Journal of King Saud University – Science (2013) **25**, 57–62

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

The G'/G-expansion method for solutions of evolution equations from isothermal magnetostatic atmospheres

H. Jafari [a], **N. Kadkhoda** [a], **Anjan Biswas** [b,*]

[a] *Department of Mathematics, University of Mazandaran, Babolsar, Iran*
[b] *Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA*

Received 23 January 2012; accepted 25 February 2012
Available online 5 March 2012

KEYWORDS
$(\frac{G'}{G})$-Expansion method;
Magnetostatic equilibria;
Nonlinear evolution equations;
Travelling waves

Abstract The equations of magnetohydrostatic equilibria for plasma in a gravitational field are investigated analytically. An investigation of a family of isothermal magneto static atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The distributed current in the model J is directed along the x-axis where x is the horizontal ignorable coordinate. These equations transform to a single nonlinear elliptic equation for the magnetic vector potential u. This equation depends on an arbitrary function of u that must be specified with choices of different arbitrary functions, we obtain analytical nonlinear solutions of the elliptic equation using the $(\frac{G'}{G})$-expansion method. Finally, the hyperbolic versions of these equations will be solved by the travelling wave hypothesis method.

1. Introduction

The equations of magnetostatic equilibria have been used extensively to model the solar magnetic structure (Aslan, 2010; Heyvaerts et al., 1982; Khater et al., 2000, 2008; Kudryashov, 1988, 1990, 1991, 2010a; Low, 1982). An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The force balance consists of the between $J \wedge B$ force (B, magnetic field induction, J, electric current density), the gravitational force, and gas pressure gradient force. However, in many models, the temperature distribution is specified a priori and direct reference to the energy equations is eliminated. In solar physics, the equations of magnetostatic equilibria have been used to model diverse phenomena, such as the slow evolution stage of solar flares, or the magnetostatic support of prominences (Khater

* Corresponding author. Tel.: +1 302 857 7913; fax: +1 302 857 7054.
E-mail address: biswas.anjan@gmail.com (A. Biswas).

Peer review under responsibility of King Saud University.
doi:10.1016/j.jksus.2012.02.002

Production and hosting by Elsevier

et al., 1997; Zwingmann, 1987). The nonlinear equilibrium problem has been solved in several cases (Khater, 1989; Lerche and Low, 1980; Webb, 1988; Webb and Zank, 1990). In this paper, we obtain the exact analytical solutions for the Liouville and sinh-Poisson equations using the $\left(\frac{G'}{G}\right)$-expansion method. Because these two models will be special case of magnetostatic atmospheres model. Also here there is force balance between different forces. Recently the $\left(\frac{G'}{G}\right)$-expansion method, first introduced by Wang et al. (2008) has become widely used to search for various exact solutions of nonlinear evolution equations (Jafari et al., in press; Kudryashov, 2010b; Li and Wang, 2009; Wang et al., 2008). The method is based on the explicit linearization of nonlinear evolution equations for travelling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simplest algebraic computation. The outline of this paper is as follows:

First we describe the $\left(\frac{G'}{G}\right)$-expansion method and the basic equations. Then we solve Liouville and sinh-Poisson equations with this method.

2. Basic idea of G'/G-expansion method

To illustrate the basic idea of this method, we consider the following nonlinear partial equation with only two independent variables x and t and a dependent variable u

$$N(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \tag{1}$$

using the travelling wave transformation

$$u = u(\xi), \quad \xi = x - ct \tag{2}$$

Eq. (1) reduces to an ordinary differential equation (ODE) in the form:

$$N(u(\xi), -cu'(\xi), u'(\xi), c^2u''(\xi), u''(\xi), \ldots) = 0. \tag{3}$$

The $\left(\frac{G'}{G}\right)$-expansion method is based on the assumption that the travelling wave solution of Eq. (3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as

$$u(\xi) = \sum_{i=0}^{n} A_i\left(\frac{G'}{G}\right)^i; \quad A_n \neq 0; \tag{4}$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G'' + \lambda G' + \mu G = 0 \tag{5}$$

