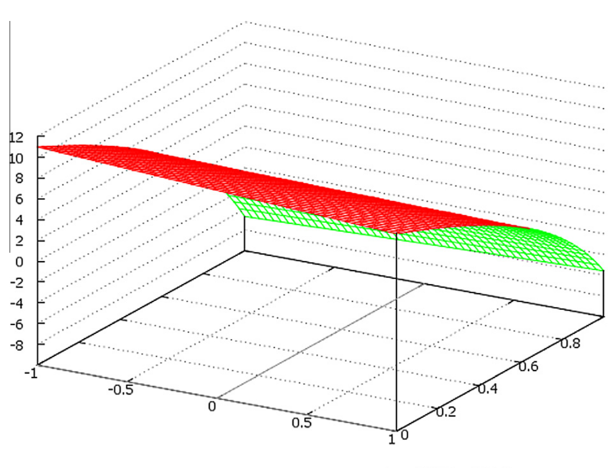

Figure 6 A *schematic* plot of the function $\sim (x - y - 10) * \{\tan(t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

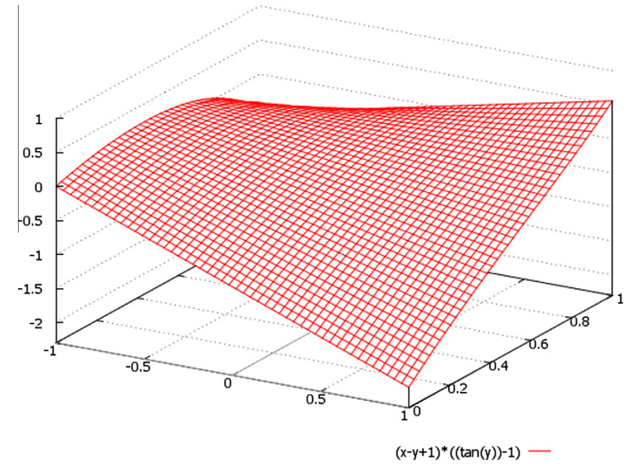

Figure 7 A *schematic* plot of the function $\sim (x - y + 1) * \{\tan(t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

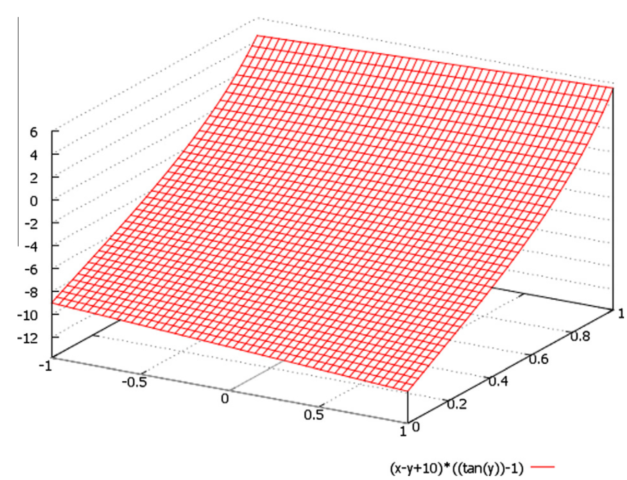

Figure 8 A *schematic* plot of the function $\sim (x - y + 10) * \{\tan(t) - 1\}$, here we designate: $x \in (-1, 1)$, $t = y \in (0, 1)$.

$$u = -b_3 \cdot x + b_3 \cdot b_6 \cdot y + \sqrt{a_3} \cdot z$$
$$v = x - b_6 \cdot y - \tfrac{\sqrt{a_3}}{b_3} \cdot z,$$
$$w = \sqrt{a_3} \cdot |b_3| \cdot x - b_6 \cdot |b_3| \cdot \sqrt{a_3} \cdot y - a_3 \cdot z,$$
$$p(x, y, z) = -b_3 \cdot b_6 \cdot x \cdot y - \sqrt{a_3} \cdot x \cdot z + b_6 \sqrt{a_3} \cdot y \cdot z + \tfrac{b_3}{2} x^2 + \left(\tfrac{b_3 \cdot (b_6)^2}{2}\right) \cdot y^2 + \tfrac{|a_3|}{2 b_3} \cdot z^2,$$

$$P(t) = -[V(t) \cdot U(t) \cdot b_6 + W(t) \cdot U(t) \cdot a_3 + U(t)^2 \cdot b_3] \cdot b_3 - [W(t) \cdot V(t) \cdot a_3 + U(t) \cdot V(t) \cdot b_3 + V^2(t) \cdot b_6] \cdot b_6 - [U(t) \cdot W(t) \cdot c_2 + V(t) \cdot W(t) \cdot c_3 + W^2(t) \cdot a_3] \cdot a_3,$$

$$W(t) = -U(t) \cdot \frac{b_3}{a_3} - V(t) \cdot \frac{b_6}{a_3}, \qquad V(t) = (b_3 \cdot b_6) \cdot U(t) + C, \quad C = const,$$

here we should choose $a_3 = \{|b_3| - b_3 \cdot |b_3| - b_3 \cdot |b_3| \cdot (b_6)^2\} \neq 0$, $\to b_3 \cdot (1 + (b_6)^2) \neq 1$, but the key function $U(t)$ should be given as below ($b_8 = b_3 \cdot b_6$):

$$\begin{cases} U(t) = \frac{\sqrt{\Delta}\cdot\tan\left(\left(\frac{\sqrt{\Delta}}{2}\right)\cdot t\right) - B}{2A}, & \Delta > 0 \\ U(t) = \frac{\sqrt{-\Delta}\cdot\tanh\left(-\left(\frac{\sqrt{-\Delta}}{2}\right)\cdot t\right) - B}{2A}, & \Delta < 0 \end{cases} \qquad \Delta = (4A \cdot D - B^2)$$

$$A = (b_8 \cdot b_3 \cdot (c_3 - b_6) + b_8 \cdot b_6 \cdot (c_2 - b_3) + b_3 \cdot (c_2 - b_3) + (b_8)^2 \cdot b_6 \cdot (c_3 - b_6))$$

$$B = C \cdot (b_3 \cdot (c_3 - b_6) + b_6 \cdot (c_2 - b_3) + 2b_8 \cdot b_6 \cdot (c_3 - b_6)), \quad D = C^2 \cdot b_6 \cdot (c_3 - b_6)$$

