Journal of King Saud University – Science 34 (2022) 102295

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

A discussion on controllability of nonlocal fractional semilinear equations of order $1 < r < 2$ with monotonic nonlinearity

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ARTICLE INFO

Article history:
Received 22 March 2022
Revised 18 May 2022
Accepted 25 August 2022
Available online 5 September 2022

2020 Subject Classification:
34K30
34K35
93C25

Keywords:
Monotone condition
Controllability
Mild solution
Fractional order system
Nonlocal conditions

ABSTRACT

In this article, we use the tool of monotone nonlinearity to present the approximate controllability discussion for fractional semilinear system with nonlocal conditions. Monotonicity is an important characteristic in many communications applications in which digital-to-analog converter circuits are used. Such applications can function in the presence of nonlinearity, but not in the presence of non-monotonicity. Therefore, it becomes quite interesting to study a problem assuming monotonicity of the nonlinear function. Also, nonlocal conditions are additional specifications for the physical measurements than classical ones and impart a finer effect on the solution. We formulate the control function for the problem and establish the controllability results. At last, we proposed two applications for the demonstration.

1. Introduction

In the modern years, fractional calculus has been receiving considerable observation from researchers because of its uses in science and engineering areas, such as fractional biological neurons, fractal theories, nonlinear oscillation of earthquake, neural network modeling, fluid dynamics, population dynamics, etc. Fractional Calculus is a very effective mechanism that has been recently incorporated to model complex biological systems with non-linear behavior and long-term memory. It came into picture with a simple question related to the derivation concept like if the first order derivative represents the slope of a function, what would a half order derivative of a function represent? Looking for such type of questions gave birth to many new problems and interesting results in the real world. For example, we cannot determine the properties of a shape memory polymers model by making use of differential frameworks of integer order because the shape varies quickly as there is a small change in the temperature. Therefore, the task of fractional calculus becomes essential here. Fractional calculus has several uses in image processing, oil reservoirs, MRI, gas transportation, damping, HIV/TB infections, etc. So, recently, many researchers have done valuable performances in electromagnetic, control theory, signal, porous media, viscoelasticity, biological, engineering problems, image processing, fluid flow, diffusion, theology, etc. For more specifics, see (Bajlekova et al., 2001; Kilbas et al., 2006; Miller et al., 1993; Podlubny et al., 1999).

On the contrary, controllability problems have fascinated several engineers, mathematicians, and physicists, and remarkable support has been made to theories as well as their application sides

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2022.102295
1018-3647/

also. The controllability of dynamical frameworks under the control of partial differential frameworks has evolved into a widely researched topic in the last three decades. It investigates the probability of driving a system from any starting position to the required terminal position employing some group of admissible controls. Naito (Naito, 1987) obtained the approximate controllability discussion for a semilinear framework governed by linear part utilizing the fixed point technique. For second-order differential systems (Shukla and Patel, 2021; Vijayakumar, 2018; Vijayakumar and Dineshkumar, 2021; Vijayakumar, 2019; Vijayakumar et al., 2017; Vijayakumar et al., 2021; Vijayakumar et al., 2021), and fractional order frameworks have been extensively analyzed in the literature, see, for instance, (Arora, 2018; Arora, 2019; Dineshkumar et al., 2021; Kavitha et al., 2022; Kavitha et al., 2021; Raja et al., 2022; Raja et al., 2021; Nisar and Vijayakumar, 2021; Shukla et al., 2015; Shukla et al., 2014; Shukla et al., 2022; Shukla et al., 2015; SSakthivel and Y.;thivel and Y.; Mahmudov, 2011; Williams et al., 2020; Zhou et al., 2017; Vijayakumar et al., 2013).

Moreover, there are several real world models which can be displayed more accurately as certain partial differential equation with nonlocal conditions. In such problems nonlocal conditions are imposed in place of classical description of the data. These conditions, given by Byszewski (1991), provide excellent results in terms of existence, uniqueness and controllability analysis. It appears mainly when the measurements are considered at regular time periods instead of historical time period. Thus, the investigation may be presented more practically than utilizing the classical condition $p(0) = p_0$ alone. The controllability problems with nonlocal conditions can be found in Vijayakumar and Dineshkumar (2021).

In SSakthivel and Y.;thivel and Y.; Mahmudov (2011), Sakthivel et al. analyzed the controllability outcomes for semilinear fractional systems by applying theory related to fractional calculus via fixed point approach. Liu and Sakthivel (2014) analyzed the approximate controllability outcomes for semilinear fractional inclusions by applying theory related to fractional calculus via a fixed point approach for multi-valued maps. Zhou (2010) analyzed the existence of fractional neutral differential systems utilizing fixed point theorems and fractional powers of operators. The fundamental theories about the cosine and sine family (CF and SF) for second-order abstract Cauchy problem were introduced by Fattorini et al. (1985) and Travis (1978).