And $A_i (i = 0, 1, 2, \ldots, n)$, λ, μ are constants to be determined later, and G is the solution of (5), the general solutions of (5) are:

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right) - \frac{\lambda}{2}; & \lambda^2 - 4\mu > 0 \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi - c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right) - \frac{\lambda}{2}; & \lambda^2 - 4\mu < 0 \end{cases} \tag{6}$$

$u(\xi)$ can be determined explicitly by using the following steps:

Step (1) By considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of $u(\xi)$ in Eq. (3), the positive integer n in (4) is determined.

Step (2) By substituting (4) with Eq. (5) into (3) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left hand side of Eq. (3) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of A_i $(i = 0, 1, 2, \ldots, n)$, c, λ, μ.

Step (3) Solving the system of algebraic equations and then substituting the results with the general solutions of Eq. (5) into (4) gives travelling wave solutions of (3).

3. Basic equations

The relevance of magnetohydrostatic equations consisting of the equilibrium equation with force balance will be as:

$$J \wedge B - \rho\nabla\Phi - \nabla P = 0 \tag{7}$$

which is coupled with Maxwells equations:

$$J = \frac{\nabla \wedge B}{\mu} \tag{8}$$

$$\nabla \cdot B = 0 \tag{9}$$

where P, ρ, μ and Φ are the gas pressure, the mass density, the magnetic permeability and the gravitational potential, respectively. It is assumed that the temperature is uniform in space and that the plasma is an ideal gas with equation of state $p = \rho R_0 T_0$, where R_0 is the gas constant and T_0 is the temperature. Then the magnetic field B can be written by the following:

$$B = \nabla u \wedge e_x + B_x e_x = \left(B_x, \frac{\partial u}{\partial z}, \frac{-\partial u}{\partial y}\right) \tag{10}$$

The form of (10) for B ensures that $\$ \cdot B = 0$, and there is no mono pole or defect structure. Eq. (7) requires the pressure and density be of the form (Low, 1977):

$$P(y,z) = P(u)e^{\frac{-z}{h}}, \quad \rho(y,z) = \frac{1}{(gh)}P(u)e^{\frac{-z}{h}} \tag{11}$$

where $h = \frac{R_0 T_0}{g}$ is the scale height and z measures height. Substituting Eqs. (8)–(11) into Eq. (7), we obtain

$$\nabla^2 u + f(u)e^{\frac{-z}{h}} = 0, \tag{12}$$

where

$$f(u) = \mu\frac{dP}{du} \tag{13}$$

Eq. (13) gives

$$P(u) = P_0 + \frac{1}{\mu}\int f(u)du \tag{14}$$

Substituting Eq. (14) into Eq. (11), we obtain

$$P(y,z) = \left(P_0 + \frac{1}{\mu}\int f(u)du\right)e^{\frac{-z}{h}} \tag{15}$$

$$\rho(y,z) = \frac{1}{gh}\left(P_0 + \frac{1}{\mu}\int f(u)du\right)e^{\frac{-z}{h}} \tag{16}$$

where P_0 is constant. With taking transformation

$$x_1 + ix_2 = e^{\frac{-z}{l}}e^{\frac{iy}{l}} \tag{17}$$

Eq. (12) reduces to

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+l^2 f(u)e^{(\frac{2}{l}-\frac{1}{h})z}=0 \tag{18}$$

These equations have been given in Khater et al. (2000).

4. Applications of the G'/G-expansion method

In this section, we will investigate the $(\frac{G'}{G})$-expansion method for solving specific forms of $f(u)$.

