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Fixed-point theorems for a probabilistic 2-metric spaces

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Abstract In this paper the notion of contraction mappings on probabilistic metric spaces and probabilistic 2-metric spaces are applied. Several fixed point theorems for such mappings are proved. One of them Theorem 1.1 is a stronger form of a result due to Sehgal and Bharucho-Reid (1972).

1. Probabilistic metric spaces

The concept of probabilistic metric spaces have been introduced by Menger (1942). In Menger's theory the concept of a distance is considered to be statistical or probabilistic, rather than deterministic. In other words, given any two points p and q of a metric space, a distribution function $F_{p,q}(t)$ is introduced. This distribution function has the probability interpreted as that distance between p and q is less than $t(t>0)$.

For more details about probabilistic metric spaces (cf. Schweizer and Sklar, 1960, 1983).

Definition 1.1 Sehgal and Bharucho-Reid, 1972. Let R be the set of real numbers. A mapping $F: R \to I = [0, 1]$ is said to be a distribution function if it is non-decreasing, left continuous with $inf\ F = 0$ and $Sup\ F = 1$. The set of all distribution functions will be denoted by D^+.

Remark 1.1. Since F is non-decreasing and $Sup\ F = 1$, then $\lim_{t\to\infty} F(t) = 1$

Definition 1.2 Sehgal and Bharucho-Reid, 1972. Let X be a nonempty set. let ξ be a mapping from $X \times X$ into D^+. (For $x, y \in X$ we write $F_{p,q}(t)$ instead of $\xi(x,y) \in D^+$.) The pair (X, ξ) is said to be a probabilistic metric space (PM-space, for short) if ξ satisfies the following axioms:

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(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$.
(PM-2) $F_{x,y}(0) = 0$.
(PM-3) $F_{x,y} = F_{y,x}$.
(PM-4) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.

Definition 1.3 Schweizer and Sklar, 1960. A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.4 Sehgal and Bharucho-Reid, 1972. A Menger space is a triple $(X, \xi, *)$ where (X, ξ) is PM-space and $*$ is a t-norm satisfying the following triangle inequality:

(PM-4′) $F_{x,z}(t + s) \geqslant F_{x,y}(t) * F_{y,z}(s)$ for all $x, y, z \in X$ and for all $s \geqslant 0, t \geqslant 0$.

Schweizer and Sklar (1983) have proved that if $(X, \xi, *)$ is a Menger PM-space with a continuous t-norm *, then $(X, \xi, *)$ is a Hausdorff topological space with a topology τ induced by the family of neighborhoods $\{U_p(\epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0\}$, where $U_p(\epsilon, \lambda) = \{x \in X : F_{x,p}(\epsilon) > 1 - \lambda\}$. In this topology a sequence $\{x_n\}$ in X converges to a point $x \in X$ (written $x_n \to x$) if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M(\epsilon, \lambda)$ such that $x_n \in U_x(\epsilon, \lambda)$ for all $n \geqslant M(\epsilon, \lambda)$ i.e, $F_{x_n,x}(\epsilon) > 1 - \lambda$, whenever $n \geqslant M(\epsilon, \lambda)$. The sequence $\{x_n\}$ in X will be called a Cauchy sequence if for each $\epsilon > 0, \lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$, whenever $n, m \geqslant M(\epsilon, \lambda)$.

Definition 1.5 Schweizer and Sklar, 1983. A Menger space $(X, \xi, *)$ is said to be complete if each Cauchy sequence in X converges to a point of X.

Viored Radu (2002) proposed the following form for the contraction condition for self-mapping T of a Menger space $(X, \xi, *)$:

$$F_{Tx,Ty}(K^r t) \geqslant \frac{F_{x,y}(t)}{F_{x,y}(t) + K^{1-r}(1 - F_{x,y}(t))} \quad \forall t > 0, \quad \forall x, y \in X$$

for some $r \in [0, 1]$ and some $k \in (0, 1)$.

We denote that the above contraction condition by K_r-contraction condition.

