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Efficient techniques on bipolar parametric ν-metric space with application

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ABSTRACT

This work aims to motivate the bipolar parametric metric space introduced by Pasha et al. We introduce the concept of bipolar parametric ν-metric space. Afterward, we state and investigate new fixed-point theorems. Suitable examples are given based on our outcomes. An application is provided to strengthen the findings we obtained.

1. Introduction

In many branches of mathematics, applications of fixed point theory are important. In fixed point theory, finding fixed points (FPs) of generalized contraction maps became recognized as an interesting area of research. Numerous authors have published many articles on FP theorems and with different applications. The Banach contraction concept in FP theory is generalized to FPs in bipolar metric spaces (BPMS). In 2016, Mutlu and Gürdal (2016) proposed the concepts of BPMS, and they have proved FP and coupled FP theorem for covariant and contravariant maps. Mutlu et al. (2017) proved a coupled FP theorem on BPMS. Gürdal et al. (2020) proposed FP results in $\alpha-\psi$ contractive mappings in BPMS. Common FP theorem in BPMS was proven by Kishore et al. (2018) using Caristi-type contraction. Kishore et al. (2019a) proved a common coupled FP theorem in BPMS. Kishore et al. (2019) proved a coupled FP theorem in partially ordered BPMS. Rao et al. (2018) proposed a common coupled FP theorem in BPMS by using Geraghty-type contraction. Kishore et al. (2021) proposed a coupled FP theorem in BPMS in three covariant mappings. Mutlu et al. (2020) proved a FP theorem in BPMS by using local and weakly contractive mappings. Gaba et al. (2021) proved FP theorems in BPMS. Mani et al. (2022) proposed the concept of $C^\star$-algebra valued BPMS and proved coupled FP theorems. Mani et al. (2023b) proved a FP theorem in bipolar-controlled metric space. Mani et al. (2023a) proved a FP theorem in $C^\star$ -algebra valued BPMS using Banach and Kannan type contraction. Ramaswamy et al. (2022) proved a FP theorem in $C^\star$ -algebra-valued BPMS using covariant and contravariant mappings. FP theorems in parametric metric spaces was proven by Hussain et al. (2014) . Rao et al. (2014) proved a common FP theorem in parametric S-metric spaces. In 2016, Krishnakumar and Sanatammappa (2016), extended complete parametric b-metric spaces to prove FP theorem on continuous mappings. Tas and Ozgur (2018) given the parametric N_b-metric spaces and proved FP theorems, and Ozgur proposed parametric N_b-metric space and proved the fixed-circle theorem. Younis and Bahuguna (2023) proposed the controlled graphical metric type spaces, with extended b-metric type spaces, graphical type spaces, and integrated controlled metric type spaces. In 2023, Mudasir et al. (2023) established a FP theorem in graphical spaces to propose solving boundary value problems with two points in the fourth order that express the deformations of elastic beams. Smarandache et al. (2020)

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have proposed quadruple neutrosophic theory. Younıs et al. (2024) proposed a FP theorem in graphical bipolar metric spaces. Ahmad et al. (2023) introduced a FP theorem in graphical bipolar b-metric spaces and applied it in covariant and contravariant maps. Bartwal et al. (2020) introduced fuzzy bipolar metric space and established new FP techniques. Riaz and Tehrim (2019) proposed bipolar fuzzy soft set and bipolar fuzzy soft mapping to diagnose bipolar disorder and its various forms accurately Riaz and Tehrim (2020) introduced bipolar fuzzy soft topology based on bipolar fuzzy soft set. The crisp topology is a generalization of the bipolar fuzzy soft topology. Mani et al. (2024) introduced Menger probabilistic bipolar metric space and proved FP theorems. Kumar et al. (2024) established FP theorems and presented the concept of binary operations at the point of non-negative parameter $\mathfrak{t}$ to generate parametric metric space, which is a generalization of parametric metric space. The new FP results in parametric b-metric space was established by Hussain et al. (2015). Pasha et al. (2024) proposed a FP theorem in bipolar parametric metric space. Motivated by the previous work done in Pasha et al. (2024), in this study, we prove FP theorems on BPPMS and introduce the concept of BPPνMS (bipolar parametric ν-metric space) without binary operation.

2. Preliminaries

We outline some fundamental definitions in this part. Bipolar metric spaces were proposed and fixed point theorems were proven by Mutlu and Gürdal (2016).

Definition 2.1 (*Hussain et al., 2015*)**.** Let Φ be a nonempty set and $\pi : \Phi \times \Phi \times (0,\infty) \to \mathbb{R}^+$ be a function s.t.(such that):

(a) If $\pi(\mathfrak{l},\mathfrak{m},\mathfrak{t}) = 0$ for all $\mathfrak{t} > 0$ then $\mathfrak{l} = \mathfrak{m}$.
(b) If $\mathfrak{l} = \mathfrak{m}$, then $\pi(\mathfrak{l},\mathfrak{m},\mathfrak{t}) = 0$, for all $\mathfrak{t} > 0$.
(c) $\pi(\mathfrak{l},\mathfrak{m},\mathfrak{t}) = \pi(\mathfrak{m},\mathfrak{l},\mathfrak{t})$, for all $\mathfrak{t} > 0$,
(d) $\pi(\mathfrak{l},\mathfrak{m},\mathfrak{t}) \le \nu(\pi(\mathfrak{l},\mathfrak{h},\mathfrak{t}) + \pi(\mathfrak{h},\mathfrak{m},\mathfrak{t}))$, for all $\mathfrak{t} > 0$ and $\mathfrak{l},\mathfrak{m},\mathfrak{h} \in \Phi$, where $\nu \ge 1$.

The pair (Φ,π) is called a parametric ν-metric space.

Definition 2.2 (*Mutlu and Gürdal, 2016*)**.** Let Φ and Ξ be nonempty sets and $\pi : \Phi \times \Xi \to \mathbb{R}^+$ be a function s.t.(such that):

(a) If $\pi(\varrho,\varpi) = 0$ then $\varrho = \varpi$, for all $(\varrho,\varpi) \in \Phi \times \Xi$.
(b) If $\varrho = \varpi$, then $\pi(\varrho,\varpi) = 0$, for all $(\varrho,\varpi) \in \Phi \times \Xi$.
(c) $\pi(\varrho,\varpi) = \pi(\varpi,\varrho)$, for all $\varrho,\varpi \in \Phi \cap \Xi$.
(d) $\pi(\varrho,\varpi) \le \pi(\varrho,\mathfrak{z}) + \pi(\alpha,\mathfrak{z}) + \pi(\alpha,\varpi)$, for all $\varrho,\alpha \in \Phi$ and $\mathfrak{z},\varpi \in \Xi$.

The triplet (Φ,Ξ,π) is said to be BPMSs.

Next, we present the idea of BPPνMSs.

Definition 2.3. Let Φ and Ξ be nonempty sets and $\pi : \Phi\times\Xi\times(0,\infty) \to \mathbb{R}^+$ be a function s.t.:

(a) If $\pi(\varrho,\varpi,\mathfrak{t}) = 0$ for all $\mathfrak{t} > 0$ then $\varrho = \varpi$, for all $(\varrho,\varpi) \in \Phi \times \Xi$.
(b) If $\varrho = \varpi$, then $\pi(\varrho,\varpi,\mathfrak{t}) = 0$, for all $\mathfrak{t} > 0$ and $(\varrho,\varpi) \in \Phi \times \Xi$.
(c) $\pi(\varrho,\varpi,\mathfrak{t}) = \pi(\varpi,\varrho,\mathfrak{t})$, for all $\mathfrak{t} > 0$ and $\varrho,\varpi \in \Phi \cap \Xi$.
(d) $\pi(\varrho,\varpi,\mathfrak{t}) \le \nu(\pi(\varrho,\mathfrak{z},\mathfrak{t})+\pi(\alpha,\mathfrak{z},\mathfrak{t})+\pi(\alpha,\varpi,\mathfrak{t}))$, for all $\mathfrak{t} > 0$, $\varrho,\alpha \in \Phi$ and $\mathfrak{z},\varpi \in \Xi$, where $\nu \ge 1$.