4.1. Liouville equation

We first consider Liouville equation and the following equation will be special case of equation (18). Let us assume $f(u)$ has the form (Dungey, 1953; Low, 1975):

$$f(u)=-\alpha^2 A_0 e^{-\frac{A}{A_0}} \tag{19}$$

where A_0 and α^2 are constants. Hence

$$P(y,z)=\left(P_0+\frac{\alpha^2 A_0^2}{2\mu}e^{\frac{-2A}{A_0}}\right)e^{\frac{-z}{h}} \tag{20}$$

Inserting Eq. (19) into Eq. (18) we obtain

$$\nabla^2 A/A_0=l^2\alpha^2 e^{-2\frac{A}{A_0}+\left(\frac{2}{l}-\frac{1}{h}z\right)} \tag{21}$$

where $\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$. Let us set

$$\frac{A}{A_0}=\frac{z}{L}+w(y,z) \tag{22}$$

where L is a constant. Then Eq. (21) becomes

$$\nabla^2 w-l^2\alpha^2 e^{-2w-(\frac{2}{L}+\frac{1}{h}-\frac{2}{l})z} \tag{23}$$

Let us identify l by

$$\frac{2}{l}=\frac{2}{L}+\frac{1}{h} \tag{24}$$

And inserting Eq. (24) into Eq. (23) we obtain a Liouville type equation

$$\phi_{x_1x_1}+\phi_{x_2x_2}-\alpha^2 l^2 e^{-2\phi}=0 \tag{25}$$

In order to apply the $\frac{G'}{G}$, we use the wave transformation $\xi=x_1-cx_2$ and change Eq. (25) into the form

$$(1+c^2)\phi''=\alpha^2 l^2 e^{-2\phi} \tag{26}$$

we next use the transformation

$$v=e^{-2\phi} \tag{27}$$

we obtain

$$(1+c^2)vv''-(1+c^2)(v')^2+2\alpha^2 l^2 u^3=0 \tag{28}$$

with balancing according to step (1) we get $n=2$, therefore the solution of (28) can be expressed by polynomial in $\frac{G'}{G}$ as follows:

$$v(\xi)=A_0+A_1\frac{G'}{G}+A_2\left(\frac{G'}{G}\right)^2 \tag{29}$$

Substituting Eq. (29) along with (5) into (28) and setting the coefficients of all powers of $\frac{G'}{G}$ to zero, we obtain a system of nonlinear algebraic equations for A_0, A_1, A_2. Solving the resulting system with the help of Mathematica, we have the following sets of solutions:

$$\begin{cases} A_0=\frac{-\mu(1+c^2)}{l^2\alpha^2}; \\ A_1=\frac{-\lambda(1+c^2)}{l^2\alpha^2}; \\ A_2=\frac{-(1+c^2)}{l^2\alpha^2}; \end{cases} \tag{30}$$

where $\xi=x_1-cx_2$, λ, α, l are constants. Therefore, substituting (30) into (29), we have

$$v(\xi)=-\frac{\mu(1+c^2)}{l^2\alpha^2}-\frac{\lambda(1+c^2)}{l^2\alpha^2}\left(\frac{G'}{G}\right)-\frac{(1+c^2)}{l^2\alpha^2}\left(\frac{G'}{G}\right)^2 \tag{31}$$

substituting the general solution (6) into (31), according to Eq. (5), we obtain two types of travelling wave solutions of (28) as follows:

where $\lambda^2-4\mu>0$, we obtain the general hyperbolic function solutions of (28)

$$v_1(\xi)=-\frac{(\lambda^2-4\mu)(1+c^2)}{l^2\alpha^2}\left[\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2-1\right] \tag{32}$$

where c_1 and c_2 are arbitrary constants, and $\xi=x_1-c\,x_2$.

In particular, if we choose $c_2\neq 0, c_1^2<c_2^2$, then the solution (32) gives the solitary wave solution:

$$v_1(\xi)=\frac{(\lambda^2-4\mu)(1+c^2)}{l^2\alpha^2}\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right) \tag{33}$$

where $\tanh\xi_0=\frac{c_1}{c_2}$, and when $\lambda^2-4\mu<0$, the general trigonometric function solutions of (28) will be:

$$v_2(\xi)=-\frac{(4\mu-\lambda^2)(1+c^2)}{l^2\alpha^2}\left[\left(\frac{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2+1\right] \tag{34}$$

where c_1 and c_2 are arbitrary constants, and $\xi=x_1-cx_2$.