Theorem 1.1. *Let $(X, \xi, *)$ be a complete Menger space. Let T be a K_r-contraction mapping from X into itself. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X and $x_n = T^n x_0$ for all $n \in N$. Since T is a K_r-contraction mapping, we have

$$F_{x_1,x_2}(t) = F_{Tx_0,Tx_1}(t) \geqslant \frac{F_{x_0,x_1}\left(\frac{t}{K^r}\right)}{F_{x_0,x_1}\left(\frac{t}{K^r}\right) + K^{1-r}\left(1 - F_{x_0,x_1}\left(\frac{t}{K^r}\right)\right)} = \alpha^{(1)}_{x_0,x_1}\left(\frac{t}{K^r}\right)$$

$$F_{x_2,x_3}(t) = F_{Tx_1,Tx_2}(t) \geqslant \frac{F_{x_1,x_2}\left(\frac{t}{K^r}\right)}{F_{x_1,x_2}\left(\frac{t}{K^r}\right) + K^{1-r}\left(1 - F_{x_1,x_2}\left(\frac{t}{K^r}\right)\right)}$$

$$\geqslant \frac{\dfrac{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)\right)}}{\dfrac{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)\right)} + K^{1-r}\left[1 - \dfrac{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)\right)}\right]}$$

$$= \frac{\alpha^{(1)}_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)}{\alpha^{(1)}_{x_0,x_1}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left[1 - \alpha^{(1)}_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)\right]} = \alpha^{(2)}_{x_0,x_1}\left(\frac{t}{K^{2r}}\right)$$

and so on we get by a simple induction the following

$$F_{x_n,x_{n+1}}(t) \geqslant \alpha^{(2)}_{x_0,x_1}\left(\frac{t}{K^{nr}}\right) \quad \forall n \in N, \quad t > 0 \tag{1.1}$$

Using (PM-4′), for any positive integer p we have,

$$F_{x_n,x_{n+p}}(t) \geqslant F_{x_n,x_{n+1}}\left(\frac{t}{p}\right) * F_{x_n,x_{n+2}}\left(\frac{t}{p}\right) * \ldots * F_{x_{n+p-1},x_{n+p}}\left(\frac{t}{p}\right)$$

Using (1.1), we have

$$F_{x_n,x_{n+p}}(t) \geqslant \alpha^{(n)}_{x_0,x_1}\left(\frac{t}{pK^{nr}}\right) * \alpha^{(n+1)}_{x_0,x_1}\left(\frac{t}{pK^{(n+1)r}}\right) * \ldots * \alpha^{(n+P-1)}_{x_0,x_1}\left(\frac{t}{pK^{(n+P-1)r}}\right)$$

Since $\lim_{t\to\infty} F_{x,y}(t) = 1$, consequently $\lim_{t\to\infty} \alpha^{(n)}_{x_0,x_1}(t) = 1$, then $\lim_{t\to\infty} F_{x_n,x_{n+p}}(t) \geqslant 1 * 1 * \ldots * 1 = 1$,

i.e.,

$$\lim_{t\to\infty} F_{x_n,x_{n+p}}(t) = 1$$

It is follows that for all $\lambda \in (0, 1)$, there exists an integer $M(t, \lambda)$ such that

$$F_{x_n,x_{n+p}}(t) > 1 - \lambda \quad \forall n, p \in N, \quad n > M(t, \lambda)$$

Consequently, the sequence $\{x_n\}$ is a Cauchy sequence. Since $(X, \xi, *)$ is complete, then there exists a point $x^* \in X$ such that the sequence $\{x_n\}$ converges to x^* i.e.,

$$\forall \lambda \in (0, 1) \exists \text{ an integer } M(t, \lambda) \text{ such that } F_{x_n,x^*}(t) > 1 - \lambda \quad \forall n \geqslant M(t, \lambda) \tag{1.2}$$

Now we need to prove that $Tx^* = x^*$. For this we need to prove that the sequence $\{x_n\}$ converges to Tx^*.

