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Original article

Results on Atangana-Baleanu fractional semilinear neutral delay integro-differential systems in Banach space

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The main focus of this manuscript is centered around Atangana-Baleanu semilinear neutral fractional integro-differential equations with finite delay. The main outcomes are demonstrated using the Mönch fixed point theorem along with its results when the measure of non-compactness collaborates. Eventually, a demonstration example is proposed.

1. Introduction

Fractional calculus has procured expansive importance during the past long time since the fractional derivative gives an imminent execution to the depiction of the memory and characteristic properties of different strategies. Lately, researchers center around fractional derivatives, especially when a couple of utilizations in biology, financial aspects, science, and engineering have displayed up (Francesco, 2010; Mainardi, 1996; Williams et al., 2020; Bedi et al., 2021; Bedi et al., 2020; Devi et al., 2021; Bedi et al., 2019; Bedi et al., 2020; Bedi et al., 2021). For Researchers, fractional derivatives have recently become a more interesting area, particularly since numerous possible methods emerged in biology, economics, science, and engineering. Fractional derivative definitions were offered in both local and nonlocal forms. Nonlocal derivatives are more interesting because the majority of these applications are dependent on the function's history.

Furthermore, integrodifferential equations are employed in a range of scientific domains where an aftereffect or delay must be taken into account, such as biology, control theory, ecology, and medicine. In general, integrodifferential equations are always employed to represent a model with hereditary characteristics, as the researcher's works (Mohan Raja et al., 2020; Dineshkumar et al., 2021; Kavitha et al., 2021) demonstrate. Neutral systems arise in a wide range of applied mathematics domains, including electronics, fluid dynamics, biological models, and chemical kinetics, and as a result, this type of equation has received a lot of attention in recent years, one can refer (Bedi et al., 2020; Bedi et al., 2021; Kavitha et al., 2021; Mallika Arjunan et al., 2021).

Fractional differential equations (FDEs) in several physical phenomena are difficult to handle via singular kernels. Subsequently,

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fractional derivatives were created involving non-singular kernels. Hence a new fractional derivative was proposed by Caputo and Fabrizio having exponential kernel in 2015. Caputo's and Riemann–Liouville's fractional derivative is the most well-known amongst the distinct fractional order differential operators with a singular kernel. To counter this, a new derivative was formulated by Atangana and Baleanu through the generalization of Mittag–Leffler function involving a non-singular kernel (Atangana and Baleanu, 2016) because non-singular kernel models can depict actuality in explicit ways when contrasted with the standard fractional calculus involving singular kernel, such as the Keller-Segel model (Atangana and Alqahtani, 2018). Refer (Atangana and Koca, 2016; Owolabi and Atangana, 2019; Ravichandran et al., 2019; Saad et al., 2018; Saad et al., 2018; Kumar and Pandey, 2020; Baleanu and Fernandez, 2018; Fernandez et al., 2019) for a list of Atangana-Baleanu derivative applications in several fields. Atangana and Baleanu (2016) proposed the Atangana-Baleanu (AB) fractional derivative in both the Riemann–Liouville and Caputo senses in recent years. This derivative includes the generalised Mittag–Leffler function as a kernel. The nonlocal behaviour of the generalised Mittag–Leffler function allows for a more realistic explanation of the macroscopic behaviour and memory effects of systems with non-local exchanges. Authors developed a new strategy for calculating the global conduct of difference equations with delay of threshold dynamics of difference equations for the SEIR model lately in Bentout et al. (2021). Bentout et al., 2021; Bentout et al., 2021; Djilali and Bentout, 2021; Djilali and Ghanbari, 2020; Khan et al., 2021; Zeb et al., 2021 for more information.