Several authors have focused the approximate controllability discussion about semilinear fractional systems by applying Lipschitz continuity on the nonlinear term with the fixed point technique. Suppose the nonlinear part fulfills monotone condition, then one may get finer conclusions as monotonicity is the main feature of many communication problems. George et al. (1995) and George et al. (1995) established the approximate controllability results for non autonomous semi linear and nonlinear evolution systems with the use of monotone and integral contractor condition on the nonlinear term. Arora (2019) discussed the controllability outcomes for fractional frameworks of order $r \in (1,2)$ with nonlinear term possess integral contractor. But no results are dealing with the nonlocal fractional semi linear control system of order $1 < r < 2$ using monotone condition on the nonlinear term.

Let $H = L_2[0,b;\mathbb{P}]$ and $V = L_2\left[0,b;\widehat{\mathbb{P}}\right]$ be the function spaces defined on $J = [0,b], 0 \leqslant b < \infty$, where $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are two Hilbert spaces. Consider the following nonlocal fractional semi linear control system:

$$
\begin{aligned}
{}^{c}D^{r}_{\varrho}p(\varrho) &= Ap(\varrho) + B\upsilon(\varrho) + \kappa(\varrho, p(\varrho)) \text{for} \varrho \in (0,b], && (1.1)\\
p(0) &= p_0 + h(p), && (1.2)\\
p'(0) &= p_1 + h_1(p), && (1.3)
\end{aligned}
$$

where

1. ${}^{c}D^{r}_{\varrho}$ is the Caputo fractional derivative of order $r \in (1,2)$.
2. $p(\varrho)$ represents the state having values in Hilbert space $\mathbb{P}$.
3. Control function υ is defined from $[0,b] \to \widehat{\mathbb{P}}$.
4. $B : \widehat{\mathbb{P}} \to \mathbb{P}$ is a bounded linear operator.
5. The map $\kappa : [0,b] \times \mathbb{P} \to \mathbb{P}$ produces nonlinearity in the system.
6. The operator $A : dom(A) \subseteq \mathbb{P} \to \mathbb{P}$ is linear, closed where $dom(A)$ is a dense subset of $\mathbb{P}$.
7. h and h_1 are continuous functions from $C(J,\mathbb{P}) \to \mathbb{P}$.

The fractional linear system with nonlocal conditions associated with (1.1)–(1.3) along with control u is presented as

$$
\begin{aligned}
{}^{c}D^{r}_{\varrho}q(\varrho) &= Aq(\varrho) + Bu(\varrho) \text{for } \varrho \in (0,b], && (1.4)\\
q(0) &= p_0 + h(p), && (1.5)\\
q'(0) &= p_1 + h_1(p). && (1.6)
\end{aligned}
$$

We now concentrate on a summary of this article: Section 2 presents some basic results related to fractional calculus and control theory. Section 3 provides controllability results for the proposed system assuming monotone nonlinearity. Section 4 presents two applications for drawing the concept of the primary outcomes.

2. Auxiliary results

We recollect known essential facts, lemmas, elementary definitions, remarks, and outcomes related to fractional calculus, semigroup theory of linear operators, and control theory.

We assume that $C([0,b];\mathbb{P}), C^1([0,b];\mathbb{P})$ stand for the space of functions $\kappa : [0,b] \to \mathbb{P}$ that are continuous and 1-time continuously differentiable. Here $\mathbb{L}(\mathbb{P})$ denotes the set of bounded linear operators from the Hilbert space $\mathbb{P}$ to $\mathbb{P}$.

Definition 2.1. (Kilbas et al., 2006) "Provided that $p(\varrho) \in L_1([0,b];\mathbb{P})$, then the Riemann–Liouville (R-L) fractional integral of order $r > 0$ is presented as

$$J^{r}_{\varrho}p(\varrho) = \frac{1}{\Gamma(r)} \int_0^{\varrho} (\varrho - s)^{r-1} p(s) ds.$$

In the above, $\Gamma(r) = \int_0^{\infty} e^{-\varrho} \varrho^{r-1} d\varrho$."

Definition 2.2. (Kilbas et al., 2006) "The R-L fractional derivative for $p(\varrho) \in L_1([0,b];\mathbb{P})$ of order $r \in (1,2)$ is presented as

$$D^{r}_{\varrho}p(\varrho) = D^2 J^{2-r}_{\varrho} p(\varrho) = \frac{1}{\Gamma(2-r)} \frac{d^2}{d\varrho^2} \int_0^{\varrho} (\varrho - s)^{1-r} p(s) ds.$$

Definition 2.3. (Kilbas et al., 2006) "The Caputo fractional derivative of order $r \in (1,2)$ is presented as

$${}^{c}D^{r}_{\varrho}p(\varrho) = J^{2-r}_{\varrho} D^2 p(\varrho) = \frac{1}{\Gamma(2-r)} \int_0^{\varrho} (\varrho - s)^{1-r} \left[\frac{d^2}{ds^2} p(s)\right] ds,$$

where $p(\varrho) \in L_1([0,b];\mathbb{P}) \cap C^1([0,b];\mathbb{P})$."