For example, if we choose $c_3 = b_6$ $(C \neq 0)$ it should simplify the expression for $U(t)$:

$$U(t) = \frac{\sqrt{-\Delta}\cdot\tanh\left(-\left(\frac{\sqrt{-\Delta}}{2}\right)\cdot t\right) - B}{2A}, \quad \Delta = -B^2$$

$$A = b_3 \cdot (c_2 - b_3) \cdot (1 + (b_6)^2), \quad B = C \cdot b_6 \cdot (c_2 - b_3), \quad D = 0$$

and, if we additionally choose $c_2 = 2b_3$, $b_3 = 2/C$ $(C \neq 0)$, $b_6 = 1$, we should obtain (see Figs. 1–5):

$$U(t) = \frac{C^2}{8} \cdot (\tanh(-t) - 1), \quad \Delta = -4, \quad A = (b_3)^2 \cdot 2,$$

$$B = 2, \quad D = 0$$

We assume at Figs. 1–5 that in the expressions $(x - y - 10)$, $(x - y + 1)$, $(x - y + 10)$, $(x - y + 30)$ set of meanings $\{-10, 1, 10, 30\}$ is varying according to the varying of the range of variable z; besides, the factor $\{\tanh(-t) - 1\}$ could be schematically presented (for imagination of the plots of solutions) by the changing of parameter t to variable y, for example.

At Figs. 6–8 we schematically imagined solutions for the case $\Delta > 0$.

Also, we should note that since some solutions are unbounded (see for instance Eq. (5.8) for $\Delta > 0$), such a solution should be defined within the limited range of the meanings of time-parameter t (it should be given by the initial conditions).

Besides, we should additionally note that the only periodic (and unbounded) solutions are the ones given by $U(t)$ for $\Delta > 0$ (Figs. 6–8), since the hyperbolic tangent is a non periodic but bounded function in this case (Figs. 1–5).

7. Conclusion

A new presentation of non-stationary solutions of 3D Euler equations for incompressible inviscid flow is considered here. Such a solution is the product of 2 separated parts: - spatial and the time-dependent parts.

Spatial part of a solution could be determined if we substitute such a solution to the equations of motion (equation of momentum) under the demand of *scale-similarity* in regard to the proper component of spatial velocity. So, the time-dependent part of equations of momentum should depend on the time-parameter only.

The main result, which should be outlined, is that the governing (time-dependent) ODE-system consists of 2 *Riccati*-type equations in regard to each other, which has no solution in general case. But we obtain conditions when each component of time-dependent part is proved to be determined by the proper *elliptical* integral in regard to the time-parameter t, which is a generalization of the class of inverse periodic functions. Thus, by the proper obtaining of re-inverse dependence of a solution from time-parameter we could present the expression for each component of motion as a set of periodic cycles.

Acknowledgements

I am thankful to unknown esteemed Reviewers for valuable advices in preparing of this manuscript. Especially I appreciate Dr. Rizwan Irshad for his kind help and assistance during navigating in submission process of my articles to JKSUS.

References

Christianto, V., Smarandache, F., 2008. An exact mapping from Navier–Stokes equation to Schrodinger equation. Prog. Phys. 1, 38–39.

Lawden D., 1989. Elliptic Functions and Applications. Springer-Verlag. See also: <http://mathworld.wolfram.com/EllipticIntegral.html>.

Drazin, P.G., Riley, N., 2006. The Navier–Stokes Equations: A Classification of Flows and Exact Solutions. Cambridge University Press, Cambridge.

Ershkov, S.V., Schennikov, V.V., 2001. Self-similar solutions to the complete system of Navier–Stokes equations for axially symmetric swirling viscous compressible gas flow. Comput. Math. Math. Phys. J. 41 (7), 1117–1124.

Kamke, E., 1971. Hand-book for Ordinary Differential Eq. Science, Moscow.

Ladyzhenskaya, O.A., 1969. The Mathematical Theory of viscous Incompressible Flow, 2nd ed. Gordon and Breach, New York.

Landau, L.D., Lifshitz, E.M., 1987. Fluid mechanics, Course of Theoretical Physics 6 (2nd revised ed.), Pergamon Press, ISBN 0-08-033932-8.

Lighthill, M.J., 1986. An Informal Introduction to Theoretical Fluid Mechanics. Oxford University Press, ISBN 0-19-853630-5.

Bender, Carl M., Orszag, Steven A., 1999. Advanced Mathematical Methods for Scientists and Engineers. Origin published by McGraw Hill, 1978, XIV. pp. 20–22.

Milne-Thomson, L.M., 1950. Theoretical Hydrodynamics. Macmillan.

Rosu, H.C., Cornejo-Pérez, O., Ojeda-May, P., 2012. Traveling kinks in cubic nonlinear Ginzburg–Landau equations. Phys. Rev. E 85, 037102.

Saffman, P.G., 1995. Vortex Dynamics. Cambridge University Press.