Furthermore, as shown in (Atangana and Koca, 2016; Mallika Arjunan et al., 2021; Owolabi and Atangana, 2019; Balasubramaniam, 2021), the Atangana-Baleanu (AB) fractional derivative retains all of the properties of previously known fractional derivatives. In a recent paper (Ravichandran et al., 2019), the authors used a fixed point approach to investigate the existence of AB fractional integro-differential and neutral systems. The authors of Mallika Arjunan et al. (2021) employed the fixed point approach given in Ravichandran et al. (2019) to show that the Atangana-Baleanu fractional neutral integro-differential and Volterra systems with or without delay exist. Motivated by these papers, the authors of Williams and Vijayakumar (2021) utilised fractional calculus, non-instantaneous impulses, the integro-differential equation, and the Darbo fixed point approach to cover the controllability and existence outcomes for fractional neutral impulsive Atangana-Baleanu integro-differential systems with delay. Recently, in Aimene et al. (2019), authors investigated the controllability of Atangana-Baleanu semilinear differential equations of fractional order with impulses and delay through semigroup theory and Darbo fixed point theorem along with measures of non-compactness. One can also refer (Mallika Arjunan et al., 2021; Williams and Vijayakumar, 2021) for the result of Atangana-Baleanu with delay.

For analysing nonlinear system existence of mild solutions, the fixed point technique can be regarded a useful and valuable tool. The fixed point technique appears to be appropriate for the solution of many problems in existence of solutions, since it is constructive and incorporates a convergence theory. The fixed point approach has yet to be widely used to stochastic impulsive control systems, despite its widespread use in both theory and numerical aspects of differential equations. Using this method, the problem is turned into a fixed point problem in a function space for an appropriate nonlinear operator. This technique relies heavily on the existence of a fixed point for the appropriate operator. The fixed point approach is the most successful method for examining the existence and controllability of differential systems with integer and fractional orders. Because of its usefulness, a number of academics have used various sorts of fixed point theorems to investigate the issues provided by evolution equations. The Mönch fixed point theorem is used to study the existence of mild solutions for Atangana-Baleanu semilinear neutral fractional integro-differential equations with finite delay.

Roused by the works above, we think about the accompanying issue of fractional semilinear differential equations in Banach space of the type

$$\begin{cases} {}^{ABC}D^{\zeta}[w(\delta)-\mathscr{S}_1(\delta,w_\delta)] = \hat{\mathscr{U}}w(\delta)+\mathscr{S}_2\Big(\delta,w_\delta,\int_0^\delta \mathscr{R}(\delta,\sigma,w_\sigma)d\sigma\Big), \\ \delta\in\mathscr{J}=[0,\mathscr{P}], \\ w_0(\delta)=\Theta(\delta)\in\boldsymbol{\mathscr{U}},\ \delta\in[-q,0] \end{cases} \tag{1}$$

${}^{ABC}D^{\zeta}$ is the Atangana-Baleanu-Caputo derivative of fractional order $0<\zeta<1$. The infinitesimal generator $\hat{\mathscr{U}}:D\big(\hat{\mathscr{U}}\big)\subset\mathscr{X}\to\mathscr{X}$ of an ζ-resolvent family $(Q_\zeta(\delta))_{\delta\geqslant 0},(P_\zeta(\delta))_{\delta\geqslant 0}$ is solution operator defined on a complex Banach space $(\mathscr{X},\|\cdot\|)$. Additionally, $\mathscr{S}_1:\boldsymbol{\mathscr{J}}\times\boldsymbol{\mathscr{U}}\to\mathscr{X};\mathscr{S}_2:\boldsymbol{\mathscr{J}}\times\boldsymbol{\mathscr{U}}\times\mathscr{X}\to\mathscr{X};\mathscr{R}:\Lambda\times\boldsymbol{\mathscr{U}}\to\mathscr{X}$ where $\Lambda=\{(\delta,\sigma):0\leqslant\sigma\leqslant\delta\leqslant\mathscr{P}\}$. $\mathscr{J}:=[0,\mathscr{P}],\mathscr{P}>0$ is a constant, $0<\delta_1<\delta_2<\ldots<\delta_m<\delta_{m+1}:=\mathscr{P},w_0\in\mathscr{X}$. Historically, w_δ represents the function $w_\delta:(-q,0]\to\mathscr{X}$ defined by $w_\delta(\rho)=w(\delta+\rho)$ for $\delta\in[0,\mathscr{P}]$ and $\rho\in[-q,0]$.