Assume that the subsequent fractional evolution system:

$${}^{c}D^{r}_{\varrho}p(\varrho) = Ap(\varrho),\ p(0) = \eta\ p'(0) = 0, \tag{2.1}$$

where $r \in (1,2); A : D(A) \subseteq \mathbb{P} \to \mathbb{P}$ is a densely defined and closed in $\mathbb{P}$. Apply R-L fractional integral of order r on (2.1), one can obtain

$$p(\varrho) = \eta + \frac{1}{\Gamma(r)} \int_0^{\varrho} (\varrho - s)^{r-1} Ap(s) ds. \tag{2.2}$$

Definition 2.4. (Bajlekova et al., 2001) "Assume that $r \in (1,2)$. A family $\{\Upsilon_r(\varrho)\}_{\varrho \geqslant 0} \subset \mathbb{L}(\mathbb{P})$ is said to be a solution operator (or a strongly continuous r-order fractional CF) for (2.1) provided that the subsequent characteristics are fulfilled:

1. $\Upsilon_r(\varrho)$ is strongly continuous for $\varrho \geqslant 0$ and $\Upsilon_r(0) = I$.
2. $\exists\, L \geqslant 1$ such that $\|\Upsilon_r(\varrho)\| \leqslant L$.
3. $\Upsilon_r(\varrho)D(A) \subset D(A)$ and $A\Upsilon_r(\varrho)\eta = \Upsilon_r(\varrho)A\eta, \forall\, \eta \in D(A), \varrho \geqslant 0$.
4. $\Upsilon_r(\varrho)\eta$ is a solution for (2.1), $\forall\, \eta \in D(A), t \geqslant 0$."

A is said to be infinitesimal generator of $\Upsilon_r(\varrho)$. The strongly continuous r-order fractional CF is also said to be r-order CF.

Definition 2.5. The fractional SF $\Psi_r : [0,\infty) \to \mathbb{L}(\mathbb{P})$ connected with Υ_r is presented as

$$\Psi_r(\varrho) = \int_0^{\varrho} \Upsilon_r(s)ds, \;\; \varrho \geqslant 0.$$

Definition 2.6. The fractional R-L family $\chi_r : [0,\infty) \to \mathbb{L}(\mathbb{P})$ connected with Υ_r is presented as

$$\chi_r(\varrho) = J_{\varrho}^{r-1}\Upsilon_r(\varrho).$$

Now, using the definition of R-L integral, we obtain for $\varrho \in [0,b]$,

$$\begin{aligned}\|\chi_r(\varrho)\| = &\; \|J_{\varrho}^{r-1}\Upsilon_r(\varrho)\| \\ = &\; \| \textstyle\int_0^{\varrho} \frac{(\varrho-\tau)^{r-2}}{\Gamma(r-1)}\Upsilon_r(\tau)d\tau\| \\ \leqslant &\; \frac{\|\Upsilon_r(\tau)\|}{\Gamma(r-1)} \| \textstyle\int_0^{\varrho} (\varrho-\tau)^{r-2}d\tau\| \\ \leqslant &\; \frac{L}{\Gamma(r-1)} | \frac{(\varrho-\tau)^{r-1}}{-(r-1)} |_0^{\varrho} \\ \leqslant &\; \frac{Lb^{r-1}}{\Gamma(r)}.\end{aligned}$$

Next, we describe the mild solution for (1.1)–(1.3) and its corresponding linear system.

Definition 2.7. The mild solution of (1.1)–(1.3) is defined by $p(\cdot) \in \mathbb{P}$ which satisfy the following integral equation.

$$\begin{aligned} p(\varrho) = &\; \Upsilon_r(\varrho)(p_0 + h(p)) + \Psi_r(\varrho)(p_1 + h_1(p)) \\ &+ \textstyle\int_0^{\varrho} \chi_r(\varrho-\tau)\{B\upsilon(\tau) + \kappa(s, p(\tau))\}d\tau, \; \varrho \in (0,b], \\ p(0) = &\; p_0 + h(p), \\ p'(0) = &\; p_1 + h_1(p), \end{aligned}$$

and the mild solution for (1.4)–(1.6) is described by the following integral equation

$$\begin{aligned} q(\varrho) = &\; \Upsilon_r(\varrho)(p_0 + h(p)) + \Psi_r(\varrho)(p_1 + h_1(p)) \\ &+ \textstyle\int_0^{\varrho} \chi_r(\varrho-\tau)Bu(\tau)d\tau, \; \varrho \in (0,b], \\ q(0) = &\; p_0 + h(p), \\ q'(0) = &\; p_1 + h_1(p). \end{aligned}$$

Definition 2.8. (Curtain et al., 1995) "The system (1.1)–(1.3) is said to be **approximately controllable** in $[0,b]$, provided that for given starting position and required final position p_F and $\epsilon > 0, \exists$ a control function $\upsilon \in V$ such that the solution of (1.1)–(1.3) fulfills

$$\|p(b) - p_F\| < \epsilon,$$

where $p(b)$ is the state value of (1.1)–(1.3) at time $\varrho = b$."

Remark 2.1. Suppose $p(b) = p_F$, then the system is called **exactly controllable**.

Controllability can also be interpreted in terms of reachable set which is defined in the following manner.