From (1.2) we have, for all $\lambda \in (0, 1)$ there exist an integer $M(t, \lambda)$ such that

$$F_{x_n,Tx^*}(t) = F_{Tx_{n-1},Tx^*}(t) \geqslant \frac{F_{x_{n-1},x^*}\left(\frac{t}{K^r}\right)}{F_{x_{n-1},x^*}\left(\frac{t}{K^r}\right) + K^{1-r}\left[1 - F_{x_{n-1},x^*}\left(\frac{t}{K^r}\right)\right]} > \frac{1 - \lambda}{(1 - \lambda) + K^{1-r}(\lambda)} > 1 - \lambda \ \forall n \in M(t, \lambda)$$

Then, the sequence $\{x_n\}$ converges to Tx^*. By the uniqueness of the limit, hence $Tx^* = x^*$.

Now we prove the uniqueness of the fixed point

Suppose that, there exist $y^* \in X$ such that $x^* \neq y^*$, $Tx^* = x^*$ and $Ty^* = y^*$. By (PM-1) there exists real number $t > 0$ and δ with $0 \leqslant \delta < 1$ such that $F_{x^*,y^*}(t) = \delta$.

One may notice that $Tx^* = x^*$ and $Ty^* = y^*$, implies that $T^n x^* = T^{n-1} x^* = \ldots = Tx^* = x^*$ and $T^n y^* = T^{n-1} y^* = \ldots = Ty^* = y^*$. It is follows that for each positive integer n

we have,
and so on we get by a simple induction the following

$$\delta = F_{x^*,y^*}(t) = F_{T^n x^*,T^n y^*}(t) \geqslant \frac{F_{T^{n-1}x^*,T^{n-1}y^*}\left(\frac{t}{K^r}\right)}{F_{T^{n-1}x^*,T^{n-1}y^*}\left(\frac{t}{K^r}\right) + K^{(1-r)}\left[1 - F_{T^{n-1}x^*,T^{n-1}y^*}\left(\frac{t}{K^r}\right)\right]} = \alpha^{(1)}_{T^{n-1}x^*,T^{n-1}y^*}\left(\frac{t}{K^r}\right)$$

$$\geqslant \frac{\dfrac{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)+K^{(1-r)}F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}}{\dfrac{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)+K^{(1-r)}F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)} + K^{1-r}\left[1 - \dfrac{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}{F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)+K^{1-r}F_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}\right]}$$

$$= \frac{\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)}{\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)} = \alpha^{(2)}_{T^{n-2}x^*,T^{n-2}y^*}\left(\frac{t}{K^{2r}}\right)$$

$$\delta \geqslant \alpha^{(n)}_{x^*,y^*}\left(\frac{t}{K^{nr}}\right)$$

Since $\lim\limits_{t\to\infty} \alpha^{(n)}_{x^*,y^*}\left(\frac{t}{K^{nr}}\right) = 1$, then $\delta \geqslant 1$. This contradicts the selection of δ. Therefore, the fixed point is unique. □

If we let $r = 1$ in K_r-contraction condition we have the contraction condition due Sehgal and Bharucho-Reid (1972) as in the following definition.

Definition 1.6 Sehgal and Bharucho-Reid, 1972. A mapping T of a PM-space $(X, \xi, *)$ into itself is said to be contraction mapping if there exists a constant $k \in (0, 1)$, such that for each $x, y \in X$,

$$F_{Tx,Ty}(Kt) \geqslant F_{x,y}(t)$$

The expression $F_{Tx,Ty}(Kt) \geqslant F_{x,y}(t)$ means that the probability that the distance between the image points Tx, Ty is less than Kt is at less equal to the probability that the distance between the points x, y is less than t.

If we let $r = 1$ in Theorem 1.1 we get the following theorem.

Theorem 1.2. *Let $(X, \xi, *)$ be a complete Menger space. Let T be a mapping from X into itself satisfy the following contraction condition $F_{Tx,Ty}(Kt) \geqslant F_{x,y}(t)$ for each $x, y \in X$. Then T has a unique fixed point.*

Theorem 1.2 is a stronger form of the following theorem.

