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Original article

The variational Adomian decomposition method for solving nonlinear two- dimensional Volterra-Fredholm integro-differential equation

F.A. Hendi [a], M.M. Al-Qarni [b,*]

[a] Department of Mathematics Faculty of Science, King Abdul Aziz University, Jeddah, Saudi Arabia
[b] Department of Mathematics Faculty of Science, King Khalid University, Abha, Saudi Arabia

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ABSTRACT

This paper outlines the coupling of variational iteration method (VIM) with Adomian decomposition method (ADM) for solving nonlinear mixed Volterra-Fredholm integro-diffierential equation (V-FIDE), this method is called variational Adomian decomposition method (VADM). Some numerical examples are introduced to verify that the method handles the difficulty of nonlinear term, red reduces the computational size and accelerates the convergence.

1. Introduction

In recent years, there has been a clear interest in the integro-differential equations which are a combination of differential and Volterra-Fredholm integral equations. Integro-differential equations play an important role in many branches of linear and nonlinear functional analysis and their applications. The mentioned integro- differential equations are usually difficult to solve analytically, so approximation strategies are required to obtain the solution of the linear and nonlinear integro-differential equations (Huesin et al., 2008).

Several strategies are proposed to achieve this goal. The ADM is introduced by G. Adomian. It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using the Adomian polynomials (Adomian, 1994). Many researchers studied and discussed the linear V-FIDE, Babolian et al. (2008) and Al-Jubory (2010) introduced some approximation method for solving Volterra-Fredholm integral and integro-differential equations. Dadkhah et al. (2010) utilized numerical solution of nonlinear V-FIDEs utilizing Legendre wavelets. Rabbani and Kiasodltani (2011) studied finding solution of nonlinear system of V-FIDE by utilizing discrete collocation method. Gherjalar and Mohammadikia, 2012 solved integral and integro-differential equations by utilizing B-splines function. The VIM (He, 2007, 1999) is a powerful device for solving various kinds of equations, linear and nonlinear. The technique has successfully been applied to many situations. For instance, He (2007) utilized the strategy to solve some integro-differential equations where he chose initial approximate solution in the form of exact solution with unknown constants.

In this work, the two-dimensional nonlinear V-FIDE is solved by the VADM. The two-dimensional nonlinear V-FIDE is given by

$$\sum_{j=0}^{k} P_j(x_1,\jmath_1)u^j(x_1,\jmath_1) = \dot{f}(x_1,\jmath_1) + \int_a^{x_1}\int_\Omega F(x_1,\jmath_1,y,\tau)\gamma(u^l(y,\tau))dyd\tau,$$
$$(x_1,\jmath_1)\in \acute{J} = [a,x_1]\times\Omega \tag{1}$$

with initial conditions

$$u^r(a,\jmath_1) = g_r, r = 0,1,\ldots,k-1, \quad \Omega = [a,b] \tag{2}$$

where $u^j(x_1,\jmath_1) = \frac{d^j u}{dx_1^j}$. The functions $\dot{f}(x_1,\jmath_1)$, $F(x_1,\jmath_1,y,\tau)$ and $\gamma(u^l(y,\tau)), l \geqslant 0$ are analytic functions on $\acute{J}$, and functions $P_j(x_1,\jmath_1), j = 0,1,\ldots,k, P_k(x_1,\jmath_1) \neq 0$ are given.

* Corresponding author.
E-mail addresses: falhendi@kau.edu.sa (F.A. Hendi), malqrni@kku.edu.sa (M.M. Al-Qarni).