Definition 2.9. (Curtain et al., 1995) "Reachable set is the collection of all the possible final positions corresponding to the control $\upsilon \in V$. The set defined by

$$\begin{aligned}\mathfrak{R}_b(\kappa) = &\; \{p(b) \in \mathbb{P} : p(\varrho) \text{is a mild solution of the} \\ &\; \text{system corresponding to control} \upsilon \in V\},\end{aligned}$$

is the reachable set for (1.1)–(1.3) and $\mathfrak{R}_b(0)$ is the reachable set for (1.4)–(1.6)."

Definition 2.10. (Curtain et al., 1995) Definition of Approximate Controllability in terms of Reachable Set: "A control system is said to be approximately controllable on an interval $[0,b]$, iff the corresponding reachable set is dense in $\mathbb{P}$. Therefore the given system (1.1)–(1.3) is approximately controllable iff

$$\overline{\mathfrak{R}_b(\kappa)} = \mathbb{P},$$

where $\overline{\mathfrak{R}_b(\kappa)}$ denotes the closure of $\mathfrak{R}_b(\kappa)$.

Remark 2.2. If $\mathfrak{R}_b(\kappa) = \mathbb{P}$, then the given system is said to exactly controllable."

3. Controllability results

Here we mainly focus on the controllability for the second order fractional semilinear control system with nonlocal conditions considering the monotone nonlinearity of the nonlinear term.

The following conditions are taken into account to obtain the approximate controllability results of (1.1)–(1.3):

(T_1) $\exists$ a constant $\alpha > 0$ such that

$$\langle Np, p\rangle_H \geqslant \alpha\|Np\|_H^2 \;\; \forall \;\; p \in H,$$

where $N : H \to H$ is the operator defined by

$$(Np)(\varrho) = \int_0^{\varrho} \chi_r(\varrho-\tau)p(\tau)d\tau.$$

(T_2) Linear fractional system with nonlocal conditions (1.4)–(1.6) is approximately controllable, that is, the corresponding reachable set $\mathfrak{R}_b(0)$ is dense in $\mathbb{P}$.

(T_3) Monotone condition is satisfied by the nonlinear function κ, that is, $\exists$ constant $\gamma > 0$ such that

$$\langle \kappa(\varrho, p_1) - \kappa(\varrho, p_2), p_1 - p_2\rangle_{\mathbb{P}} \leqslant -\gamma\|p_1 - p_2\|_{\mathbb{P}}^2.$$

(T_4) $\|Kp\|_H \leqslant b_1 + b_2\|p\|_H$, where b_1 and b_2 are constants and $K : H \to H$ is Nemytskii operator defined as

$$(Kp)(\varrho) = \kappa(\varrho, p(\varrho)) \quad p \in H.$$

(T_5) $Range(K) \subseteq \overline{Range(B)}$, that is, for any given $\epsilon_1 > 0, \exists$ a w in $L_2\left[0,b;\widehat{\mathbb{P}}\right]$ such that

$$\|Kq - Bw\|_H < \epsilon_1. \tag{3.1}$$

(T_6) There exist constants M_h and M_{h_1} such that $\|h(p)\| \leqslant M_h$ and $\|h_1(p)\| \leqslant M_{h_1} \;\forall\; p \in dom(A)$.

Next, we prove a lemma before establishing the main result which focuses on the controllability of the semilinear system.

Lemma 1. The solution $p(\varrho)$ of the nonlocal fractional system (1.1)–(1.3) connected with the control $v = u - w$ fulfills the subsequent

$$\|p(\varrho)\|_{\mathbb{P}} \leqslant \Bigg[\left(1 + b_2 \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\right) L\big(\|p_0\| + \|p_1\| + M_h + M_{h_1}\big) + \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\left\{\left(1 + \frac{Lb^{r-\frac{1}{2}}b_2}{\Gamma(r)}\right)\|Bu\|_H + b_1 + \epsilon_1\right\} + \frac{Lb^r b_1}{\Gamma(r)}\Bigg] e^{\frac{Lb^r b_2}{\Gamma(r)}},$$

where $\|\Upsilon_r(\varrho)\| \leqslant L, \|\Psi_r(\varrho)\| \leqslant L, \|\chi_r(\varrho)\| \leqslant \frac{Lb^{r-1}}{\Gamma(r)}$ for each $\varrho \in [0, b]$.

Proof. Let $p(\varrho)$ be the mild solution of (1.1)–(1.3) associated with the control $v = u - w$. Therefore, one can consider $p(\varrho)$ in the following way:

$$p(\varrho) = \Upsilon_r(\varrho)(p_0 + h(p)) + \Psi_r(\varrho)(p_1 + h_1(p)) + \int_0^{\varrho} \chi_r(\varrho - \tau)B(u - w)(\tau)d\tau + \int_0^{\varrho} \chi_r(\varrho - \tau)\kappa(\tau, p(\tau))d\tau.$$