In particular, if we choose $c_2\neq 0$, $c_1^2<c_2^2$, then the solution Eq. (28) gives the solitary wave solution:

$$v_2(\xi)=-\frac{(4\mu-\lambda^2)(1+c^2)}{l^2\alpha^2}\sec^2\left(-\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_1\right) \tag{35}$$

where $\tan\xi_1=\frac{c_1}{c_2}$, when $\lambda^2-4\mu>0$ using with transformation $v=e^{-2\phi}$ we get:

$$\phi_1(\xi)=\frac{-1}{2}\times\ln\left[-\frac{(\lambda^2-4\mu)(1+c^2)}{l^2\alpha^2}\left(\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2-1\right)\right] \tag{36}$$

where c_1 and c_2 are arbitrary constants, and $\xi=x_1-cx_2$.

In particular, if we choose $c_2\neq 0$, $c_1^2<c_2^2$, then the solution (36) gives the solitary wave solution:

$$\phi_1(\xi)=\frac{-1}{2}\times\ln\left[\frac{(\lambda^2-4\mu)(1+c^2)}{l^2\alpha^2}\left(\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)\right)\right] \tag{37}$$

where $\tanh\xi_0=\frac{c_1}{c_2}$, when $\lambda^2-4\mu<0$ using with transformation $v=e^{-2\phi}$ we get:

$$\phi_2(\xi)=\frac{-1}{2}\times\ln\left[-\frac{(4\mu-\lambda^2)(1+c^2)}{l^2\alpha^2}\left(\left(\frac{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2+1\right)\right] \tag{38}$$

where c_1 and c_2 are arbitrary constants, and $\xi = x_1 - cx_2$.

In particular, if we choose $c_2\neq 0,\ c_1^2 < c_2^2$, then the solution (38) give the solitary wave solution:

$$\phi_2(\xi)=\frac{-1}{2}\times\ln\left[-\frac{(4\mu-\lambda^2)(1+c^2)}{l^2\alpha^2}\sec^2\left(-\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_1\right)\right] \tag{39}$$

where $\tan\xi_1=\frac{c_1}{c_2}$.

4.2. Sinh-Poisson equation

Secondly we consider sinh-Poisson equation which plays an important role in the soliton model with BPS bound. Also, this equation will be special case of Eq. (18). If we assume

$$f(u)=-\frac{\beta^2}{4}\left(\frac{A_0}{h}\right)\sinh(\phi) \tag{40}$$

As same as above we have

$$\phi_{x_1x_1}+\phi_{x_2x_2}=\beta^2\sinh(\phi) \tag{41}$$

where $l = 2h$. In order to apply the $\frac{G'}{G}$, we use the wave transformation $\xi = x_1 - cx_2$ and change Eq. (41) into the form

$$(1+c^2)\phi''=\beta^2\sinh(\phi) \tag{42}$$

we next use the transformation

$$\begin{cases} v=e^{\phi}\\ \sinh(\phi)=\frac{e^{\phi}-e^{-\phi}}{2}\end{cases} \tag{43}$$

we obtain

$$2(1+c^2)vv''-2(1+c^2)(v')^2-\beta^2(v^3-v)=0 \tag{44}$$

with balancing according to step (1) we get $n = 2$, therefore the solution of (44) can be expressed by polynomial in $\frac{G'}{G}$ as follow:

$$v(\xi)=A_0+A_1\frac{G'}{G}+A_2\left(\frac{G'}{G}\right)^2 \tag{45}$$

Where G is the solution of (5) that was displayed in (6), as same as the previous section, we obtain a system of nonlinear algebraic equations for A_0, A_1, A_2, c, λ, μ. By solving the resulting system we obtain the following solution:

$$\begin{cases} A_0=\frac{\lambda^2(1+c^2)}{\beta^2};\\ A_1=\frac{4\lambda(1+c^2)}{\beta^2};\\ A_2=\frac{4(1+c^2)}{\beta^2};\end{cases} \tag{46}$$

where $\xi = x_1 - cx_2$, λ, β are constants.