Theorem 1.3. (Sehgal and Bharucho-Reid, 1972). *Let $(X, \xi, *)$ be a complete Menger space, where * is a continuous t-norm satisfy the additional condition: $x * x \geqslant x$ for each $x \in [0, 1]$.*

Let T be a mapping from X into itself satisfy the following contraction condition.

$F_{Tx,Ty}(Kt) \geqslant F_{x,y}(t)$ for each $x, y \in X$. Then T has a unique fixed point.

2. Fixed-point theorems in probabilistic 2-metric spaces

Gähler (1963) investigate the concept of 2-metric space is a natural generalization of a metric space. Some fixed-point theorems in 2-metric spaces are obtained in Iseki (1975), Rhoades (1979). The probabilistic 2-metric spaces where first introduced in Golet (1988a), Golet (1988b) study a fixed point theorem in probabilistic 2-metric spaces. In this section we introduce some fixed-point theorems in probabilistic 2-metric space by using K_r-contraction condition in 2-metric spaces.

Definition 2.1 Gähler, 1963. A 2-metric space is an ordered pair (X, d) where X is an abstract set and d is a mapping from X^3 into the positive real numbers, i.e., $d : X^3 \to R^+$, d associates a real number $d(x, y, z)$ with every triple (x, y, z). The mapping d is assumed to satisfy the following conditions:

(1) For distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,
(2) $d(x, y, z) = 0$ if at least two of x, y and z are equal,
(3) $d(x, y, z) = d(x, z, y) = d(y, z, x) \quad \forall x, y, z \in X$,
(4) $d(x, y, z) \leqslant d(x, y, u) + d(x, u, z) + d(u, y, z)$ $\forall x, y, z, u \in X$.

The function d is called a 2-metric for the space X and the pair (X, d) denotes a 2-metric space. It has shown by Gähler (1963) that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric d which is continuous in all of its arguments is said to be continuous. Geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y and z.

Definition 2.2 Golet, 1995. A probabilistic 2-metric space (P2M-space, for short) is an order pair (X, ξ) where X is an abstract set and ξ is a mapping from X^3 into D^+. In other words, $\xi(x, y, z) \in D^+$, for all $(x, y, z) \in X^3$, We shall denote the distribution function $\xi(x, y, z)$ by $F_{x,y,z}$, where the symbol $F_{x,y,z}(t)$ will denote the value of $F_{x,y,z}$ at the real number t. The function $F_{x,y,z}$ are assumed to satisfy the following conditions.

(P2M-1) $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of the three points x, y, z are equal, $F_{x,y,z}(t) = 0$ for all $t \leqslant 0$ $\forall x, y, z \in X$.
(P2M-2) For distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ if $t > 0$.
(P2M-3) $F_{x,y,z} = F_{x,z,y} = F_{y,z,x}$.
(P2M-4) $F_{x,y,w}(t_1) = 1, F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$.

Example 2.1.

Let $\xi(x,y,z)(t) = F_{x,y,z}(t) = \begin{cases} \frac{t}{t+min\{|x-y|,|x-z|,|y-z|\}}, & \text{if} \quad t > 0 \\ 0, & \text{i} \quad t \leqslant 0 \end{cases}$, for all $(x,y,z) \in X^3$. Then (X,ξ) is P2M-space.

Definition 2.3 Golet, 1995. A mapping $* : [0,1]^3 \to [0,1]$ is said to be 2-t-norm if

(2T-1) $a * 1 * 1 = a$,
(2T-2) $a * b * c = a * c * b = c * b * a$,
(2T-3) $a * b * c \leqslant d * e * f$, if $a \leqslant d, b \leqslant e$ and $c \leqslant f$,
(2T-4) $(a * b * c) * d * e = a * (b * c * d) * e = a * b * (c * d * e)$ for all $a,b,c,d,e \in [0,1]$.