2. Preliminaries

Definition 2.1 Podlubny, 1999. The Riemann–Liouville fractional integral of order $\epsilon\in\mathbb{R}^+$: If there exists a function $h:\mathbb{R}^+\to\mathbb{R}$ then

$$I^{\epsilon}_{0+}h(\delta)=\frac{1}{\Gamma(\epsilon)}\int_0^\delta(\delta-\imath)^{\epsilon-1}h(\imath)d\imath,\quad \delta>0,$$

where the RHS is pointwise on $\mathbb{R}^+$, where Γ is a gamma function.

Definition 2.2 Podlubny, 1999. The Caputo fractional derivative of order $\epsilon\in(n-1,n]$: If there exists a continuous function $h:\mathbb{R}^+\to\mathbb{R}$, then

$${}^{C}D^{\epsilon}_{0+}h(\delta)=\frac{1}{\Gamma(n-\epsilon)}\int_0^\delta(\delta-\imath)^{n-1-\epsilon}h^{(n)}(\imath)d\imath,\ \delta>0,$$

where the integrals (2.1) and (2.2) are taken in Bochner's sense.

Definition 2.3. The Riemann–Liouville fractional derivative of order $\epsilon\in(n-1,n]$: If there exists any function $h:\mathbb{R}^+\to\mathbb{R}$, then

$${}^{RL}D^{\epsilon}_{0+}h(\delta)=\frac{1}{\Gamma(n-\epsilon)}\int_0^\delta(\delta-\imath)^{n-1-\epsilon}h(\imath)d\imath,\ \delta>0,$$

where the function h has absolutely continuous derivatives up to order $n-1$.

Definition 2.4. Atangana and Baleanu, 2016. The Caputo sense of A-B fractional derivative: For $\rho\in T^1(e,P),e<P$ and at $\delta\in(e,P)$ of order ζ we have

$${}^{ABC}D^{\zeta}_{e+}\rho(\delta)=\frac{B(\zeta)}{1-\zeta}\int_e^\delta\rho(\imath)\mathscr{H}_\zeta\Big(-\beta(\delta-\imath)^{\zeta}\Big)d\imath, \tag{2.1}$$

where the function $\beta=\zeta/(1-\zeta),\mathscr{H}_\zeta(\cdot)$ is Mittag Leffler, and $B(\zeta)=(1-\zeta)+\zeta/\Gamma(\zeta)$ is the normalization function fulfilling $B(0)=B(1)=1$.

Definition 2.5 Atangana and Baleanu, 2016. The Riemann–Liouville sense of A-B fractional derivative: For $\rho\in T^1(e,P),e<P$ and at $\delta\in(e,P)$ of order ζ we have

$$^{ABR}D^{\zeta}_{e^+}\rho(\delta) = \frac{B(\zeta)}{1-\zeta}\frac{d}{d\delta}\int_e^{\delta}\rho(\imath)\mathscr{H}_{\zeta}\left(-\beta(\delta-\imath)^{\zeta}\right)d\imath. \quad (2.2)$$

For $\zeta = 1$ in (2.1), let ∂_δ be the classical derivative.

The fractional integral order related to the A-B derivative is given by

$$^{AB}I^{\zeta}_{a^+} = \frac{1-\zeta}{B(\zeta)}\rho(\delta) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)}\int_a^{\delta}(\delta-\imath)^{(\zeta-1)}\rho(\imath)d\imath. \quad (2.3)$$

Definition 2.6 Pazy, 1983. The resolvent set is given by $\zeta(A) = \{\varpi \in \mathbb{C}; (\varpi - A) : \mathbb{D}(A) \to \mathscr{H}$ is invertible$\}$, the through closed graph theorem, $R(\varpi, A) = (\varpi - A)^{-1}$, is the bounded operator for $\varpi \in \zeta(A)$ on $\mathscr{H}$ which is known to be the resolvent of A at ϖ. Hence, $AR(\varpi, A) = \varpi R(\varpi, A) - I, \forall\ \varpi \in \zeta(A)$.