The triplet (Φ,Ξ,π,ν) is called a BPPνMSs.

Example 2.1. Let $\Phi = (-\infty,0]$ and $\Xi = [0,\infty)$ be equipped with $\pi(\varrho,\varpi,\mathfrak{t}) = \mathfrak{t}|\varrho-\varpi|^2$ for all $\varrho \in \Phi$, $\varpi \in \Xi$ and $\mathfrak{t} > 0$. Easily one can check that conditions (a)–(c). Next, we check the condition (d). For this,

$$\begin{aligned}\pi(\varrho,\varpi,\mathfrak{t}) = \mathfrak{t}|\varrho-\varpi|^2 &= \mathfrak{t}|\varrho-\mathfrak{z}+\alpha-\mathfrak{z}-\alpha+\varpi|^2\\ &\le 3\mathfrak{t}(|\varrho-\mathfrak{z}|^2+|\alpha-\mathfrak{z}|^2+|\alpha-\varpi|^2)\\ &= 3(\pi(\varrho,\mathfrak{z},\mathfrak{t})+\pi(\alpha,\mathfrak{z},\mathfrak{t})+\pi(\alpha,\varpi,\mathfrak{t})).\end{aligned}$$

Then, (Φ,Ξ,π,ν) is a complete BPPνMS with $\nu = 3$. But it is not a bipolar parametric metric space (BPPMSs).

Remark 2.2. If we take $\nu = 1$, then we get BPPMSs in Pasha et al. (2024).

We introduce covariant mapping, contravariant mapping, contraction mapping, convergent sequence, continuous mapping and Cauchy sequence as follows:

Definition 2.4.

(A1) Let (Φ,Ξ,π,ν) be a BPPνMSs. Afterward, the set points Φ, Ξ and $\Phi\cap\Xi$ are referred to as left, right, and central points, and any sequence on (Φ,Ξ,π,ν) that solely consists of left (or right, or central) points is considered to be a left (or right, or central) sequence.
(A2) Let $(\Phi_1,\Xi_1,\pi_1,\nu_1)$ and $(\Phi_2,\Xi_2,\pi_2,\nu_2)$ be BPPνMSs and $\Theta : \Phi_1\cup\Xi_1 \to \Phi_2\cup\Xi_2$ be a function. If $\Theta(\Phi_1) \subseteq \Phi_2$ and $\Theta(\Xi_1) \subseteq \Xi_2$, then Θ is called a covariant map, or a map from $(\Phi_1,\Xi_1,\pi_1,\nu_1)$ to $(\Phi_2,\Xi_2,\pi_2,\nu_2)$ and this is written as $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$. If $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Xi_2,\Phi_2,\overline{\pi}_2,\nu_2)$ be a map, furthermore Θ is said to be contravariant map from $(\Phi_1,\Xi_1,\pi_1,\nu_1)$ *to* $(\Phi_2,\Xi_2,\pi_2,\nu_2)$ and is referred as $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightleftarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$.

Definition 2.5. Let (Φ,Ξ,π,ν) be a BPPνMSs. A left sequence $\{\varrho_\omega\}$ converges to a right point ϖ iff for every $\ell > 0$ we can find an $\omega_0 \in \mathbb{N}$ s.t. $\pi(\varrho_\omega,\varpi,\mathfrak{t}) < \ell$ for all $\omega \ge \omega_0$ and $\mathfrak{t} > 0$. In a similar way, a right sequence $\{\varpi_\omega\}$ tends to a left point ϱ iff, for each $\ell > 0$ one can finds an $\omega_0 \in \mathbb{N}$ satisfying, whenever $\omega \ge \omega_0, \mathfrak{t} > 0, \pi(\varrho,\varpi_\omega,\mathfrak{t}) < \ell$.

Definition 2.6. Let (Φ,Ξ,π,ν) be a BPPνMSs.

(i) A sequence $(\{\varrho_\omega\},\{\varpi_\omega\})$ on the set $\Phi\times\Xi$ is said to be bisequence on (Φ,Ξ,π).
(ii) Both $\{\varrho_\omega\}$ and $\{\varpi_\omega\}$ is convergent, then the bisequence $(\{\varrho_\omega\},\{\varpi_\omega\})$ is said to be convergent. If $\{\varrho_\omega\}$ and $\{\varpi_\omega\}$ both converge to a same point $u \in \Phi\cap\Xi$, then the bisequence is said to be biconvergent.
(iii) A bisequence $(\{\varrho_\omega\},\{\varpi_\omega\})$ on (Φ,Ξ,π,ν) is said to be a C-biseq, if for each $\ell > 0$, we can find a number $\omega_0 \in \mathbb{N}$, satisfies the positive integers $\omega,\rho \ge \omega_0, \mathfrak{t} > 0, \pi(\varrho_\omega,\varpi_\rho,\mathfrak{t}) < \ell$.

Definition 2.7. Let $(\Phi_1,\Xi_1,\pi_1,\nu_1)$ and $(\Phi_2,\Xi_2,\pi_2,\nu_2)$ be BPPνMSs.

(i) The mapping $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$ is said to be left continuous at $\varrho_0 \in \Phi_1$ if each sequence $\{\varpi_\omega\} \subset \Xi_1$ with $\varpi_\omega \to \varrho_0$ we have $\Theta(\varpi_\omega) \to \Theta(\varrho_0)$ on $(\Phi_2,\Xi_2,\pi_2,\nu_2)$.
(ii) The mapping $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$ is said to be right continuous at $\varpi_0 \in \Xi_1$ if each sequence $\{\varrho_\omega\} \subset \Phi_1$ with $\varrho_\omega \to \varpi_0$ we have $\Theta(\varrho_\omega) \to \Theta(\varpi_0)$ on $(\Phi_2,\Xi_2,\pi_2,\nu_2)$.
(iii) A contravariant map $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \leftrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$ is continuous iff it is continuous as a covariant map $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Phi_2,\Xi_2,\bar{\pi}_2,\nu_2)$.

Definition 2.8. Let $(\Phi_1,\Xi_1,\pi_1,\nu_1)$ and $(\Phi_2,\Xi_2,\pi_2,\nu_2)$ be BPPνMSs and $\phi > 0$. A covariant map $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \rightrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$ s.t.

$\pi_2(\Theta(\varrho),\Theta(\varpi),\mathfrak{t}) \le \phi\pi_1(\varrho,\varpi,\mathfrak{t})$ for all $\mathfrak{t} > 0, \varrho \in \Phi_1, \varpi \in \Xi_1$,

or a contravariant map $\Theta : (\Phi_1,\Xi_1,\pi_1,\nu_1) \leftrightarrows (\Phi_2,\Xi_2,\pi_2,\nu_2)$ s.t.

$\pi_2(\Theta(\varpi),\Theta(\varrho),\mathfrak{t}) \le \phi\pi_1(\varrho,\varpi,\mathfrak{t})$ for all $\mathfrak{t} > 0, \varrho \in \Phi_1, \varpi \in \Xi_1$,

is called Lipschitz continuous. If $\phi = 1$, then covariant or contravariant map is called non-expansive, and if it is obtained for a $\phi \in (0,1)$, it is called a contraction.