Definition 2.4 Golet, 1995. A 2-Menger space is a triple $(X,\xi,*)$ where (X,ξ) is P2M-space, and $*$ is a 2-t-norm satisfying the following triangle inequality:

(P2M-4′) $F_{x,y,z}(t_1 + t_2 + t_3) \geqslant F_{x,y,w}(t_1) * F_{x,w,z}(t_2) * F_{w,y,z}(t_3) \ \forall x,y,z,w \in X$

Definition 2.5. Let $(X,\xi,*)$ be a 2-Menger with a continuous 2-t-norm $*$. The sequence $\{x_n\}$ in X is said to be converges to a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M(\epsilon,\lambda)$ such that $F_{x_n,x,a}(\epsilon) > 1 - \lambda$, whenever $n \geqslant M(\epsilon,\lambda)$

Definition 2.6. The sequence $\{x_n\}$ in X is said to be Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M(\epsilon,\lambda)$ such that $F_{x_n,x_m,a}(\epsilon) > 1 - \lambda$, whenever $n,m \geqslant M(\epsilon,\lambda)$

Definition 2.7. A 2-Menger space $(X,\xi,*)$ will be complete if each Cauchy sequence in X converges to a point of X.

Now, we introduce some fixed-point theorems in P2M-space analogous to Theorem 1.1 as the following.

Theorem 2.1. *Let $(X,\xi,*)$ be a complete 2-Menger space. Let T be a mapping from X into itself. satisfying the following condition:*

$$F_{Tx,Ty,a}(K^r t) \geqslant \frac{F_{x,y,a}(t)}{F_{x,y,a}(t) + K^{1-r}(1 - F_{x,y,a}(t))} \quad \forall t > 0, \quad \forall x,y,a \in X$$

for some $r \in [0,1]$ and some $k \in (0,1)$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X and $x_n = T^n x_0$ for all $n \in N$. From the given condition, we have

$$F_{x_1,x_2,a}(t) = F_{Tx_0,Tx_1,a}(t) \geqslant \frac{F_{x_0,x_1,a}\left(\frac{t}{K^r}\right)}{F_{x_0,x_1,a}\left(\frac{t}{K^r}\right) + K^{1-r}\left(1 - F_{x_0,x_1,a}\left(\frac{t}{K^r}\right)\right)} = \alpha^{(1)}_{x_0,x_1,a}\left(\frac{t}{K^r}\right)$$

$$F_{x_2,x_3,a}(t) = F_{Tx_1,Tx_2,a}(t) \geqslant \frac{F_{x_1,x_2,a}\left(\frac{t}{K^r}\right)}{F_{x_1,x_2,a}\left(\frac{t}{K^r}\right) + K^{1-r}\left(1 - F_{x_1,x_2,a}\left(\frac{t}{K^r}\right)\right)}$$

$$\geqslant \frac{\dfrac{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)\right)}}{\dfrac{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)\right)} + K^{1-r}\left[1 - \dfrac{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)}{F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left(1 - F_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)\right)}\right]}$$

$$= \frac{\alpha^{(1)}_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)}{\alpha^{(1)}_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\left[1 - \alpha^{(1)}_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)\right]} = \alpha^{(2)}_{x_0,x_1,a}\left(\frac{t}{K^{2r}}\right)$$

and so on we get by a simple induction the following

$$F_{x_n,x_{n+1},a}(t) \geqslant \alpha^{(2)}_{x_0,x_1,a}\left(\frac{t}{K^{nr}}\right) \quad \forall n \in N, t > 0 \tag{2.1}$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. For any $t > 0$ and a natural number $p \geqslant 2$. One can write $t = t_0 + t_1 + t_2 + t_3 + \ldots + t_{2p-2} + t_{2p-1} + tk^p$ where $t_0 = t(1 - k^{\frac{1}{2}})$, $t_1 = tk^{\frac{1}{2}}(1 - k^{\frac{1}{2}})$, $t_2 = tk(1 - k^{\frac{1}{2}})$, $t_3 = tk^{\frac{3}{2}}(1 - k^{\frac{1}{2}})$, $\ldots$, $t_{2p-2} = tk^{p-1}\left(1 - k^{\frac{1}{2}}\right)$ and $t_{2p-1} = tk^{\frac{2p-1}{2}}\left(1 - k^{\frac{1}{2}}\right)$, i.e., $t = t(1 - k^{\frac{1}{2}}) + tk^{\frac{1}{2}}\left(1 - k^{\frac{1}{2}}\right) + tk\left(1 - k^{\frac{1}{2}}\right) + tk^{\frac{3}{2}}\left(1 - k^{\frac{1}{2}}\right) + \ldots + tk^{p-1}\left(1 - k^{\frac{1}{2}}\right) + tk^{\frac{2p-1}{2}}\left(1 - k^{\frac{1}{2}}\right) + tk^p$. Using (P2M-4′), for any positive integer p we have,

