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# On fractional modelling of dye removal using fractional derivative with non-singular kernel

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ABSTRACT

Several studies showed that the adsorption process for the dye removal has its unique fractional property. In line with this, we focused on the fractional model to study the transport of direct textile industry wastewater. We used the recently introduced fractional derivative without singular kernel, the Caputo-Fabrizio fractional derivative to obtain the analytic solution for the adsorption transport equation and plot the concentration that corresponds to the dye with a given initial conditions.



## 1. Introduction

There are many industries that contribute to the pollution of our ecological system. One of these industries is the textile industry in which large amount of polluted substances have been discharged directly into the water bodies and ground waters. If these phenomenon continued to occur then it may cause much damage to the quality of waters and can significantly affect the environment. In this study, we focus on the adsorption process of the direct textile industry where we follow the normally adopted transport equation given by Ardejani et al. (2007) of the form

$$R\frac{\partial C}{\partial t} + KS\rho_d = 0 \tag{1}$$

where

$C$ is the concentration of solution.
$S$ is the quantity of mass adsorbed on the solid surface.
$R$ is the retardation factor.
$K$ is the dalay constant.
$\rho_d$ is the bulk density of the medium.

Using the Langmuir isotherm which gives the relation between $C$ and $S$, we have

$$S = \frac{Q_o K_l C}{1 + K_l C} \tag{2}$$

where $Q_o$ is the maximum adsorption capacity, and $K_l$ is the corresponding Langmuir constant. Substituting Eq. (2) to Eq. (1), we have

$$R\frac{\partial C}{\partial t} + \frac{KK_l Q_o \rho_d C}{1 + K_l C} = 0 \tag{3}$$

with the condition $C(0) = C_o$. Recently, it was found that the adsorption process has a fractional property (Quiroga et al., 2013; He and Li, 2016). Hence, fractional derivative can be an effective tool.

In the review work of Yagub et al. (2014), they extensively reviewed the information about dyes, its classification and toxicity, various treatment methods, and dye adsorption characteristic by various adsorbent. One of their objectives is to organise the scattered available information on various aspects on awide range of potentially effective adsorbents in the removal of dyes. It was found that the amount of adsorption for dye removal is highly dependent on the initial dye concentration and which the effect depends on the relation between the concentration of the dye available site on the adsorbent surface (Yagub et al., 2014). Hence, this study aims to provide another mathematical method or model of the adsorption for dye removal concentration.

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## 2. Fractional derivative without singular kernel

In contrast to the study of He and Li (2016), where they used the Reimann-Liouville type fractional derivative, we use the recently introduced fractional derivative without singular kernel called Caputo-Fabrizio fractional derivative. In this section, we will review some definition and properties of the Caputo-Fabrizio fractional derivative.

Application of the Caputo-Fabrizio fractional derivative was proven to be efficient in real world fractional phenomena problem. Among its applications are modelling of fractional electrical circuits (Atangana and Alkahtani, 2015); analysis on logistic equation (Kumar et al., 2017); fractional descriptor continuous-time linear systems (Kaczorek and Borawski, 2016); fractional Maxwell fluid (Gao and Yang, 2016); heat transfer analysis (Shah and Khan, 2016); heat transfer in magnetohydrodynamic (Abro and Solangi, 2017).

**Definition 2.1.** The Caputo-Fabrizio fractional derivative on a function $f \in H(a,b)$ is defined (Caputo and Fabrizio, 2015) as

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t f'(s)\exp\left\{-\frac{\alpha}{1-\alpha}(t-s)\right\}ds \tag{4}$$

where $\alpha \in (0,1)$ and $M(\alpha)$ is the normalization function. The main difference of this new fractional derivative is that the new kernel (which is the exponential term) has no singularity if $t = s$. On the other hand, Eq. (4) is not yet in a formal form and it was Losada and Nieto (2015) who thoroughly investigated the Caputo-Fabrizio fractional derivative by solving the normalization function.

**Definition 2.2.** Let $0 < \alpha < 1$, then the Caputo-Fabrizio fractional derivative of order $\alpha$ of a function $f(t)$ is given by

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{1}{1-\alpha}\int_0^t f'(s)\exp\left\{-\frac{\alpha}{1-\alpha}(t-s)\right\}ds \tag{5}$$

In the next section, we will solve the analytic solution for the adsorption process given by the Eq. (1) using the Caputo-Fabrizio fractional derivative.

## 3. Application of Caputo-Fabrizio fractional derivative in modeling dye removal

Employing the definition of the Caputo-Fabrizio fractional derivative (5) on the Eq. (1), we have

$$ R\,{}^{CF}D_t^{\alpha} C + \frac{KK_lQ_o\rho_d C}{1+K_lC} = 0 \tag{6}$$

We can rewrite (6) as

$$ \frac{1}{1-\alpha}\int_0^t C'(s)\exp\left\{-\frac{\alpha}{1-\alpha}(t-s)\right\}ds + \frac{K_l}{1-\alpha}C(t)\int_0^t C'(s)\exp\left\{-\frac{\alpha}{1-\alpha}(t-s)\right\}ds + \beta C(t) = 0 \tag{7}$$

where $\beta = \frac{K\rho_d Q_o K_l}{R}$. Eq. (7) can be reduced to the simple form

$$ (1+K_lC(t))\int_0^t C'(s)\exp\left\{\frac{\alpha}{1-\alpha}(s)\right\}ds + \beta\exp\left\{\frac{\alpha}{1-\alpha}t\right\}C(t) = 0 \tag{8}$$