3. Main results

This part concerns to study Banach, Kannan's and Reich type fixed point theorems on BPPνMS with an examples. Here C-biseq means Cauchy bisequence

Theorem 3.1. *Consider (Φ, Ξ, π, ν) to be a complete BPPνMS and a covariant contraction $\Theta : (\Phi, \Xi, \pi, \nu) \rightrightarrows (\Phi, \Xi, \pi, \nu)$. Then the map $\Theta : \Phi \cup \Xi \to \Phi \cup \Xi$ has a UFP (unique fixed point).*

Proof. Consider $\varrho_0 \in \Phi$, and $\varpi_0 \in \Xi$. For all $\omega \in \mathbb{N}$, let $\Theta(\varrho_\omega) = \varrho_{\omega+1}$ and $\Theta(\varpi_\omega) = \varpi_{\omega+1}$. Then $(\{\varrho_\omega\}, \{\varpi_\omega\})$ is a bisequence on (Φ, Ξ, π, ν). Say $\Omega := \pi(\varrho_0, \varpi_0, \mathfrak{r}) + \pi(\varrho_0, \varpi_1, \mathfrak{r})$ and $\Lambda_\omega := \frac{\nu \Xi^\omega \Omega}{1-\nu\Xi}$. Then, for all $\omega, \kappa \in \mathbb{N}$,

$$\begin{aligned}\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) &= \pi(\Theta(\varrho_{\omega-1}), \Theta(\varpi_{\omega-1}), \mathfrak{r}) \\ &\le \phi \pi(\varrho_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}) \\ &\vdots \\ &\le \phi^\omega \pi(\varrho_0, \varpi_0, \mathfrak{r}), \end{aligned} \tag{3.1}$$

and also,

$$\begin{aligned}\pi(\varrho_\omega, \varpi_{\omega+1}, \mathfrak{r}) &= \pi(\Theta(\varrho_{\omega-1}), \Theta(\varpi_\omega), \mathfrak{r}) \\ &\le \phi \pi(\varrho_{\omega-1}, \varpi_\omega, \mathfrak{r}) \\ &\vdots \\ &\le \phi^\omega \pi(\varrho_0, \varpi_1, \mathfrak{r}). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned}\pi(\varrho_{\omega+\kappa}, \varpi_\omega, \mathfrak{r}) &\le \nu(\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+1}, \mathfrak{r}) + \pi(\varrho_\omega, \varpi_{\omega+1}, \mathfrak{r}) + \pi(\varrho_\omega, \varpi_\omega, \mathfrak{r})) \\ &\le \nu\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+1}, \mathfrak{r}) + \nu\phi^\omega\Omega \\ &\le \nu^2\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+2}, \mathfrak{r}) + \nu^2\pi(\varrho_{\omega+1}, \varpi_{\omega+2}, \mathfrak{r}) \\ &\quad + \nu^2\pi(\varrho_{\omega+1}, \varpi_{\omega+1}, \mathfrak{r}) + \nu\phi^\omega\Omega \\ &\le \nu^2\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+2}, \mathfrak{r}) + (\nu^2\phi^{\omega+1} + \nu\phi^\omega)\Omega \\ &\vdots \\ &\le \nu^{\kappa+1}\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+\kappa}, \mathfrak{r}) + (\nu^\kappa\phi^{\omega+\kappa-1} + \cdots + \nu^2\phi^{\omega+1} + \nu\phi^\omega)\Omega \\ &\le (\nu^{\kappa+1}\phi^{\omega+\kappa} + \cdots + \nu^2\phi^{\omega+1} + \nu\phi^\omega)\Omega \\ &\le \frac{\nu\phi^\omega\Omega}{1-\nu\phi} = \Lambda_\omega,\end{aligned}$$

and similarly $\pi(\varrho_\omega, \varpi_{\omega+\kappa}, \mathfrak{r}) \le \Lambda_\omega$. Let $\ell > 0$. Since $\phi \in (0,1)$, we can find an $\omega_0 \in \mathbb{N}$ satisfying $\Lambda_{\omega_0} = \frac{\nu\phi_0^\omega}{1-\nu\phi} < \frac{\ell}{3}$. Then

$$\begin{aligned}\pi(\varrho_\omega, \varpi_\rho, \mathfrak{r}) &\le \nu(\pi(\varrho_\omega, \varpi_{\omega_0}, \mathfrak{r}) + \pi(\varrho_{\omega_0}, \varpi_{\omega_0}, \mathfrak{r}) + \pi(\varrho_{\omega_0}, \varpi_\rho, \mathfrak{r})) \\ &\le 3\Lambda_{\omega_0} < \ell,\end{aligned}$$

and $(\{\varrho_\omega\}, \{\varpi_\omega\})$ is a C-biseq. Since $(\{\varrho_\omega\}, \{\varpi_\omega\})$ converges, (Φ, Ξ, π, ν) is complete, and it biconverges to $\aleph \in \Phi \cap \Xi$ and

$$\{\Theta(\varpi_\omega)\} = \{\varpi_{\omega+1}\} \to \aleph \in \Phi \cap \Xi,$$

guarantees that $\{\Theta(\varpi_\omega)\}$ has a unique limit. Since Θ is continuous $\Theta(\varpi_\omega) \to \Theta(\aleph)$, so $\Theta(\aleph) = \aleph$. Hence $\aleph$ is a FP of Θ. If $\hbar$ is any FP of Θ, then $\Theta(\hbar) = \hbar$ suggests that $\hbar \in \Phi \cap \Xi$ and we have

$$\pi(\aleph, \hbar, \mathfrak{r}) = \pi(\Theta(\aleph), \Theta(\hbar), \mathfrak{r}) \le \pi(\aleph, \hbar, \mathfrak{r}),$$

where $0 < \phi < 1$, that implies $\pi(\aleph, \hbar, \mathfrak{r}) = 0$, and so $\aleph = \hbar$. □

Example 3.2. Let $\Phi = [-2, 0]$ and $\Xi = [0, 2]$ be equipped with $\pi(\varrho, \varpi, \mathfrak{r}) = \mathfrak{r}|\varrho - \varpi|^2$ for all $\varrho \in \Phi$, $\varpi \in \Xi$ and $\mathfrak{r} > 0$. Furthermore, (Φ, Ξ, π, ν) is a complete BPPνMS with $\nu = 3$. Define $\Theta : \Phi \cup \Xi \rightrightarrows \Phi \cup \Xi$ given by

$$\Theta(\varrho) = \begin{cases} \frac{\varrho}{7}, & \text{if } \varrho \in [-2, 0], \\ 0, & \text{if } \varrho \in (0, 2], \end{cases}$$

$\forall\ \varrho \in \Phi \cup \Xi$. Let $\varrho \in \Phi$ and $\varpi \in \Xi$, then

$$\begin{aligned}\pi(\Theta\varrho, \Theta\varpi, \mathfrak{r}) &= \mathfrak{r}\left|\frac{\varrho}{7} - 0\right| \\ &\le \frac{\mathfrak{r}}{2}|\varrho - \varpi|.\end{aligned}$$

Hence, the axioms of Theorem 3.1 are fulfilled, and Θ owns a UFP $\varrho = 0$.

Example 3.3. Suppose that $\Xi = \{\mathcal{L}_\omega(\mathbb{R}) : \mathcal{L}_\omega(\mathbb{R})$ is an lower triangular matrices over $\mathbb{R}\}$, $\Phi = \{\mathcal{U}_\omega(\mathbb{R}) : \mathcal{U}_\omega(\mathbb{R})$ is an upper triangular matrices over $\mathbb{R}\}$, and the operator $\pi : \Phi \times \Xi \to \mathbb{R}^+$ defined by

$$\pi(\Gamma, \Delta, \mathfrak{r}) = \mathfrak{r}\sum_{\mathrm{i,j}=1}^{\omega} |\varpi_{\mathrm{ij}} - \varrho_{\mathrm{ij}}|^2,$$

for all $\mathfrak{r} > 0$, $\Gamma = (\varpi_{\mathrm{ij}})_{\omega\times\omega} \in \Phi$ and $\Delta = (\varrho_{\mathrm{ij}})_{\omega\times\omega} \in \Xi$. Furthermore (Φ, Ξ, π, ν) is a complete BPPνMS. Define $\Theta : \Phi \cup \Xi \rightrightarrows \Phi \cup \Xi$ given by

$$\Theta(\Gamma) = \left(\frac{\varpi_{\mathrm{ij}}}{7}\right)_{\omega\times\omega}$$