Taking norm on both sides, we get

$$\begin{aligned}
\|p(\varrho)\|_{\mathbb{P}} \leqslant\ & L\|p_0\| + L\|h(p)\| + L\|p_1\| + L\|h_1(p)\| \\
& + \tfrac{Lb^{r-1}}{\Gamma(r)} \textstyle\int_0^{\varrho} \|B(u-w)(\tau)\|_{\mathbb{P}} d\tau + \tfrac{Lb^{r-1}}{\Gamma(r)} \int_0^{\varrho} \|\kappa(\tau, p(\tau))\|_{\mathbb{P}} d\tau \\
\leqslant\ & L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} + \tfrac{Lb^{r-1}}{\Gamma(r)}\sqrt{b}(\|Bu\|_H + \|Bw\|_H) \\
& + \tfrac{Lb^{r-1}}{\Gamma(r)} \textstyle\int_0^{\varrho} (b_1 + b_2\|p(\tau)\|_{\mathbb{P}}) d\tau \\
\leqslant\ & L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} \\
& + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}(\|Bu\|_H + \|Kq\|_H + \epsilon_1) + \tfrac{Lb^r b_1}{\Gamma(r)} + \tfrac{Lb^{r-1}b_2}{\Gamma(r)} \textstyle\int_0^{\varrho} \|p(\tau)\|_{\mathbb{P}} d\tau \\
\leqslant\ & L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} \\
& + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}(\|Bu\|_H + b_1 + b_2\|q\|_H + \epsilon_1) \\
& + \tfrac{Lb^r b_1}{\Gamma(r)} + \tfrac{Lb^{r-1}b_2}{\Gamma(r)} \textstyle\int_0^{\varrho} \|p(\tau)\|_{\mathbb{P}} d\tau \\
\leqslant\ & L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)} \\
& \left(\|Bu\|_H + b_1 + b_2(L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\|Bu\|_H) + \epsilon_1\right) \\
& + \tfrac{Lb^r b_1}{\Gamma(r)} + \tfrac{Lb^{r-1}b_2}{\Gamma(r)} \textstyle\int_0^{\varrho} \|p(\tau)\|_{\mathbb{P}} d\tau \leqslant L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} \\
& + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\left\{\left(1 + \tfrac{Lb^{r-\frac{1}{2}}b_2}{\Gamma(r)}\right)\|Bu\|_H + b_1 + b_2L\|p_0\|\right. \\
& \left. + b_2LM_h + b_2L\|p_1\| + b_2LM_{h_1} + \epsilon_1\right\} + \tfrac{Lb^r b_1}{\Gamma(r)} + \tfrac{Lb^{r-1}b_2}{\Gamma(r)} \textstyle\int_0^{\varrho} \|p(\tau)\|_{\mathbb{P}} d\tau \\
\leqslant\ & \left(1 + b_2\tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\right) L(\|p_0\| + \|p_1\|) + L\left(1 + b_2\tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\right)(M_h + M_{h_1}) \\
& + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\left\{\left(1 + \tfrac{Lb^{r-\frac{1}{2}}b_2}{\Gamma(r)}\right)\|Bu\|_H + b_1 + \epsilon_1\right\} + \tfrac{Lb^r b_1}{\Gamma(r)} + \tfrac{Lb^{r-1}b_2}{\Gamma(r)} \textstyle\int_0^{\varrho} \|p(\tau)\|_{\mathbb{P}} d\tau.
\end{aligned}$$

Applying Gronwall's inequality, we attain

$$\|p(\varrho)\|_{\mathbb{P}} \leqslant \Bigg[\left(1 + b_2 \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\right) L\big(\|p_0\| + \|p_1\| + M_h + M_{h_1}\big) + \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\left\{\left(1 + \frac{Lb^{r-\frac{1}{2}}b_2}{\Gamma(r)}\right)\|Bu\|_H + b_1 + \epsilon_1\right\} + \frac{Lb^r b_1}{\Gamma(r)}\Bigg] e^{\frac{Lb^r b_2}{\Gamma(r)}}.$$

Hence, the result is proved.

Theorem 3.1. Suppose that (T_1)-(T_6) are fulfilled, then the fractional semilinear control system with nonlocal conditions (1.1)–(1.3) is approximately controllable on $[0, b]$.

Proof. Assume that $q(\varrho)$ be the mild solution of the fractional linear system with nonlocal conditions (1.4)–(1.6) associated with the control u. Then we can write $q(\varrho)$ as

$$q(\varrho) = \Upsilon_r(\varrho)(p_0 + h(p)) + \Psi_r(\varrho)(p_1 + h_1(p)) + \int_0^{\varrho} \chi_r(\varrho - \tau) \times (Bu)(\tau)d\tau. \tag{2.2}$$

Let $p(\varrho)$ be the mild solution of (1.1)–(1.3) associated with the control $v = u - w$. Then, we can express $p(\varrho)$ as

$$\begin{aligned} p(\varrho) =\ & \Upsilon_r(\varrho)(p_0 + h(p)) + \Psi_r(\varrho)(p_1 + h_1(p)) \\ & + \textstyle\int_0^{\varrho} \chi_r(\varrho - \tau)B(u - w)(\tau)d\tau \\ & + \textstyle\int_0^{\varrho} \chi_r(\varrho - \tau)\kappa(\tau, p(\tau))d\tau. \end{aligned} \tag{2.3}$$