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2. Adomian decomposition method

In what follows we display an outline for utilizing the ADM for solving the nonlinear V-FIDE. The Eq. (1) can be written as follows:

$$u(x_1,\jmath_1)=L^{-1}\left(\frac{\dot f(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r + L^{-1}\left(\int_a^{x_1}\int_\Omega \frac{F(x_1,\jmath_1,y,\tau)\gamma(u^l(y,\tau))}{P_k(x_1,\jmath_1)}dyd\tau\right) - L^{-1}\left(\sum_{j=0}^{k-1}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}u^j(x_1,\jmath_1)\right) \quad (3)$$

where $L^{-1}$is the multiple integration operator, and Eq. (3) takes the form

$$u(x_1,\jmath_1)=L^{-1}\left(\frac{\dot f(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r + \int_a^{x_1}\int_\Omega \frac{(x_1-\jmath_1)^k}{(k)!}\frac{F(x_1,\jmath_1,y,\tau)\gamma(u^l(y,\tau))}{P_k(x_1,\jmath_1)}dyd\tau - \sum_{j=0}^{k-1}\int_a^{x_1}\frac{(x_1-\jmath_1)^{k-1}}{(k-1)!}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}u^j(x_1,\jmath_1)d\jmath_1 \quad (4)$$

since

$$\sum_{j=0}^{k-1}L^{-1}\left(\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right)u^j(x_1,\jmath_1)=\sum_{j=0}^{k-1}\int_a^{x_1}\frac{(x_1-\jmath_1)^{k-1}}{(k-1)!}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}u^j(x_1,\jmath_1)d\jmath_1$$

The ADM introduces the following expression

$$u(x_1,\jmath_1)=\sum_{i=0}^{\infty}u_i(x_1,\jmath_1) \quad (5)$$

for the solution $u(x_1,\jmath_1)$of Eq. (1) with initial conditions (2), where the components $u_i(x_1,\jmath_1)$ will be determined recurrently. In addition, the technique defines the nonlinear function $\gamma(u^l(y,\tau))$ by an infinite series of polynomials

$$\gamma(u^l(y,\tau))=\sum_{n=0}^{\infty}\Lambda_n \quad (6)$$

where $\Lambda_n, n\geqslant 0$ are defined by

$$\Lambda_n=\frac{1}{n!}\frac{d^n}{d\alpha^n}\left[\gamma\left(\sum_{i=0}^{\infty}\alpha^i u_i\right)\right]_{\alpha=0},\quad n=0,1,2,\ldots \quad (7)$$

Substituting Eqs. (5) and (6) into Eq. (3) yields

$$\sum_{i=0}^{\infty}u_i(x_1,\jmath_1)=L^{-1}\left(\frac{\dot f(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r + L^{-1}\left(\int_a^{x_1}\int_\Omega \frac{F(x_1,\jmath_1,y,\tau)}{P_k(x_1,\jmath_1)}\sum_{i=0}^{\infty}\Lambda_i dyd\tau\right) - L^{-1}\left(\sum_{j=0}^{k-1}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}L_{i_j}\right) \quad (8)$$

To determine the components $u_0,u_1,u_2,\ldots$ of the solution $u(x_1,\jmath_1)$, we set the recurrence relation

$$u_0(x_1,\jmath_1)=L^{-1}\left(\frac{\dot f(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r,$$
$$u_{i+1}(x_1,\jmath_1)=L^{-1}\left(\int_a^{x_1}\int_\Omega \frac{F(x_1,\jmath_1,y,\tau)}{P_k(x_1,\jmath_1)}\Lambda_i dyd\tau\right) - L^{-1}\left(\sum_{j=0}^{k-1}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}L_{i_j}\right),\quad i\geqslant 0 \quad (9)$$

The nonlinear terms given as follows:

$$\gamma(u^l(y,\tau))=\sum_{i=0}^{\infty}\Lambda_i,\quad D^j(u(x_1,\jmath_1))=\sum_{i=0}^{\infty}L_{i_j}.$$

where $\left(D^j=\frac{\partial^j u(x_1,\jmath_1)}{\partial x_1^j}\text{ is derivative operator}\right)$.