for all $\Gamma = (\varpi_{\mathrm{ij}})_{\omega\times\omega} \in \mathcal{U}_\omega(\mathbb{R}) \cup \mathcal{L}_\omega(\mathbb{R})$. Now,

$$\begin{aligned}\pi(\Theta(\Gamma), \Theta(\Delta), \mathfrak{r}) &= \frac{\mathfrak{r}}{49}\sum_{\mathrm{i,j}=1}^{\omega} |\varpi_{\mathrm{ij}} - \varrho_{\mathrm{ij}}|^2 \\ &\le \frac{\mathfrak{r}}{6}\sum_{\mathrm{i,j}=1}^{\omega} |\varpi_{\mathrm{ij}} - \varrho_{\mathrm{ij}}|^2 \\ &= \frac{\mathfrak{r}}{6}\sum_{\mathrm{i,j}=1}^{\omega} |\varpi_{\mathrm{ij}} - \varrho_{\mathrm{ij}}|^2 \\ &= \phi\pi(\Delta, \Gamma, \mathfrak{r}),\end{aligned}$$

for all $\Gamma = (\varpi_{\mathrm{ij}})_{\omega\times\omega} \in \Phi$ and $\Delta = (\varrho_{\mathrm{ij}})_{\omega\times\omega} \in \Xi$. Thus, the conditions of Theorem 3.1 are fulfilled with $\phi = \frac{1}{6}$, and Θ owns a UFP $(0_{\omega\times\omega}, 0_{\omega\times\omega}) \in \mathcal{U}_\omega(\mathbb{R}) \cup \mathcal{L}_\omega(\mathbb{R})$ where $0_{\omega\times\omega}$ is the null matrix.

Theorem 3.4. *Let (Φ, Ξ, π, ν) be a complete BPPνMS and a contravariant contraction $\Theta : (\Phi, \Xi, \pi, \nu) \leftrightarrows (\Phi, \Xi, \pi, \nu)$. Then the map $\Theta : \Phi \cup \Xi \to \Phi \cup \Xi$ has a UFP.*

Proof. Let $\varrho_0 \in \Phi$. For each $\omega \in \mathbb{N}$, define $\Theta(\varrho_\omega) = \varpi_\omega$ and $\Theta(\varpi_\omega) = \varrho_{\omega+1}$. Then $(\{\varrho_\omega\}, \{\varpi_\omega\})$ is a bisequence on (Φ, Ξ, π). Say

$$\Lambda_\omega = \frac{\nu\phi^{2\omega}}{1-\nu\phi}\pi(\varrho_0, \varpi_0, \mathfrak{r}).$$

Then for all $\omega, \kappa \in \mathbb{Z}^+$,

$$\begin{aligned}\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) &= \pi(\Theta(\varpi_{\omega-1}), \Theta(\varrho_\omega), \mathfrak{r}) \\ &\le \pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r}) \\ &= \phi\pi(\Theta(\varpi_{\omega-1}), \Theta(\varrho_{\omega-1}), \mathfrak{r}) \\ &\le \phi^2\pi(\varrho_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}) \\ &\vdots \\ &\le \phi^{2\omega}\pi(\varrho_0, \varpi_0, \mathfrak{r}) \\ &= (1-\phi)K_\omega \\ &\le \Lambda_\omega.\end{aligned} \tag{3.3}$$

$$\begin{aligned}\pi(\varrho_{\omega+1}, \varpi_\omega, \mathfrak{r}) &= \pi(\Theta(\varpi_\omega), \Theta(\varrho_\omega), \mathfrak{r}) \\ &\le \phi\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) \\ &\le \phi^{2\omega+1}\pi(\varrho_0, \varpi_0, \mathfrak{r}).\end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we get

$$\begin{aligned}\pi(\varrho_{\omega+\kappa}, \varpi_\omega, \mathfrak{r}) &\le \nu(\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+1}, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_{\omega+1}, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_\omega, \mathfrak{r})) \\ &\le \nu\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+1}, \mathfrak{r}) + (\nu\phi^{2\omega+2} + \nu\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}) \\ &\le \nu^2\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+2}, \mathfrak{r}) \\ &\quad + \nu^2\pi(\varrho_{\omega+2}, \varpi_{\omega+2}, \mathfrak{r}) + \nu^2\pi(\varrho_{\omega+2}, \varpi_{\omega+1}, \mathfrak{r}) \\ &\quad + (\nu\phi^{2\omega+2} + \nu\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}) \\ &\le \nu^2\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+2}, \mathfrak{r})\end{aligned}$$

$$
\begin{aligned}
&+ (v^2\phi^{2\omega+4} + v^2\phi^{2\omega+3} + v\phi^{2\omega+2} + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&\vdots\\
&\leq v^{\kappa-1}\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+\kappa-1}, \mathfrak{r})\\
&\quad + (v^{\kappa-1}\phi^{2\omega+2\kappa-2} + \cdots . + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&\leq (v^{\kappa-1}\phi^{2\omega+2\omega-1} + v^{\kappa-1}\phi^{2\omega+2\kappa-2} + v^{\kappa-1}\phi^{2\omega+2\kappa-3} + \cdots .\\
&\quad + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&\leq \frac{v\phi^{2\omega+1}}{1 - v\Xi}\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&= \phi\Lambda_\omega\\
&< \Lambda_\omega.
\end{aligned}
$$

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_{\omega+\kappa}, \mathfrak{r}) &\leq v(\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_\omega, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_{\omega+\kappa}, \mathfrak{r}))\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}) + v\pi(\varrho_{\omega+1}, \varpi_{\omega+\kappa}, \mathfrak{r})\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}) + v^2\pi(\varrho_{\omega+1}, \varpi_{\omega+1}, \mathfrak{r})\\
&\quad + v^2\pi(\varrho_{\omega+2}, \varpi_{\omega+1}, \mathfrak{r})\\
&\quad + v^2\pi(\varrho_{\omega+2}, \varpi_{\omega+\kappa}, \mathfrak{r})\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1} + v^2\phi^{2\omega+2} + v^2\phi^{2\omega+3})\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&\quad + v^2\pi(\varrho_{\omega+2}, \varpi_{\omega+\kappa}, \mathfrak{r})\\
&\vdots\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1} + \cdots + v^\kappa\phi^{2\omega+2\kappa-1})\pi(\varrho_0\varpi_0, \mathfrak{r})\\
&\quad + v^\kappa\pi(\varrho_{\omega+\kappa}, \varpi_{\omega+\kappa}, \mathfrak{r})\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1} + \cdots + v^\kappa\phi^{2\omega+2\kappa-1}\\
&\quad + v^\kappa\phi^{2\omega+2\kappa})\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&\leq \frac{v\phi^{2\omega}}{1 - v\phi}\pi(\varrho_0, \varpi_0, \mathfrak{r})\\
&= \Lambda_\omega.
\end{aligned}
$$

Now, since $0 < \phi < 1$, for any $\ell > 0$, we can find an integer ω_0 satisfying

$$\Lambda_{\omega_0} = \frac{v\phi^{2n_0+1}}{1 - v\phi}\pi(\varrho_0, \varpi_0, \mathfrak{r}) < \frac{\ell}{3}.$$