Using (2.2) and (2.3), one can obtain

$$q(\varrho) - p(\varrho) = \int_0^{\varrho} \chi_r(\varrho - \tau)Bw(\tau)d\tau - \int_0^{\varrho} \chi_r(\varrho - \tau)\kappa(\tau, p(\tau))d\tau. \tag{2.4}$$

Writing the above equation in operator theoretic form, we get

$$\begin{aligned} q - p =\ & NBw - NKp \\ =\ & N(Bw - Kq) + (NKq - NKp). \end{aligned}$$

Applying the inner product on both sides with $Kq - Kp$, one can obtain

$$\begin{aligned} & \langle q - p, Kq - Kp\rangle_H = \langle N(Bw - Kq) + (NKq - NKp), Kq - Kp\rangle_H \\ & = \langle N(Bw - Kq), Kq - Kp\rangle_H + \langle (NKq - NKp), Kq - Kp\rangle_H. \end{aligned} \tag{2.5}$$

Using condition (T_3), we get that the left hand side of the Eq. (2.5) is $\leqslant -\gamma\|q - p\|^2$ and the second term of the R.H.S is nonnegative from assumption (T_1).

If we can show that $\langle N(Bw - Kq), Kq - Kp\rangle_H$ is negligbly small, then it will indicate that $\|q - p\|_H$ is also arbitrary small from the Eq. (2.5).

Therefore, we show that $\langle N(Bw - Kq), Kq - Kp\rangle_H$ is arbitrarily small.

Using Cauchy Schwartz Inequality, we have

$$\begin{aligned} & |\langle N(Bw - Kq), Kq - Kp\rangle_H| \leqslant \|N(Bw - Kq)\|_H \|Kq - Kp\|_H \\ & \leqslant \frac{Lb^{r-1}}{\Gamma(r)}.b\|Bw - Kq\|_H\{\|Kq\|_H + \|Kp\|_H\} \\ & \leqslant \frac{Lb^r\epsilon_1}{\Gamma(r)}\{b_1 + b_2\|q\|_H + b_1 + b_2\|p\|_H\}. \end{aligned} \tag{2.6}$$

Using Lemma 1, we get

$$|\langle N(Bw - Kq), Kq - Kp\rangle_H| \leqslant \frac{LGb^r\epsilon_1}{\Gamma(r)}. \tag{2.7}$$

where

$$\begin{aligned} G =\ & 2b_1 + b_2\left[L\|p_0\| + LM_h + L\|p_1\| + LM_{h_1} + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\|Bu\|_H\right] \\ & + b_2\Bigg[\Bigg\{\left(1 + b_2\tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\right)L(\|p_0\| + \|p_1\| + LM_h + LM_{h_1}) \\ & + \tfrac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\left\{\left(1 + \tfrac{Lb^{r-\frac{1}{2}}b_2}{\Gamma(r)}\right)\|Bu\|_H + b_1 + \epsilon_1\right\} + \tfrac{Lb^r b_1}{\Gamma(r)}\Bigg\} e^{\frac{Lb^r b_2}{\Gamma(r)}}\Bigg]. \end{aligned}$$

Therefore, G is finite for given u and ϵ_1. It indicates that $\langle N(Bw - Kq), Kq - Kp\rangle_H$ is arbitrary small as ϵ_1 is arbitrarily small.

Now, using Eqs. (2.5), (2.6)condition (T_4), we get that $\|q - p\| < \epsilon_2$, for some $\epsilon_2 > 0$.

Next, we determine that $\|q(\varrho) - p(\varrho)\|_{\mathbb{P}}$ is arbitrary small.

Consider,

$$\begin{aligned} q(\varrho) - p(\varrho) =\ & NBw(\varrho) - N(Kp)(\varrho) \\ =\ & N\{Bw(\varrho) - (Kq)(\varrho)\} + N\{(Kq)(\varrho) - (Kp)(\varrho)\}. \end{aligned} \tag{2.8}$$

Taking norm on both sides, we get

$$\begin{aligned}\|q(\varrho)-p(\varrho)\|_{\mathbb{P}} &= \|N\{Bw(\varrho)-(Kq)(\varrho)\}+N\{(Kq)(\varrho)-(Kp)(\varrho)\}\|_{\mathbb{P}}\\ &\leqslant \|N\{Bw(\varrho)-(Kq)(\varrho)\}\|_{\mathbb{P}}\\ &+\|N\{(Kq)(\varrho)-(Kp)(\varrho)\}\|_{\mathbb{P}}.\end{aligned}$$