Definition 2.7 Pazy, 1983. If closed and linear operator A is a sectorial operator then $\exists$ a constant $\mathscr{T} > 0,\ \phi \in \mathbb{R}$ and $\beta \in [\frac{\pi}{2} : \pi], \ni$ the conditions

1. $\sum_{(\beta,\phi)} = \{\varpi \in \mathbb{C}; \varpi \neq \phi, |\arg(\varpi - \phi)|\beta\} \subset \zeta(\mathbb{A})$,
2. $\|R(\varpi, A)\| \leqslant \frac{\mathscr{T}}{|\varpi-\phi|}, \varpi \in \sum_{(\beta,\phi)}$,

are fulfilled.

Definition 2.8 Aimene et al., 2019. If $w : \mathscr{C}([-q, \mathscr{P}], \mathscr{X}) \to \mathscr{X}$ is a mild solution of (1) then $w_0(\rho) = \Theta(\rho), \rho \in [-q, 0]$ and

$$w(\delta) = \begin{cases} \Theta(\delta), \quad \delta = [-q, 0], \\ \mathscr{G}P_{\zeta}(\delta)[\Theta(0) - \mathscr{S}_1(0, w_0)] \\ +\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\imath)^{\zeta-1}\mathscr{S}_2\left(\imath, w_\imath, \int_0^{\imath}\mathscr{R}(\imath,\theta,w_\theta)d\theta\right)d\imath \\ +\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\imath)^{\zeta-1}\sigma^*\mathscr{S}_1(\imath, w_\imath)d\imath \\ +\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\imath)\mathscr{S}_2\left(\imath, w_\imath, \int_0^{\imath}\mathscr{R}(\imath,\theta,w_\theta)d\theta\right)d\imath \\ +\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\imath)\hat{\mathscr{U}}\mathscr{S}_1(\imath, w_\imath)d\imath, \quad \delta \in [0, \mathscr{P}], \end{cases} \quad (2.4)$$

where $\mathscr{G} = \sigma^*\left(\sigma^* I - \hat{\mathscr{U}}\right)^{-1}$; $\mathscr{K} = -\hat{\gamma}^*\hat{\mathscr{U}}\left(\sigma^* I - \hat{\mathscr{U}}\right)^{-1}$ with $\sigma^* = \frac{B(\zeta)}{1-\zeta}$, $\hat{\gamma} = \frac{\zeta}{1-\zeta}$, $\hat{\mathscr{U}} = \frac{-B(\zeta)\mathscr{K}}{\zeta\mathscr{G}}$ and

$$P_{\zeta}(\delta) = \mathscr{X}_{\zeta}\left(-\mathscr{K}\delta^{\zeta}\right) = \tfrac{1}{2\pi i}\int_{\Gamma}e^{\varpi\delta}\varpi^{\zeta-1}\left(\varpi^{\zeta}I - \mathscr{K}\right)^{-1}d\varpi.$$
$$Q_{\zeta}(\delta) = \delta^{\zeta-1}\mathscr{X}_{\zeta,\zeta}\left(-\mathscr{K}\delta^{\zeta}\right) = \tfrac{1}{2\pi i}\int_{\Gamma}e^{\varpi\delta}\left(\varpi^{\zeta}I - \mathscr{K}\right)^{-1}d\varpi$$

and $\exists\ \Gamma$ lying on $\sum_{(\tilde{\delta},\omega)}, \mathscr{S}_2 \in \mathscr{C}(\mathscr{J}, \mathscr{X})$.

Definition 2.9 (*Deimling, 2010; Heinz, 1983*). The Kuratowski measure of noncompactness: Consider a banach space $\mathscr{X}$ and $\mathscr{S}(\mathscr{X}) \subset \mathscr{X}$ is bounded then $\alpha : \mathscr{S}(\mathscr{X}) \to [0, \infty)$ is a mapping which can be specified by $\alpha(\mathscr{B}) = \inf\{\epsilon > 0 : \mathscr{B} \subseteq \cup_{i=1}^{n}\mathscr{B}_i$ and $\mathrm{diam}(\mathscr{B}_i) \leqslant \epsilon\}$, where $\mathscr{B} \in \mathscr{S}(\mathscr{X})$ and $\mathrm{diam}(\mathscr{B}_i) = \sup\{\|w - x\| : w, x \in \mathscr{B}\}$.