Hence

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_\rho, \mathfrak{r}) &\leq \pi(\varrho_\omega, \varpi_{\omega_0}, \mathfrak{r}) + \pi(\varrho_{\omega_0}, \varpi_{\omega_0}, \mathfrak{r}) + \pi(\varrho_{\omega_0}, \varpi_\rho, \mathfrak{r})\\
&\leq 3K_{\omega_0} < \ell,
\end{aligned}
$$

and $(\{\varrho_\omega\}, \{\varpi_\omega\})$ is a C-biseq. Since (Φ, Ξ, π, v) is complete BPPvMS, $(\{\varrho_\omega\}, \{\varpi_\omega\})$ converges and has convergent C-biseq and it biconverges. Let $\{\varrho_\omega\} \to \aleph, \{\varpi_\omega\} \to \aleph$, where $\aleph \in \Phi \cap \Xi$. Since the contravariant map Θ is continuous

$$\{\varrho_\mathfrak{a}\} \to \aleph,$$

which results in

$$\{\varpi_\omega\} = \{\Theta(\varrho_\omega)\} \to \Theta(\aleph),$$

and combining this with $\{\varpi_\omega\} \to \aleph$ gives $\Theta(\aleph) = \aleph$. Let $\hbar$ be a FP of Θ, then $\Theta(\hbar) = \hbar$ implies $\hbar \in \Phi \cap \Xi$ so that

$$
\begin{aligned}
\pi(\aleph, \hbar, \mathfrak{r}) &= \pi(\Theta(\aleph), \Theta(\hbar), \mathfrak{r})\\
&\leq \phi\pi(\aleph, \hbar, \mathfrak{r}),
\end{aligned}
$$

which gives $\pi(\aleph, \hbar, \mathfrak{r}) = 0$. Hence $\aleph = \hbar$. □

Example 3.5. Let $\Phi = \{0, 1, 2, 7\}$ and $\Xi = \{0, \frac{1}{4}, \frac{1}{2}, \frac{7}{4}, 3\}$ be equipped with $\pi(\varrho, \varpi, \mathfrak{r}) = \mathfrak{r}|\varrho - \varpi|^2$ for all $\varrho \in \Phi$, $\varpi \in \Xi$ and $\mathfrak{r} > 0$. Furthermore, (Φ, Ξ, π, v) is a complete BPPvMS with $v = 3$. Define $\Theta : \Phi \cup \Xi \rightleftarrows \Phi \cup \Xi$ given by

$$\Theta(\varrho) = \begin{cases} \frac{\varrho}{3}, & \text{if } \varrho \in \{0, 2, 7\},\\ 0, & \text{if } \varrho \in \{\frac{1}{2}, \frac{1}{4}, \frac{7}{4}, 1, 3\}, \end{cases}$$

$\forall\ \varrho \in \Phi \cup \Xi$. Let $\varrho \in \Phi$ and $\varpi \in \Xi$, then we can easily get

$$\pi(\Theta\varpi, \Theta\varrho, \mathfrak{r}) \leq \frac{1}{2}\pi(\varrho, \varpi, \mathfrak{r}).$$

Therefore, conditions of Theorem 3.4 are satisfied and Θ has a UFP $\varrho = 0$.

In conclusion, we construct a theorem derived from Kannan's FP result (Kannan and R, 1968).

Theorem 3.6. *Let $\Theta : (\Phi, \Xi, \pi, v) \leftrightarrows (\Phi, \Xi, \pi, v)$, where (Φ, Ξ, π, v) is a complete BPPvMS and let $\alpha \in (0, \frac{1}{2})$ satisfies*

$$\pi(\Theta\varpi, \Theta\varrho, \mathfrak{r}) \leq \alpha(\pi(\varrho, \Theta\varrho, \mathfrak{r}) + \pi(\Theta\varpi, \varpi, \mathfrak{r})) \tag{3.5}$$

holds for all $\mathfrak{r} > 0$, $\varrho \in \Phi$ and $\varpi \in \Xi$. Then the map $\Theta : \Phi\cup\Xi \to \Phi \cup \Xi$ has a UFP.

Proof. Assume that $\varrho_0 \in \Phi$, for all positive integer ω, We clarify $\varpi_\omega = \Theta\varrho_\omega$ and $\varrho_{\omega+1} = \Theta\varpi_\omega$. By (3.5), we have

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) &= \pi(\Theta\varpi_{\omega-1}, \Theta\varrho_\omega, \mathfrak{r})\\
&\leq \alpha(\pi(\varrho_\omega, \Theta\varrho_\omega, \mathfrak{r}) + \pi(\Theta\varpi_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}))\\
&= \alpha(\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) + \pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r})),
\end{aligned}
$$

for all integers $\omega \geq 1$. Then,

$$\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) \leq \frac{\alpha}{1 - \alpha}\pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r}).$$

From (3.5), we have

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r}) &= \pi(\Theta\varpi_{\omega-1}, \Theta\varrho_{\omega-1}, \mathfrak{r})\\
&\leq \alpha(\pi(\varrho_{\omega-1}, \Theta\varrho_{\omega-1}, \mathfrak{r}) + \pi(\Theta\varpi_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}))\\
&= \alpha(\pi(\varrho_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}) + \pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r})),
\end{aligned}
$$

so that

$$\pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r}) \leq \frac{\alpha}{1 - \alpha}\pi(\varrho_{\omega-1}, \varpi_{\omega-1}, \mathfrak{r}).$$

If we say $\phi := \frac{\alpha}{1-\alpha}$, then we have $\phi \in (0, 1)$ since $\alpha \in (0, \frac{1}{2})$. Now

$$\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) \leq \phi^{2\omega}\pi(\varrho_0, \varpi_0, \mathfrak{r}), \tag{3.6}$$

$$\pi(\varrho_\omega, \varpi_{\omega-1}, \mathfrak{r}) \leq \phi^{2\omega-1}\pi(\varrho_0, \varpi_0, \mathfrak{r}). \tag{3.7}$$

For all $\rho, \omega \in \mathbb{N}$, using (3.6) and (3.7), we get

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_\rho, \mathfrak{r}) &\leq v(\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_\omega, \mathfrak{r}) + \pi(\varrho_{\omega+1}, \varpi_\rho, \mathfrak{r}))\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}) + v\pi(\varrho_{\omega+1}, \varpi_\rho, \mathfrak{r})\\
&\vdots\\
&\leq (v\phi^{2\omega} + v\phi^{2\omega+1} + \cdots + v^\mathfrak{b}\phi^{2\rho})\pi(\varrho_0, \varpi_0, \mathfrak{r}),
\end{aligned}
$$

if $\rho > \omega$, and

$$
\begin{aligned}
\pi(\varrho_\omega, \varpi_\rho, \mathfrak{r}) &\leq v(\pi(\varrho_{\rho+1}, \varpi_\rho, \mathfrak{r}) + \pi(\varrho_{\rho+1}, \varpi_{\rho+1}, \mathfrak{r}) + \pi(\varrho_\omega, \varpi_{\rho+1}, \mathfrak{r}))\\
&\leq (v\phi^{2\rho+1} + v\phi^{2\rho+2})\pi(\varrho_0, \varpi_0, \mathfrak{r}) + v\pi(\varrho_\omega, \varpi_{\rho+1}, \mathfrak{r})\\
&\vdots\\
&\leq (v\phi^{2\rho+1} + v\phi^{2\rho+2} + \cdots + v^\mathfrak{a}\phi^{2\omega})\pi(\varrho_0, \varpi_0, \mathfrak{r}) + v^\mathfrak{a}\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r})\\
&< (v\phi^{2\rho+1} + v\phi^{2\rho+2} + \cdots + v^\mathfrak{a}\phi^{2\omega+1})\pi(\varrho_0, \varpi_0, \mathfrak{r}),
\end{aligned}
$$

if $\rho < \omega$. Since $\phi \in (0, 1)$. Therefore $(\{\varrho_\omega\}, \{\varpi_\rho\})$ is a C-biseq. Since (Φ, Ξ, π, v) is complete. Then $\{\varrho_\omega\} \to \aleph, \{\varpi_\rho\} \to \aleph$ and $\aleph \in \Phi\cup\Xi$. Since

$$\{\Theta\varrho_\omega\} = \{\varpi_\omega\} \to \aleph,\ \ \pi(\Theta\aleph, \Theta\varrho_\omega, \mathfrak{r}) \to \pi(\Theta\aleph, \aleph, \mathfrak{r}).$$