3. Variational iteration method

According to the VIM (He, 2007; He, 1999; He, 2000; He and Wang, 2007; Nadjafi and Tamamgar, 2008; Sadigh Behzadi, 2012) we can write the correction functional for Eq. (1) as

$$u_{i+1}(x_1,\jmath_1)=u_i(x_1,\jmath_1)+\int_0^{\jmath_1}\lambda(\varsigma)[u_i^k(x_1,\varsigma)-\frac{\dot f(x_1,\varsigma)}{P_k(x_1,\varsigma)} - \int_a^{x_1}\int_\Omega\frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma_1)}\gamma(u^l(y,\tau))dyd\tau + \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}u^j(x_1,\varsigma)]d\varsigma \quad (10)$$

where $\lambda(\varsigma)$ is a Lagrange multiplier which can be identified optimally via variational theory, u_i is the ith approximate solution, and $\delta u_i=0$.

To find the optimal λ, we proceed as follows

$$\delta u_{i+1}(x_1,\jmath_1)=\delta u_i(x_1,\jmath_1)+\delta\int_0^{\jmath_1}\lambda(\varsigma)[u_i^k(x_1,\varsigma)-\frac{\dot f(x_1,\varsigma)}{P_k(x_1,\varsigma)} - \int_a^{x_1}\int_\Omega\frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma_1)}\gamma(u^l(y,\tau))dyd\tau + \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}u^j(x_1,\varsigma)]d\varsigma=0$$

Then we apply the following stationary conditions

$$1-\lambda'=0,\quad \lambda(\varsigma=\jmath_1)=0,\quad \lambda''=0$$

This in turn gives

$$\lambda=\varsigma-\jmath_1$$

The solution is given by

$$u(x_1,\jmath_1)=\lim_{i\to\infty}u_i(x_1,\jmath_1). \quad (11)$$

4. Variational Adomian decomposition method (VADM) for solving Eqs. (1) and (2)

This modified version of VADM which is obtained by the elegant coupling of VIM and ADM.

In VIM, from the correction functional for Eq. (1) we can write

$$u_{i+1}(x_1,\jmath_1)=u_i(x_1,\jmath_1)+\int_0^{\jmath_1}\lambda(\varsigma)\left[u_i^k(x_1,\varsigma)-\frac{\dot f(x_1,\varsigma)}{P_k(x_1,\varsigma)} - \int_a^{x_1}\int_\Omega\frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma_1)}\gamma(u^l(y,\tau))dyd\tau + \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}u^j(x_1,\varsigma)\right]d\varsigma \quad (12)$$

Substituting from Eq. (6) in Eq. (12)we have

Table 1
Exact solution, approximate solution and Error by using VADM.

x_1	$\jmath_1$	Exact	App.-VADM	Err.-VADM
1.00E−02	1.00E−02	1.00000E−04	1.00041E−04	4.10000E−08
4.00E−02	4.00E−02	1.60000E−03	1.60245E−03	2.45000E−06
6.00E−02	6.00E−02	3.60000E−03	3.60790E−03	7.90000E−06
8.00E−02	8.00E−02	6.40000E−03	6.41774E−03	1.77400E−05
1.00E−01	1.00E−01	1.00000E−02	1.00326E−02	3.26000E−05

Table 2
Exact solution, approximate solution and Error by using VADM.

x_1	$\jmath_1$	Exact	App.-VADM	Err.-VADM
1.00E−03	1.00E−03	3.14152E−03	3.14300E−03	1.48000E−06
4.00E−03	4.00E−03	1.25621E−02	1.25333E−02	2.88000E−05
6.00E−03	6.00E−03	1.88350E−02	1.87290E−02	1.06000E−04
8.00E−03	8.00E−03	2.50984E−02	2.48510E−02	2.47400E−04
1.00E−02	1.00E−02	3.13487E−02	3.08400E−02	5.08700E−04

$$u_{i+1}(x_1,\jmath_1) = u_i(x_1,\jmath_1) + \int_0^{\jmath_1} \lambda(\varsigma)\left[u_i^k(x_1,\varsigma) - \frac{\dot{f}(x_1,\varsigma)}{P_k(x_1,\varsigma)} - \int_a^{x_1}\int_\Omega \frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma_1)}\Lambda_i dy d\tau + \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}u_i^j(x_1,\varsigma)\right]d\varsigma \quad (13)$$