$$\begin{aligned} F_{x_n,x_{n+p},a}(t) \geqslant\ & F_{x_n,x_{n+1},a}(t_0) * F_{x_n,x_{n+p},x_{n+1}}(t_1) * F_{x_{n+1},x_{n+2},a}(t_2) \\ & * F_{x_{n+1},x_{n+2},x_{n+p}}(t_3) * \ldots * F_{x_{n+p-2},x_{n+p-1},a}(t_{2p-2}) \\ & * F_{x_{n+p-2},x_{n+p-1},x_{n+p}}(t_{2p-1}) * F_{x_{n+p-1},x_{n+p},a}(tk^p) \end{aligned}$$

Using (2.1), we have

$$\begin{aligned} F_{x_n,x_{n+p},a}(t) \geqslant\ & \alpha^{(n)}_{x_0,x_1,a}\left(\frac{t_0}{K^{nr}}\right) * \alpha^{(n)}_{x_0,x_1,x_{n+p}}\left(\frac{t_1}{K^{nr}}\right) * \alpha^{(n+1)}_{x_0,x_1,a}\left(\frac{t_2}{K^{(n+1)r}}\right) \\ & * \alpha^{(n+1)}_{x_0,x_1,x_{n+p}}\left(\frac{t_3}{K^{(n+1)r}}\right) * \ldots * \alpha^{(n+p-2)}_{x_0,x_1,a}\left(\frac{t_{2p-2}}{K^{(n+p-2)r}}\right) \\ & * \alpha^{(n+p-2)}_{x_0,x_1,x_{n+p}}\left(\frac{t_{2p-1}}{K^{(n+p-2)r}}\right) * \alpha^{(n+p-1)}_{x_0,x_1,a}\left(\frac{tK^p}{K^{(n+p-1)r}}\right) \end{aligned}$$

Since $\lim_{t\to\infty} F_{x,y,a}(t) = 1$, consequently $\lim_{t\to\infty} \alpha^{(n)}_{x_0,x_1,a}(t) = 1$, then $\lim_{t\to\infty} F_{x_n,x_{n+p},a}(t) \geqslant 1 * 1 * \ldots * 1 = 1$,

i.e.,

$$\lim_{t\to\infty} F_{x_n,x_{n+p},a}(t) = 1$$

It is follows that for all $\lambda \in (0,1)$, there exists an integer $M(t,\lambda)$ such that

$$F_{x_n,x_{n+p},a}(t) > 1 - \lambda \quad \forall n,p \in N, \quad n > M(t,\lambda).$$

This means that, the sequence $\{x_n\}$ is a Cauchy sequence. Since the 2-Menger space $(X,\xi,*)$ is complete, then there exists a point $x^* \in X$ such that the sequence $\{x_n\}$ converges to x^* i.e.,

$$\forall \lambda \in (0,1) \exists \text{ an integer } M(t,\lambda) \text{s. t. } F_{x_n,x^*,a}(t) > 1 - \lambda \ \forall n \geqslant M(t,\lambda) \tag{2.2}$$

Now we need to prove that $Tx^* = x^*$. For this we need to prove that the sequence $\{x_n\}$ converges to Tx^*.