On the other hand,

$$\pi(\Theta\aleph, \Theta\varrho_\omega, \mathfrak{r}) \leq \alpha(\pi(\varrho_\omega, \Theta\varrho_\omega, \mathfrak{r}) + \pi(\Theta\aleph, \aleph, \mathfrak{r})) = \alpha(\pi(\varrho_\omega, \varpi_\omega, \mathfrak{r}) + \pi(\Theta\aleph, \aleph, \mathfrak{r})),$$

which in turn implies that $\pi(\Theta\aleph, \aleph, \mathfrak{r}) \leq \alpha\pi(\Theta\aleph, \aleph, \mathfrak{r})$. Hence $\Theta\aleph = \aleph$. If $\hbar$ is any FP of Θ, then $\Theta\hbar = \hbar$ implies that $\hbar$ is in $\Phi \cap \Xi$. Then

$$
\begin{aligned}
\pi(\aleph, \hbar, \mathfrak{r}) = \pi(\Theta\aleph, \Theta\hbar, \mathfrak{r}) &\leq \alpha(\pi(\aleph, \Theta\aleph, \mathfrak{r}) + \pi(\Theta\hbar, \hbar, \mathfrak{r}))\\
&= \alpha(\pi(\aleph, \aleph, \mathfrak{r}) + \pi(v, \hbar, \mathfrak{r})) = 0.
\end{aligned}
$$

Consequently $\aleph = \hbar$. □

Example 3.7. Let $\Phi = [-10, 0]$ and $\Xi = [0, 10]$ be equipped with $\pi(\varrho, \varpi, \mathfrak{t}) = \mathfrak{t}|\varrho - \varpi|^2$ for all $\varrho \in \Phi$, $\varpi \in \Xi$ and $\mathfrak{t} > 0$. Furthermore, (Φ, Ξ, π, ν) is a complete BPPνMS with $\nu = 3$. Define $\Theta : \Phi \cup \Xi \rightleftarrows \Phi \cup \Xi$ given by

$$\Theta(\varrho) = \frac{-7\varrho}{82}, \forall \varrho \in \Phi \cup \Xi.$$

Let $\varrho \in \Phi$ and $\varpi \in \Xi$, then

$$\begin{aligned}
\pi(\Theta\varpi, \Theta\varrho, \mathfrak{t}) &= \mathfrak{t}\left(\frac{7}{82}\right)^2 \left|\varpi - \varrho\right|^2 \\
&\leq \mathfrak{t}2\left(\frac{7}{82}\right)^2 \left(\left|\varpi\right|^2 + \left|\varrho\right|^2\right) \\
&= \mathfrak{t}\frac{2 \times 49}{(89)^2}\left(\frac{89}{82}\right)^2 \left(\left|\varpi\right|^2 + \left|\varrho\right|^2\right) \\
&= \frac{98}{7921}\left(\pi(\varrho, \Theta\varrho, \mathfrak{t}) + \pi(\varpi, \Theta\varpi, \mathfrak{t})\right) \\
&= \alpha\pi(\varpi, \varrho, \mathfrak{t}).
\end{aligned}$$

Hence, the axioms of Theorem 3.6 are fulfilled with $\alpha = \frac{98}{7921}$ and Θ owns a UFP $\varrho = 0$.

Finally, we prove a theorem motivated by the Reich type FP theorem (Reich, 1971).

Theorem 3.8. *Let (Φ, Ξ, π, ν) be a complete BPPνMS. Define the map as $\Theta : (\Phi, \Xi, \pi, \nu) \leftrightarrows (\Phi, \Xi, \pi, \nu)$ s.t.*

$$\pi(\Theta\varpi, \Theta\varrho, \mathfrak{t}) \leq \alpha\pi(\varpi, \varrho, \mathfrak{t}) + \kappa\pi(\varpi, \Theta\varpi, \mathfrak{t}) + \nu\pi(\Theta\varrho, \varrho, \mathfrak{t}), \tag{3.8}$$

for all $\varpi \in \Xi$ and $\varrho \in \Phi$, where $\alpha, \kappa, \nu \geq 0$ s.t. $\alpha + \kappa + \nu < 1$. Then the map $\Theta : \Phi \cup \Xi \to \Phi \cup \Xi$ has a UFP.

Proof. Let $\varpi_0 \in \Xi$. Define $\varrho_\omega = \Theta\varpi_\omega$ and $\varpi_{\omega+1} = \Theta\varrho_\omega$ for all $\omega \in \mathbb{N}$. By (3.8), we have

$$\begin{aligned}
\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) &= \pi(\Theta\varrho_{\omega-1}, \Theta\varpi_\omega, \mathfrak{t}) \\
&\leq \alpha\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) + \kappa\pi(\varpi_\omega, \Theta\varpi_\omega, \mathfrak{t}) + \nu\pi(\Theta\varrho_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}) \\
&= (\alpha + \nu)\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) + \kappa\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}),
\end{aligned}$$

for all integers $\omega \geq 1$. Then,

$$\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) \leq \left(\frac{\alpha + \nu}{1 - \kappa}\right)\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}).$$

From (3.8), we have

$$\begin{aligned}
\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) &= \pi(\Theta\varrho_{\omega-1}, \Theta\varpi_{\omega-1}, \mathfrak{t}) \\
&\leq \alpha\pi(\varpi_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}) + \kappa\pi(\varpi_{\omega-1}, \Theta\varpi_{\omega-1}, \mathfrak{t}) \\
&\quad + \nu\pi(\Theta\varrho_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}) \\
&= (\alpha + \kappa)\pi(\varpi_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}) + \nu\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}),
\end{aligned}$$

so

$$\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) \leq \left(\frac{\alpha + \kappa}{1 - \nu}\right)\pi(\varpi_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}).$$

If we say $\rho := \frac{\alpha+\kappa}{1-\nu}$ and $g := \frac{\alpha+\nu}{1-\kappa}$, then we have $\rho, g \in (0, 1)$. Now

$$\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) \leq g^{2\omega}\pi(\varpi_0, \varrho_0, \mathfrak{t}), \tag{3.9}$$

$$\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) \leq \rho^{2\omega-1}\pi(\varpi_0, \varrho_0, \mathfrak{t}). \tag{3.10}$$

For all natural numbers $\omega < \rho$, using (3.9) and (3.10), we have

$$\begin{aligned}
\pi(\varpi_\omega, \varrho_\rho, \mathfrak{t}) &\leq \nu(\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) + \pi(\varpi_{\omega+1}, \varrho_\omega, \mathfrak{t}) + \pi(\varpi_{\omega+1}, \varrho_\rho, \mathfrak{t})) \\
&\leq \nu\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) + \nu\pi(\varpi_{\omega+1}, \varrho_\omega) + \nu^2\pi(\varpi_{\omega+1}, \varrho_{\omega+1}, \mathfrak{t}) \\
&\quad + \nu^2\pi(\varpi_{\omega+2}, \varrho_{\omega+1}, \mathfrak{t}) \\
&\quad + \cdots + \nu^{\mathfrak{b}}\pi(\varpi_{\rho-1}, \varrho_{\rho-1}, \mathfrak{t}) + \nu^{\mathfrak{b}}\pi(\varpi_\rho, \varrho_{\rho-1}, \mathfrak{t}) + \nu^{\mathfrak{b}}\pi(\varpi_\rho, \varrho_\rho, \mathfrak{t}) \\
&\leq \nu g^{2\omega}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu\rho^{2\omega+1}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu^2 g^{2\omega+2}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \nu^2\rho^{2\omega+3}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \cdots + \nu^{\mathfrak{b}} g^{2\rho-2}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu^{\mathfrak{b}}\rho^{2\rho-1}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \nu^{\mathfrak{b}} g^{2\rho}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&= (\nu g^{2\omega} + \nu^2 g^{2\omega+2} + \cdots + \nu^{\mathfrak{b}} g^{2\rho})\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + (\nu\rho^{2\omega+1} + \nu^2\rho^{2\omega+3} \\
&\quad + \cdots + \nu^{\mathfrak{b}}\rho^{2\rho-1})\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\leq \nu g^{2\omega}\left(\frac{1}{1 - \nu g^2}\right)\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu\rho^{2\omega+1}\left(\frac{1}{1 - \nu\rho^2}\right)\pi(\varpi_0, \varrho_0, \mathfrak{t}).
\end{aligned}$$