Therefore, substituting (46) into (45), we have

$$v(\xi)=\frac{\lambda^2(1+c^2)}{\beta^2}+\frac{4\lambda(1+c^2)}{\beta^2}\left(\frac{G'}{G}\right)+\frac{4(1+c^2)}{\beta^2}\left(\frac{G'}{G}\right)^2 \tag{47}$$

substituting the general solution (6) into (47) according to Eq. (5), we obtain two types of travelling wave solutions of (44) as follows:

where $\lambda^2 - 4\mu > 0$, we obtain the general hyperbolic function solutions of (44):

$$v_1(\xi)=\frac{(\lambda^2-4\mu)(1+c^2)}{\beta^2}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2 \tag{48}$$

where c_1 and c_2 are arbitrary constants, and $\xi = x_1 - cx_2$.

In particular, if we choose $c_2\neq 0,\ c_1^2 < c_2^2$, then the solution (48) gives the solitary wave solution:

$$v_1(\xi)=\frac{(\lambda^2-4\mu)(1+c^2)}{\beta^2}\left[\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)\right]^2 \tag{49}$$

where $\tanh\xi_0=\frac{c_1}{c_2}$, and when $\lambda^2 - 4\mu < 0$, the general trigonometric function solutions of Eq. (44) will be:

$$v_2(\xi)=\frac{(4\mu-\lambda^2)(1+c^2)}{\beta^2}\left(\frac{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2 \tag{50}$$

where c_1 and c_2 are arbitrary constants, and $\xi = x_1 - cx_2$.

In particular, if we choose $c_2\neq 0,\ c_1^2 < c_2^2$, then the solution (50) gives the solitary wave solution:

$$v_2(\xi)=\frac{(4\mu-\lambda^2)(1+c^2)}{\beta^2}\left[\tan\left(-\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_1\right)\right]^2 \tag{51}$$

where $\tan\xi_1=\frac{c_1}{c_2}$, when $\lambda^2 - 4\mu > 0$ using with transformation $v = e^{\phi}$ we get:

$$\phi_1(\xi)=\ln\left[\frac{(\lambda^2-4\mu)(1+c^2)}{\beta^2}\left(\frac{c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)^2\right] \tag{52}$$

where c_1 and c_2 are arbitrary constants, and $\xi = x_1 - cx_2$.

In particular, if we choose $c_2\neq 0,\ c_1^2 < c_2^2$, then the solution (52) gives the solitary wave solution:

$$\phi_1(\xi)=\ln\left[\frac{(\lambda^2-4\mu)(1+c^2)}{\beta^2}\left(\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)\right)^2\right] \tag{53}$$

where $\tanh\xi_0=\frac{c_1}{c_2}$, when $\lambda^2 - 4\mu < 0$ using with transformation $v = e^{\phi}$ we get:

$$\phi_2(\xi)=\ln\left[\frac{(4\mu-\lambda^2)(1+c^2)}{\beta^2}\left(\frac{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)^2\right] \tag{54}$$

where c_1 and c_2 are arbitrary constants, and $\xi = x_1 - cx_2$.

In particular, if we choose $c_2\neq 0,\ c_1^2 < c_2^2$, then the solution (54) gives the solitary wave solution:

$$\phi_2(\xi)=\ln\left[\frac{(4\mu-\lambda^2)(1+c^2)}{\beta^2}\left(\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_1\right)\right)^2\right] \tag{55}$$

where $\tan\xi_1=\frac{c_1}{c_2}$.

5. Travelling waves

In this section the travelling wave solution of the regular Liouville equation as well as the sinh-Gordon equation will be obtained. This study is split into the following two subsections.