Then the recurrence relation is given by

$$u_0(x_1,\jmath_1) = \sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r,$$

$$u_{i+1}(x_1,\jmath_1) = u_i(x_1,\jmath_1) + \int_0^{\jmath_1} \lambda(\varsigma)\left[u_i^k(x_1,\varsigma) - \frac{\dot{f}(x_1,\varsigma)}{P_k(x_1,\varsigma)} - \int_a^{x_1}\int_\Omega \frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma_1)}\Lambda_i dy d\tau + \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}u_i^j(x_1,\varsigma)\right]d\varsigma,$$
$$i \geqslant 0 \quad (14)$$

The approximate solution is given by

$$u(x_1,\jmath_1) = \lim_{i\to\infty} u_i(x_1,\jmath_1). \quad (15)$$

To demonstrate the efficiency of the VADM we have considered the following examples.

5. Numerical examples

Example (5.1.) (Aghazadeh and Khajehnasiri, 2013; Poorfattah and Shaerlar, 2015; Darania et al., 2011): Consider the nonlinear integro - differential equation

$$\frac{\partial^2 u(x_1,\jmath_1)}{\partial x_1^2} + sin(x_1\jmath_1)u(x_1,\jmath_1) - \int_0^{x_1}\int_0^1 x_1\jmath_1\frac{\partial u(y,\tau)}{\partial x_1}dy d\tau = f(x_1,\jmath_1) \quad (16)$$

where

$$f(x_1,\jmath_1) = (x_1\jmath_1)sin(x_1\jmath_1) - \frac{1}{3}\jmath_1^3. \quad (17)$$

with the initial conditions

$$I.Cs: \begin{cases} u(0,\jmath_1) = 0, \\ \frac{\partial u(0,\jmath_1)}{\partial x_1} = \jmath_1, \end{cases} \quad (18)$$

which has exact solution $u^*(x_1,\jmath_1) = x_1\jmath_1$.

This example is solved by the VADM formule (14), the results are shown in (Table 1).

Example (5.2) (Aghazadeh and Khajehnasiri, 2013; Poorfattah and Shaerlar, 2015; Darania et al., 2011): Consider the nonlinear integro-differential equation

$$u(x_1,\jmath_1)\frac{\partial^2 u(x_1,\jmath_1)}{\partial \jmath_1^2} - 4u(x_1,\jmath_1)\frac{\partial^2 u(x_1,\jmath_1)}{\partial x_1^2} + 4\int_0^{x_1}\int_0^1 u^2(y,\tau)dy d\tau = \dot{f}(x_1,\jmath_1), \quad x_1 \in [0,1], \quad (19)$$

where

$$\dot{f}(x_1,\jmath_1) = \left(x_1 - \frac{1}{2\pi}sin(2\pi x_1)\right)\left(\jmath_1 - \frac{1}{4\pi}sin(4\pi\jmath_1)\right). \quad (20)$$

with the boundary conditions

$$B.Cs: u(0,\jmath_1) = u(1,\jmath_1) = 0, \quad (21)$$

and the initial conditions

$$I.Cs: \begin{cases} u(x_1,0) = sin(\pi x_1), & 0 \leqslant x_1 \leqslant 1 \\ \frac{\partial u(x_1,0)}{\partial \jmath_1} = 0, & 0 \leqslant x_1 \leqslant 1. \end{cases} \quad (22)$$

which has exact solution $u^*(x_1,\jmath_1) = sin(\pi x_1)cos(2\pi\jmath_1)$.

This example is solved by the VADM formule (14), the results are written in (Table 2).

6. Conclusion

In this work, we applied the VADM for solving nonlinear mixed V-FIDE. The approximate solutions of V-FIDE are obtained by two powerful methods VIM and ADM in VADM. The given numerical examples showed the efficiency and accuracy of the introduced method, it reduces the size of computation without the restrictive assumption to handle nonlinear terms.

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