For all natural numbers $\rho < \omega$, we have

$$\begin{aligned}
\pi(\varpi_\omega, \varrho_\rho, \mathfrak{t}) &\leq \nu(\pi(\varpi_{\rho+1}, \varrho_\rho, \mathfrak{t}) + \pi(\varpi_{\rho+1}, \varrho_{\rho+1}, \mathfrak{t}) + \pi(\varpi_\omega, \varrho_{\rho+1}, \mathfrak{t})) \\
&\leq \nu\pi(\varpi_{\rho+1}, \varrho_\rho, \mathfrak{t}) + \nu\pi(\varpi_{\rho+1}, \varrho_{\rho+1}, \mathfrak{t}) + \nu^2\pi(\varpi_{\rho+2}, \varrho_{\rho+1}, \mathfrak{t}) \\
&\quad + \nu^2\pi(\varpi_{\rho+2}, \varrho_{\rho+2}, \mathfrak{t}) \\
&\quad + \cdots + \nu^{\mathfrak{a}}\pi(\varpi_{\omega-1}, \varrho_{\omega-1}, \mathfrak{t}) + \nu^{\mathfrak{a}}\pi(\varpi_\omega, \varrho_{\omega-1}, \mathfrak{t}) + \nu^{\mathfrak{a}}\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) \\
&\leq \nu\rho^{2\rho+1}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu g^{2\rho+2}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu^2\rho^{2\rho+3}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \nu^2 g^{2\omega+4}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \cdots + \nu^{\mathfrak{a}} g^{2\omega-2}\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu^{\mathfrak{a}}\rho^{2\omega-1}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\quad + \nu^{\mathfrak{a}} g^{2\omega}\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&= (\nu g^{2\rho+2} + \nu^2 g^{2\omega+4} + \cdots + \nu^{\mathfrak{a}} g^{2\omega})\pi(\varpi_0, \varrho_0, \mathfrak{t}) + (\nu\rho^{2\rho+1} + \nu^2\rho^{2\rho+3} \\
&\quad + \cdots + \nu^{\mathfrak{a}}\rho^{2\omega-1})\pi(\varpi_0, \varrho_0, \mathfrak{t}) \\
&\leq \nu g^{2\rho+2}\left(\frac{1}{1 - \nu g^2}\right)\pi(\varpi_0, \varrho_0, \mathfrak{t}) + \nu\rho^{2\rho+1}\left(\frac{1}{1 - \nu\rho^2}\right)\pi(\varpi_0, \varrho_0, \mathfrak{t}).
\end{aligned}$$

Therefore, $(\{\varpi_\omega\}, \{\varrho_\rho\})$ is a C-biseq. Since (Φ, Ξ, π, ν) is complete BPPνMS, $\{\varpi_\omega\} \to \mathfrak{e}, \{\varrho_\rho\} \to \mathfrak{e}$ where $\mathfrak{e} \in \Phi \cup \Xi$. Since

$$\{\Theta\varpi_\omega\} = \{\varrho_\omega\} \to \mathfrak{e}, \ \pi(\Theta\mathfrak{e}, \Theta\varpi_\omega, \mathfrak{t}) \to \pi(\Theta\mathfrak{e}, \mathfrak{e}, \mathfrak{t}).$$

On the other hand,

$$\begin{aligned}
\pi(\Theta\mathfrak{e}, \Theta\varpi_\omega, \mathfrak{t}) &\leq \alpha\pi(\varpi_\omega, \mathfrak{e}, \mathfrak{t}) + \kappa\pi(\varpi_\omega, \Theta\varpi_\omega, \mathfrak{t}) + \nu\pi(\Theta\mathfrak{e}, \mathfrak{e}, \mathfrak{t}) \\
&= \alpha\pi(\varpi_\omega, \mathfrak{e}, \mathfrak{t}) + \kappa\pi(\varpi_\omega, \varrho_\omega, \mathfrak{t}) + \nu\pi(\Theta\mathfrak{e}, \mathfrak{e}, \mathfrak{t}).
\end{aligned}$$

Therefore, $\pi(\Theta\mathfrak{e}, \mathfrak{e}, \mathfrak{t}) \leq \nu\pi(\Theta\mathfrak{e}, \mathfrak{e}, \mathfrak{t})$. Hence $\Theta\mathfrak{e} = \mathfrak{e}$. If $\mathfrak{r}$ is any FP of Θ, then $\Theta\mathfrak{r} = \mathfrak{r}$, shows that $\mathfrak{r} \in \Phi \cap \Xi$. Then

$$\begin{aligned}
\pi(\mathfrak{e}, \mathfrak{r}, \mathfrak{t}) = \pi(\Theta\mathfrak{e}, \Theta\mathfrak{r}, \mathfrak{t}) &\leq \alpha\pi(\mathfrak{r}, \mathfrak{e}, \mathfrak{t}) + \kappa\pi(\mathfrak{e}, \Theta\mathfrak{e}, \mathfrak{t}) + \nu\pi(\Theta\mathfrak{r}, \mathfrak{r}, \mathfrak{t}) \\
&= \alpha\pi(\mathfrak{r}, \mathfrak{e}, \mathfrak{t}) \\
&< \pi(\mathfrak{e}, \mathfrak{r}, \mathfrak{t}).
\end{aligned}$$

Consequently $\mathfrak{e} = \mathfrak{r}$. □

Example 3.9. Consider $\Phi = [0, 1]$, and $\Xi = [1, 2]$ be gifted with $\pi(\varrho, \varpi, \mathfrak{t}) = \mathfrak{t}|\varrho - \varpi|^2$ for all $\varrho \in \Phi$, $\varpi \in \Xi$ and $\mathfrak{t} > 0$. Furthermore, (Φ, Ξ, π, ν) is a complete BPPνMS with $\nu = 3$. Define $\Theta : \Phi \cup \Xi \rightleftarrows \Phi \cup \Xi$ given by

$$\Theta(\varrho) = \frac{(\sqrt{3} + 1) - \varrho}{\sqrt{3}}, \forall \varrho \in \Phi \cup \Xi.$$

Let $\varrho \in \Phi$ and $\varpi \in \Xi$, then

$$\pi(\Theta\varpi, \Theta\varrho, \mathfrak{t}) = \frac{\mathfrak{t}}{3}|\varrho - \varpi|^2 = \frac{\mathfrak{t}}{3}|\varpi - \varrho|^2 \leq \frac{1}{2}\pi(\varpi, \varrho, \mathfrak{t}) = \alpha\pi(\varpi, \varrho, \mathfrak{t}).$$

Thus, the axioms of Theorem 3.8 are fulfilled with $\alpha = \frac{1}{2}$, $\kappa = \nu = 0$ and Θ owns a UFP unique $\varrho = 1$.

4. Fractional differential equation's application

Fractional differential equations (FDEs) can be used to model and examine physical systems that exhibit constant interactions or distributions. In engineering study, they are frequently employed to derive correlations between numbers or to offer a more thorough explanation of phenomena than differential equations (Thabet et al., 2023b). They provide a structure to grasp the intricate interactions and behaviors encountered in a variety of engineering systems. Implicit differential equations (FDEs) possess a multitude of uses in engineering research.

In this section, we demonstrate that the FDE has a unique solution. In engineering, this type of differential equation is commonly used. They are essential for research in magnetic field assessment for radars, control mechanisms, structural evaluation, digital circuit analysis, material science, heat exchange, fluid circulation simulation, data processing operations, and mechanical design fatigue. And which are useful to medical imaging, non-destructive testing, inverse and geophysics problems related to sound waves for spreading of waves, ophthalmology and diffraction studies. These formulas offer an adaptable structure for interactions, comprehending and evaluating continuous distributions in a range of engineering domains. Younis and Singh (2022) examined the necessary conditions for the existence of solutions to a specific category of fractional differential equations and Hammerstein integral equations. The presence of positive solutions and their abundance for nonlinear fractional differential equations were examined by Bai and Lü (2005). Unbounded solution of multi-order ϱ-Hilfer fractional implicit pantograph system established by Thabet et al. (2023a). For more details about fractional operators typos one can see in Abdeljawad et al. (2023), Podlubny (1999).