These travelling wave solutions are going to be very useful in various situations and circumstances. For example, in the context of plasma Physics, these travelling waves very easily study the behaviour of the weakly nonlinear ion acoustic waves in the presence of an uniform magnetic field. Thus, these solutions will be extremely useful for problems that are related to nonlinear quantum ion-acoustic waves in magnetized plasma containing cold ions and hot isothermal electrons.

5.1. Liouville equation

The form of the Liouville equation that will be studied in this section is given by Wazwaz (2009)

$$q_{tt} - k^2 q_{xx} + ae^{2q} = 0 \tag{56}$$

Occasionally, this is referred to as the hyperbolic Liouville equation. The travelling wave assumption that is going to be made is given by

$$q(x,t) = g(x - vt) \tag{57}$$

where $g(s)$ is the wave profile and v is the velocity of the wave and

$$s = x - vt \tag{58}$$

Substituting the hypothesis given by Eq. (50) into Eq. (49) yields

$$g'' + \frac{a}{v^2 - k^2} e^{2g} = 0 \tag{59}$$

Now, multiplying both sides of (52) by g' and integrating yields

$$(g')^2 = \frac{a}{v^2 - k^2}(1 - e^{2g}) \tag{60}$$

and on separating variables this leads to

$$\int \frac{dg}{\sqrt{1 - e^{2g}}} = \sqrt{\frac{a}{v^2 - k^2}} \int ds \tag{61}$$

Eq. (54) integrates to

$$\tanh^{-1}(\sqrt{1 - e^{2g}}) = \sqrt{\frac{a}{v^2 - k^2}}(x - vt) \tag{62}$$

which yields, after simplification,

$$q(x,t) = \frac{1}{2}\ln(\text{sech}[B(x - vt)]) \tag{63}$$

where

$$B = \sqrt{\frac{a}{v^2 - k^2}} \tag{64}$$

In order for the travelling wave to exist, it is necessary that the constraint condition given by

$$a(v^2 - k^2) > 0 \tag{65}$$

must hold, that follows from Eq. (57).

5.2. Sinh-Gordon equation

In this subsection, the travelling wave hypothesis will be applied to solve the hyperbolic version of the sinh-Poisson equation that is also known as the sinh-Gordon equation. The equation of study is therefore going to be (Wazwaz, 2009)

$$q_{tt} - k^2 q_{xx} - b\sinh q = 0 \tag{66}$$

The starting hypothesis stays the same as in Eq. (50), which when substituted into Eq. (59) gives after simplification

$$(g')^2 = \frac{2b}{v^2 - k^2}(1 + \cosh g) \tag{67}$$

Now, separation of variables imply

$$\int \frac{dg}{\sqrt{1 + \cosh g}} = \sqrt{\frac{2b}{v^2 - k^2}} \int ds \tag{68}$$

which integrates to

$$4\tan^{-1}\left(e^{\frac{g}{2}}\right) = \sqrt{\frac{2b}{v^2 - k^2}}(x - vt) \tag{69}$$

Simplification leads to the travelling wave solution as

$$q(x,t) = 2\ln(\text{sech}[B(x - vt)]) \tag{70}$$

where

$$B = \frac{1}{2}\sqrt{\frac{b}{v^2 - k^2}} \tag{71}$$

Similarly, the constraint condition is given by

$$b(v^2 - k^2) > 0 \tag{72}$$

that follows from Eq. (64).

6. Concluding remarks

This study shows that the $\left(\frac{G'}{G}\right)$-expansion method is quite efficient and practically well suited for use in finding exact solutions for the Liouville and sinh-Poisson equations. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. In this paper, the $\left(\frac{G'}{G}\right)$-expansion method has been successfully used to obtain some exact travelling wave solutions for the Liouville and sinh-Poisson equations. These exact solutions include the hyperbolic function solutions and trigonometric function solutions. When the parameters are taken as special values, the solitary wave solutions are derived from the hyperbolic function solutions. Finally, the travelling wave solutions of the hyperbolic Liouville equation, or the regular Liouville equation as well as the sinh-Gordon equation are also obtained.