From (2.2) we have, for all $\lambda \in (0,1)$ there exist an integer $M(t,\lambda)$ such that

$$\begin{aligned} F_{x_n,Tx^*,a}(t) &= F_{Tx_{n-1},Tx^*,a}(t) \\ &\geqslant \frac{F_{x_{n-1},x^*,a}\left(\frac{t}{K^r}\right)}{F_{x_{n-1},x^*,a}\left(\frac{t}{K^r}\right) + K^{1-r}\left[1 - F_{x_{n-1},x^*,a}\left(\frac{t}{K^r}\right)\right]} \\ &> \frac{1 - \lambda}{(1 - \lambda) + K^{1-r}(\lambda)} > 1 - \lambda \ \forall n \in M(t,\lambda) \end{aligned}$$

Then, the sequence $\{x_n\}$ converges to Tx^*. By the uniqueness of the limit, then $Tx^* = x^*$.

Now we prove the uniqueness of the fixed point.

Suppose that, there exist $y^* \in X$ such that $x^* \neq y^*$, $Tx^* = x^*$ and $Ty^* = y^*$.

By (P2M-2) there exists real number $t > 0$ and δ with $0 \leqslant \delta < 1$ such that $F_{x^*,y^*,a}(t) = \delta \ \forall \ a \neq x^*$ and $a \neq y^*$.

One may notice that $Tx^* = x^*$ and $Ty^* = y^*$, implies that $T^n x^* = T^{n-1}x^* = \ldots = Tx^* = x^*$ and $T^n y^* = T^{n-1}y^* = \ldots = Ty^* = y^*$. It is follows that for each positive integer n we have,

$$\begin{aligned}
\delta &= F_{x^*,y^*,a}(t) = F_{T^n x^*, T^n y^*, a}(t) \\
&\geqslant \frac{F_{T^{n-1}x^*,T^{n-1}y^*,a}\left(\frac{t}{K^r}\right)}{F_{T^{n-1}x^*,T^{n-1}y^*,a}\left(\frac{t}{K^r}\right) + K^{(1-r)}\left[1 - F_{T^{n-1}x^*,T^{n-1}y^*,a}\left(\frac{t}{K^r}\right)\right]} \\
&= \alpha^{(1)}_{T^{n-1}x^*,T^{n-1}y^*,a}\left(\frac{t}{K^r}\right) \\
&= \frac{\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*,a}\left(\frac{t}{K^{2r}}\right)}{\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*,a}\left(\frac{t}{K^{2r}}\right) + K^{1-r}\alpha^{(1)}_{T^{n-2}x^*,T^{n-2}y^*,a}\left(\frac{t}{K^{2r}}\right)} \\
&= \alpha^{(2)}_{T^{n-2}x^*,T^{n-2}y^*,a}\left(\frac{t}{K^{2r}}\right)
\end{aligned}$$

and so on we get by a simple induction the following

$$\delta \geqslant \alpha^{(n)}_{x^*,y^*,a}\left(\frac{t}{K^{nr}}\right)$$

Since $\lim\limits_{t\to\infty} \alpha^{(n)}_{x^*,y^*,a}\left(\frac{t}{K^{nr}}\right) = 1$, then $\delta \geqslant 1$. This contradicts the selection of δ. Therefore, the fixed point is unique. □

If we let $r = 1$ in Theorem 2.1 we get the following theorem.

Theorem 2.2. *Let $(X, \xi, *)$ be a complete 2-Menger space. Let T be a mapping from X into itself satisfy the following contraction condition*

$$F_{Tx,Ty,a}(Kt) \geqslant F_{x,y,a}(t) \ \forall t > 0, \ \forall x, y, a \in X \text{ for some } K \in (0, 1).$$

Then T has a unique fixed point.

3. Conclusion

Fixed-point theorems have proved to be a useful instrument in many applied areas such as mathematical economics, non-cooperative game theory, dynamic optimization and stochastic games, functional analysis, variational calculus and etc. However, for many practical situations, the conditions in the fixed-point theorems are too strong, so there is then no guarantee that a fixed point exists. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric space has developed in many directions. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic 2-metric space corresponds to situations when we do not know exactly the distance between three points, but we know probabilities of possible values of this distance. Such a probabilistic generalization of 2-metric spaces appears to be interest in the investigation of physical quantities and physiological threshold. It is also of fundamental importance in probabilistic functional analysis, non-linear analysis and applications (Chang et al., 1996, 2001; Khamsi and Kreinovich, 1996; Schweizer et al., 1998).

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