We study several significant definitions of fractional calculus theory. For a function $\varpi \in \mathbb{C}[0,1]$, the Riemann–Liouville derivative of fractional order $\delta > 0$ is defined as

$$\frac{1}{\Gamma(\mathfrak{u}-\delta)}\frac{d^{\mathfrak{u}}}{d\mathfrak{u}^{\mathfrak{u}}}\int_0^{\mathfrak{u}}\frac{\varpi(x)dx}{(\mathfrak{u}-x)^{\delta-\mathfrak{u}+1}} = \mathcal{D}^{\delta}\varpi(\mathfrak{u}),$$

where $\mathfrak{u} \in [0,1]$, $[\delta]$ is the integer part of δ, and Γ is the well known gamma function.

Now, let us assume the fractional differential equation as given by

$$\begin{aligned} &{}^{c}\mathcal{D}^{\varrho}\varpi(\mathfrak{u}) + g(\mathfrak{u},\varpi(\mathfrak{u})) = 0, \quad 0 \le \mathfrak{u} \le 1, \quad 1 < \varrho \le 2, \\ &\varpi(0) = \varpi(1) = 0, \end{aligned} \tag{4.1}$$

such that $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ represents a continuous function, and ${}^{c}\mathcal{D}^{\varrho}$ is the Caputo derivative of fractional order ϱ and it is given as

$${}^{c}\mathcal{D}^{\varrho} = \frac{1}{\Gamma(\mathfrak{u}-\varrho)}\int_0^{\zeta}\frac{\varpi^{\mathfrak{u}}(x)dx}{(\mathfrak{u}-x)^{\varrho-\mathfrak{u}+1}}$$

Let $\Phi = (\mathbb{C}[0,1],[0,\infty)) = \{g : [0,1] \to [0,\infty) \text{ be a continuous function}\}$.
Let $\Xi = (\mathbb{C}[0,1],(-\infty,0]) = \{g : [0,1] \to (-\infty,0] \text{ be a continuous function}\}$.
Define $\pi : \Phi\times\Xi \to \mathbb{R}^{+}$ is given by

$$\pi(\varpi,\varpi',\mathfrak{r}) = \mathfrak{r}\sup_{\mathfrak{u}\in[0,1]}|\varpi(\mathfrak{u})-\varpi'(\mathfrak{u})|^2$$

for all $(\varpi,\varpi') \in \Phi\times\Xi$ and $\mathfrak{r} > 0$. From definition conditions of (a), (b) and (c) are fulfilled. Next, we prove definition conditions of (d). Regarding this,

$$\begin{aligned}\mathfrak{r}|\varpi(\mathfrak{u})-\varpi'(\mathfrak{u})|^2 &= \mathfrak{r}|\varpi(\mathfrak{u})-\Theta(\mathfrak{u})-\varrho(\mathfrak{u})+\Theta(\mathfrak{u})+\varrho(\mathfrak{u})-\varpi'(\mathfrak{u})|^2 \\ &\le \mathfrak{r}|\varpi(\mathfrak{u})-\Theta(\mathfrak{u})|^2+\mathfrak{r}|\varrho(\mathfrak{u})-\Theta(\mathfrak{u})|^2+\mathfrak{r}|\varrho(\mathfrak{u})-\varpi'(\mathfrak{u})|^2.\end{aligned}$$

Taking the supremum on both sides, we get

$$\pi(\varpi,\varpi',\mathfrak{r}) \le \pi(\varpi,\Theta,\mathfrak{r})+\pi(\varrho,\Theta,\mathfrak{r})+\pi(\varrho,\varpi',\mathfrak{r}),$$

for all $\mathfrak{r} > 0$, $\varrho,\varpi \in \Phi$ and $\Theta,\varpi' \in \Xi$. Then (Φ,Ξ,π,ν) is a complete BPPνMS.

Theorem 4.1. *Consider the fractional boundary value problem (4.1). Assume that the subsequent conditions are satisfies as follows:*

(i) there are $\mathfrak{u} \in [0,1]$, $\phi \in (0,1)$ *and* $(\varpi,\varpi') \in \Phi\times\Xi$ *s.t.*

$$|g(\mathfrak{u},\varpi)-g(\mathfrak{u},\varpi')| \le \sqrt{\phi}|\varpi(\mathfrak{u})-\varpi'(\mathfrak{u})|;$$

(ii) $\int_0^1 |\mathcal{G}(\mathfrak{u},x)|^2dx \le 1.$

Then, Eq. (4.1) owns one solution in $\Phi\cup\Xi$.

***Proof*.** The main problem (4.1) is similar to the succeeding integral identity

$$\varpi(\mathfrak{u}) = \int_0^1 \mathcal{G}(\mathfrak{u},x)g(\mathfrak{u},\varpi(x))dx,$$

where

$$\mathcal{G}(\mathfrak{u},x) = \begin{cases}\frac{[\mathfrak{u}(1-x)]^{\varrho-1}-(\mathfrak{u}-x)^{\varrho-1}}{\Gamma(\varrho)}, & 0 \le x \le \mathfrak{u} \le 1,\\ \frac{[\mathfrak{u}(1-x)]^{\varrho-1}}{\Gamma(\varrho)}, & 0 \le \mathfrak{u} \le x \le 1.\end{cases}$$

Now, we introduce the covariant mapping $\Theta : \Phi\cup\Xi \to \Phi\cup\Xi$ as given by

$$\Theta\varpi(\mathfrak{u}) = \int_0^1 \mathcal{G}(\mathfrak{u},x)g(\mathfrak{u},\varpi(x))dx.$$

Now

$$\begin{aligned}\mathfrak{r}|\Theta\varpi(\mathfrak{u})-\Theta\varpi'(\mathfrak{u})|^2 &= \mathfrak{r}\left|\int_0^1 \mathcal{G}(\mathfrak{u},x)g(\mathfrak{u},\varpi(x))dx - \int_o^1 \mathcal{G}(\mathfrak{u},x)g(\mathfrak{u},\varpi'(x))dx\right|^2 \\ &\le \mathfrak{r}\int_0^1|\mathcal{G}(\mathfrak{u},x)|^2dx\int_0^1\left|g(\mathfrak{u},\varpi(x))-g(\mathfrak{u},\varpi'(x))\right|^2dx \\ &\le \phi\mathfrak{r}\left|\varpi(\mathfrak{u})-\varpi'(\mathfrak{u})\right|^2.\end{aligned}$$

Taking the supremum on both sides, we get

$$\pi(\Theta\varpi,\Theta\varpi',\mathfrak{r}) \le \phi\pi(\varpi,\varpi',\mathfrak{r}).$$

Thus, the axioms of a Theorem 3.1 are verified, then problem (4.1) owns one solution. □

5. Conclusion

In this article, we introduced BPPνMS and proved Banach, Kannan's, Reich type FP theorems. And we have given suitable examples for our obtained outcomes. An illustrative application to a fractional differential equation is presented. Kumar et al. (2024) introduced generalized parametric bipolar metric space and proved FP theorems. It is an interesting open problem to introduce generalized parametric bipolar ν-metric space and prove FP theorems.

CRediT authorship contribution statement

Gunaseelan Mani: Writing – original draft, Validation, Methodology, Investigation, Formal analysis. **Subramanian Chinnachamy:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Sugapriya Palanisamy:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis. **Sabri T.M. Thabet:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis. **Imed Kedim:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis. **Miguel Vivas-Cortez:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Funding acquisition, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and materials

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