Now, applying integration by parts on the integral term, we have the quadratic solution for the adsorption process given by

$$ K_lC^2(t) + C(t)\left(\frac{1}{K_l} + \frac{\beta(1-\alpha)^2}{K_l(1-2\alpha)}\right) = C_o\exp\left\{-\frac{\alpha}{1-\alpha}t\right\}\left(1+\frac{1}{K_l}\right) \tag{9}$$

To finally obtain the solution, we can use the completing the square method, thus giving us the analytic solution in the sense of Caputo-Fabrizio fractional derivative

$$ C(t) = \sqrt{C_o\exp\left\{-\frac{\alpha}{1-\alpha}t\right\}\left(1+\frac{1}{K_l}\right) + \frac{1}{4}\left(\frac{1}{K_l} + \frac{\beta(1-\alpha)^2}{K_l(1-2\alpha)}\right)^2} - \frac{1}{2}\left(\frac{1}{K_l} + \frac{\beta(1-\alpha)^2}{K_l(1-2\alpha)}\right) \tag{10}$$

The figure shown below was the simulated results of the solution (10) with the parameters. $K_l = 1, \rho_d = 0.001, 0.8 \times 10^{-9}$ and $K = 0.1$. Observe that the concentration decays faster for small initial and takes long time to decay for large initial depending on the fractional order. In addition, for small $\alpha$, concentration decreases slower than those of having large $\alpha$ (see Figs. 1 and 2).

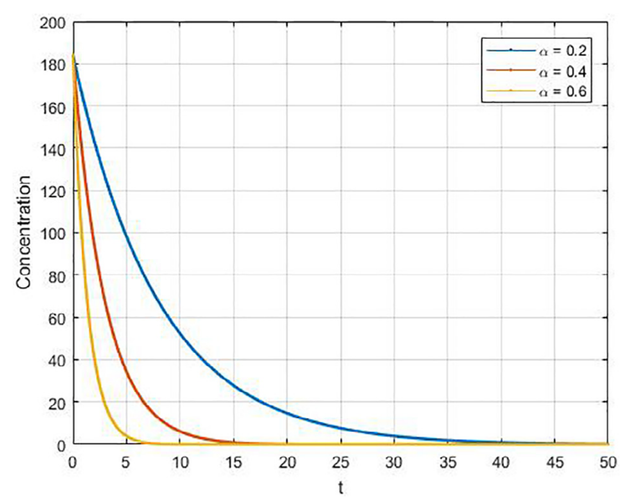


**Fig. 1.** Simulation for the solution of the adsorption process with different fractional order with $C_o = 1.7 \times 10^4$.

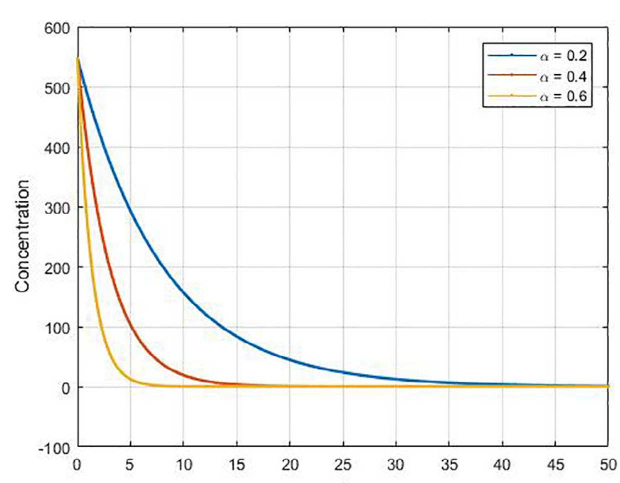


**Fig. 2.** Simulation for the solution of the adsorption process with different fractional order with $C_o = 1.5 \times 10^5$.

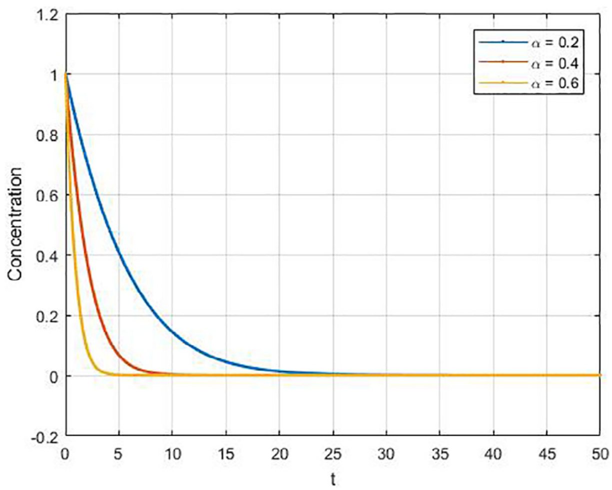


**Fig. 3.** Simulation for the solution of the adsorption process with different fractional order with $C_o = 1$.

## 4. Summary and conclusion

In this study, we solve the solution for the transport equation corresponding to the adsorption process where we utilized the Caputo-Fabrizio fractional derivative and we also give graphical illustration for different initial concentration. It was found in the model that the concentration varies directly with time and exponentially decreasing to $t = 0$. In addition, for the adsorption process, we have found the relation $C > 0$ for $C_o \exp\left\{-\frac{\alpha}{1-\alpha}t\right\}(K_l + 1) > \frac{1}{2}\left(1 + \frac{\beta(1-\alpha)^2}{1-2\alpha}\right)$ and $C = 0$ as $t \to \infty$ (see Fig. 3).

## Conflict of interest

None.


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