References

Aslan, I., 2010. A note on the $\left(\frac{G'}{G}\right)$-expansion method again. Applied Mathematics and Computation 217 (2), 937–938.

Dungey, J.W., 1953. A family of solutions of the magneto-hydrostatic problem in a conducting atmosphere in a gravitational field. Monthly Notices of the Royal Astronomical Society 113, 180–187.

Heyvaerts, J., Larsy, J.M., Schatzman, M., Witomsky, P., 1982. Blowing up of two dimensional magnetohydrostatic equilibria by an increase of electric current or pressure. Astronomy and Astrophysics 111 (1), 104–112.

Jafari, H., Salehpour, E., Kadkhoda, N., in press. Application of $\frac{G'}{G}$-expansion method to nonlinear Lienard equation. Indian Journal of Science and Technology.

Khater, A.H., 1989. Analytical solutions for some nonlinear two-dimensional magnetostatic equilibria. Astrophysics and Space Science 162, 151–157.

Khater, A.H., Callebaut, D.K., Ibrahim, R.S., 1997. Backlund transformations and Painlev analysis, exact solutions for the nonlinear isothermal magnetostatic atmospheres. Physics of Plasmas 4, 2853–2864.

Khater, A.H., Callebaut, D.K., El-Kalawy, O.H., 2000. Backlund transformations and exact solutions for a nonlinear elliptic equation modelling isothermal magentostatic atmosphere. IMA Journal of Applied Mathematics 65, 97–108.

Khater, A.H., Callebaut, D.K., Kamel, E.S., 2008. Nonlinear periodic solutions for isothermal magnetostatic atmospheres. Physics of Plasmas 15, 122903.

Kudryashov, N.A., 1988. Exact soliton solutions of the generalized evolution equation of wave dynamics. Applied Mathematics and Mechanics 52 (3), 361–365.

Kudryashov, N.A., 1990. Exact solutions of the generalized Kuramoto–Sivashinsky equation. Physics Letters A 147, 287–291.

Kudryashov, N.A., 1991. On types nonlinear nonintegrable differential equations with exact solutions. Physics Letters A 155, 269–275.

Kudryashov, N.A., 2010a. Meromorphic solutions of nonlinear ordinary differential equations. Communications in Nonlinear Science and Numerical Simulation 15 (10), 2778–2790.

Kudryashov, N.A., 2010b. A note on the G'/G-expansion method. Applied Mathematics and Computation 217 (4), 1755–1758.

Lerche, I., Low, B.C., 1980. On the equilibrium of a cylindrical plasma supported horizontally by magnetic fields in uniform gravity. Solar Physics 67, 229–243.

Li, X., Wang, M., 2009. The $\left(\frac{G''}{G}\right)$-expansion method and traveling wave solutions for a higher-order nonlinear Schrodinger equation. Applied Mathematics and Computation 208, 440–445.

Low, B.C., 1975. Nonisothermal magnetostatic equilibria in a uniform gravity field. I. Mathematical formulation. Astrophysical Journal 197, 251.

Low, B.C., 1977. Evolving force-free magnetic fields. I, the development of the pre flare stage. Astrophysical Journal 212, 234–242.

Low, B.C., 1982. Nonlinear force-free magnetic fields. Reviews of Geophysics 20, 145–159.

Webb, G.M., 1988. Isothermal magnetostatic atmospheres. II – similarity solutions with current proportional to the magnetic potential cubed. Astrophysical Journal 327, 933–949.

Webb, G.M., Zank, G.P., 1990. Application of the sine-Poisson equation in solar magnetostatics. Solar Physics 127, 229–252.

Wang, M., Li, X., Zhang, J., 2008. The $\left(\frac{G'}{G}\right)$-expansion method and traveling wave and solutions of nonlinear evolution equations in mathematical physics. Physics Letters A 372, 417–423.

Wazwaz, A.M., 2009. Partial Differential Equations and Solitary Waves Theory. Springer-Verlag.

Zwingmann, W., 1987. Theoretical study of onset conditions for solar eruptive processes. Solar Physics 111, 309–331.