Now,

$$\begin{aligned}\|N\{Bw(\varrho)-(Kq)(\varrho)\}\|_{\mathbb{P}} = &\ \|\int_0^\varrho \chi_r(\varrho-\tau)\{Bw(\tau)-(Kq)(\tau)\}d\tau\|_{\mathbb{P}}\\ \leqslant &\ \frac{Lb^{r-1}}{\Gamma(r)}\int_0^\varrho \|\{Bw(\tau)-(Kq)(\tau)\}\|_{\mathbb{P}}d\tau\\ \leqslant &\ \frac{Lb^{r-1}}{\Gamma(r)}\sqrt{b}\|Bw-Kq\|_H\\ < &\ \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\epsilon_1,\end{aligned}$$

and

$$\begin{aligned}\|N\{(Kq)(\varrho)-(Kp)(\varrho)\}\|_{\mathbb{P}} = &\ \|\int_0^\varrho \chi_r(\varrho-\tau)\{(Kq)(\tau)-(Kp)(\tau)\}d\tau\|_{\mathbb{P}}\\ \leqslant &\ \frac{Lb^{r-\frac{1}{2}}}{\Gamma(r)}\sqrt{b}\|Kq-Kp\|_H.\end{aligned} \tag{2.9}$$

Since K is continuous on H and $\|q-p\|<\epsilon_2$, therefore, the right hand side of the above defined equation can be made arbitrarily small.

Thus, we get

$\|q(\varrho)-p(\varrho)\|<\epsilon\ \forall\ \varrho\in[0,b]$andforgiven$\epsilon>0$.

Thus, $\|q(\varrho)-p(\varrho)\|$ may be formed arbitrarily small by selecting appropriate w.

$\Rightarrow$ $\mathfrak{R}_b(\kappa)$(reachable set for the fractional semilinear control system with nonlocal conditions (1.1)–(1.3) is dense in $\mathfrak{R}_b(0)$(corresponding fractional linear system with nonlocal conditions (1.4)–(1.6).

But $\mathfrak{R}_b(0)$ is dense in $\mathbb{P}$ due to the condition (T_2) as the corresponding fractional linear system with nonlocal conditions is approximately controllable. Therefore, $\mathfrak{R}_b(\kappa)$ is dense in $\mathbb{P}$, which implies that (1.1)–(1.3) is approximately controllable.

4. Applications

4.1. Abstract system

Assume the fractional differential system has the form:

$$^cD^r_\varrho p(\varrho,y)=p_{yy}(\varrho,y)+\eta(\varrho,y)+\delta(\varrho,p(\varrho,y));$$

$$\varrho\in[0,b],0\leqslant y\leqslant\pi, \tag{4.1}$$

$$p(\varrho,0)=p(\varrho,\pi)=0;\ \ for\ \ \varrho>0, \tag{4.2}$$

$$p(0,y)+\sum_{i=1}^{n}\alpha_i p(t_i,y)=p_0(y), \tag{4.3}$$

$$p_\varrho(0,y)+\sum_{i=1}^{k}b_i p(s_i,y)=p_1(y), \tag{4.4}$$

where $r\in(1,2)$ and let $\mathbb{P}=L_2[0,\pi]$ and define $A: dom(A)\to\mathbb{P}$ by

$A\zeta=\zeta''$; $\zeta\in dom(A)$,

where $dom(A)=\{\zeta(.)\in\mathbb{P}:\zeta(0)=\zeta(\pi)=0\}$. A has spectrum of eigen values $-n^2, n=1,2,3,\cdots$ with the corresponding eigen functions $\zeta_n(s)=(2/\pi)^{1/2}sin(ns)\quad n=1,2,3,\cdots$. The operator representation of A is

$$A\zeta=\sum_{n=1}^{\infty}(-n^2)(\zeta,\zeta_n)\zeta_n,\quad \zeta\in dom(A).$$

The cosine function $\{\Upsilon(\varrho)\}$ and the associated sine function $\{\Psi(\varrho)\}$ are defined by

$$\Upsilon(\varrho)\zeta=\sum_{n=1}^{+\infty}cos\ n\varrho(\zeta,\zeta_n)\zeta_n,\quad \zeta\in\mathbb{P},$$

and

$$\Psi(\varrho)\zeta=\sum_{n=1}^{+\infty}\frac{1}{n}sin\ nt(\zeta,\zeta_n)\zeta_n,\quad \zeta\in\mathbb{P}.$$

respectively.

Because A is the infinitesimal generator of a strongly continuous CF $\Upsilon(\varrho)$ for $r\in(1,2)$. Now, using the subordinate theorem (Bajlekova et al., 2001), which gives A is the infinitesimal generator of a strongly continuous exponentially bounded fractional CF $\Upsilon_r(\varrho)$ such that $\Upsilon_r(0)=I$, also

$$\Upsilon_r(\varrho)=\int_0^\infty\varphi_{\varrho,r/2}(s)\Upsilon(s)ds,\ \ \varrho>0,$$

where $\varphi_{\varrho,r/2}(s)=\varrho^{-r/2}P_{r/2}(s\varrho^{-r/2})$, and

$$P_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)},\ \ 0<\gamma<1.$$

Let $\upsilon:[0,b]\to\widehat{\mathbb{P}}$ be defined by

$(\upsilon(\varrho))(y)=\eta(\varrho,y);\ \ y\in[0,\pi]$,

where $\eta:[0,b]\times[0,\pi]\to[0,\pi]$ be a continuous control function.

Let $\kappa:[0,b]\times\mathbb{P}\to\mathbb{P}$ be defined by

$\kappa(\varrho,p)(y)=\delta(\varrho,p(y));\ \ p\in\mathbb{P},y\in[0,\pi]$.

where $\delta:[0,b]\times[0,\pi]\to[0,\pi]$ is a nonlinear function.

We now define $h,h_1:C(J,\mathbb{P})\to\mathbb{P}$ in the following way:

$$h(p)=\sum_{i=1}^{n}\alpha_i p(t_i,y)\ \text{and}\ h_1(p)=\sum_{i=1}^{k}\beta_i p(s_i,y)$$

for $0<t_i,s_i<b, y\in[0,\pi]$.

Here, the nonlinear function κ can be considered satisfying the conditions (T_3)-(T_5). The nonlocal functions $h(p)$ and $h_1(p)$ can be taken satisfying the assumption (T_6).

The problem (4.1)–4.4 can be rewritten as

$$\begin{aligned}&^cD^r_\varrho p(\varrho)=Ap(\varrho)+B\upsilon(\varrho)+\kappa(\varrho,p(\varrho));\ \ \varrho\in[0,b],\\ &p(0)=p_0+h(p),\\ &p'(0)=p_1+h_1(p).\end{aligned}$$

Therefore, by Theorem 3.1, the fractional differential system (4.1)–(4.4) is approximately controllable.

4.2. Filter system

Digital filters (DFs) play an extremely important role in the field of Digital Signal Processing (DSP). The execution of DFs is phenomenal; each of the critical factors that DSP has grown highly regarded. Filters are commonly classified as having two main applications: signal separation and signal restoration.

If a transmission is influenced by agitation, sound disturbance, or other signals, the use of filters in signal separation is essential. For instance, if one gadget is used to calculate the electrical operation of a baby's heart (EKG) while still in the womb. Using a mother's breath and pulse as a coarse indicator could be humiliating. To separate these signals from the target, a filter might be used.

We provide our filter system Fig. 1 as a response to the Filter systems presented in Zahoor et al. (2017) and Chandra et al. (2016). Fig. 1 depicts the block diagram's rough layout, which aids in improving the solution's utility while using the smallest amount

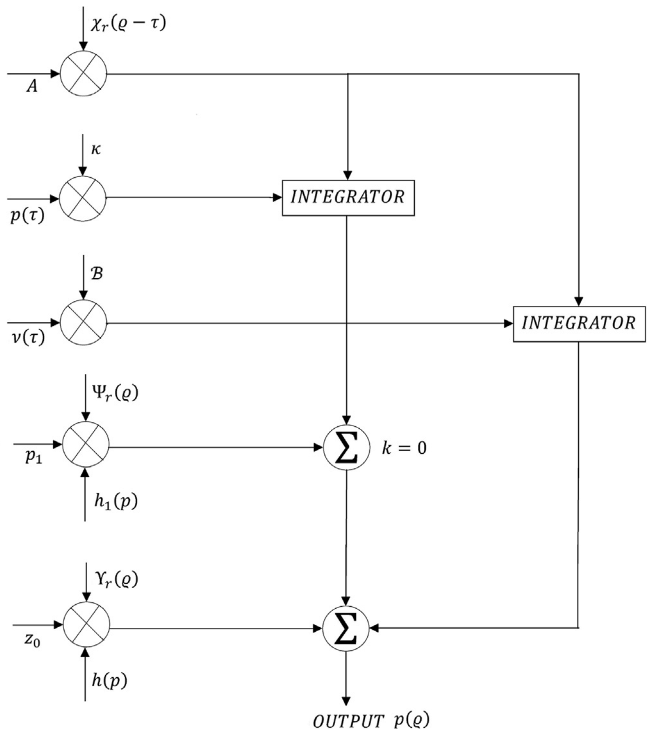

Fig. 1. Filter System.

of inputs possible. (i) Product modulator (PM)-1 accepts the input $\chi_r(\varrho - \tau)$ and A gives an output of $A\chi_r(\varrho - \tau)$. Likewise, (ii) PM-2 accepts $p(\tau)$ and κ produces $\kappa(\tau, p(\tau))$, (iii) PM-3 accepts $v(\tau)$ and B gives $Bv(\tau)$, (iv) PM-4 accepts $p_1, h_1(p)$ and $\Psi_r(\varrho)$ at time $\varrho = 0$, produces $\Psi_r(\varrho)(p_1 + h_1(p))$, (v) PM-5 accepts $p_0, h(p)$ and $\Upsilon_r(\varrho)$ at time $\varrho = 0$, produces $\Upsilon_r(\varrho)(p_0 + h(p))$, respectively. The integrators execute the integral of $\chi_r(\varrho - \tau)[Bv(\tau) + \kappa(s, p(\tau)))]$, over the period ϱ.

Finally, we move the outputs from the integrators to the summer network. Therefore, the output of $p(\varrho)$ is attained, it is bounded and approximately controllable.

5. Conclusion

In the present manuscript, we have established the approximate controllability results for nonlocal fractional semilinear system with order $r \in (1, 2)$. The results have been determined by assuming the monotone condition on the nonlinear term.

The method discussed in the present article can be further utilized to discuss the approximate controllability results for nonlocal impulsive fractional system after suitable modifications.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

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