Definition 2.10 Ji et al., 2011. Let $\mathscr{D}^+$ be the positive cone of an ordered Banach space $(\mathscr{D}, \leqslant)$. A function E defined on the set of all bounded subsets of Banach space $\mathscr{Z}$ with values in $\mathscr{D}^+$ is called a measure of noncompactness(MNC) on $\mathscr{Z}$ if and only if $E(\overline{co}\mathscr{T}) = E(\mathscr{T})$ for all bounded subsets $\mathscr{T} \subseteq \mathscr{Z}$, where $\overline{co}\mathscr{T}$ stands for the closed convex hull of $\mathscr{T}$. The MNC of E is said to be:

1. monotone if and only if for all subsets $\mathscr{T}_1, \mathscr{T}_2$ of $\mathscr{Z}$, we have
   $(\mathscr{T}_1 \subseteq \mathscr{T}_2) \Rightarrow (E(\mathscr{T}_1) \leqslant E(\mathscr{T}_2))$;
2. nonsingular if and only if $E(\{a\} \cup \mathscr{T}) = E(\mathscr{T})$ for every $a \in \mathscr{Z}, \mathscr{T} \subset \mathscr{Z}$;
3. regular if and only if $E(\mathscr{T}) = 0$ if and only if $\mathscr{T}$ is relatively compact in $\mathscr{Z}$. One of the many examples of MNC is the noncompactness measure of Hausdroff σ defined on each bounded subset $\mathscr{T}$ of $\mathscr{Z}$ by
   $\sigma(\mathscr{T}) = \inf\{\epsilon > 0;\ \mathscr{T}$ can be covered by a finite number of balls of radii smaller than $\epsilon\}$.

   For all bounded subsets $\mathscr{T}, \mathscr{T}_1, \mathscr{T}_2$ of $\mathscr{Z}$,
4. $\sigma(\mathscr{T}_1 + \mathscr{T}_2) \leqslant \sigma(\mathscr{T}_1) + (\mathscr{T}_2)$, where $\mathscr{T}_1 + \mathscr{T}_2 = \{z + y : z \in \mathscr{T}_1, y \in \mathscr{T}_2\}$;
5. $\sigma(\mathscr{T}_1 \cup \mathscr{T}_2) \leqslant \max\{\sigma(\mathscr{T}_1), \sigma(\mathscr{T}_2)\}$;
6. $\sigma(\lambda\mathscr{T}) \leqslant |\lambda|\sigma(\mathscr{T})$ for any $\lambda \in \mathbb{R}$;
7. If $\mathscr{Q} : \mathscr{D}(\mathscr{Q}) \subseteq \mathscr{Z} \to \mathscr{Y}$ is lipschitz continuous with constant w, then $\sigma_Y(\mathscr{D}\mathscr{T}) \leqslant w\sigma(\mathscr{T})$ for any bounded subset $\mathscr{T} \subseteq \mathscr{D}(\mathscr{Q})$, where Y is Banach space.

Lemma 2.11 (*Deimling, 2010; Heinz, 1983*). Consider $\mathscr{X}$ as a Banach space, suppose $\mathscr{B}$ is bounded and equicontinuous in $\mathscr{C}([c, d], \mathscr{X})$ we get $\alpha(\mathscr{B}(\delta))$ is continuous on $[c, d]$, along with $\alpha(\mathscr{B}) = \sup_{\delta\in U}\alpha(\mathscr{B}(\delta)), \delta \in [c, d]$, where $\mathscr{B}(\delta) = \{w(\delta) : w \in \mathscr{B}\} \subset \mathscr{X}$.

Lemma 2.12 (*Deimling, 2010; Heinz, 1983*). If $\mathscr{B}$ is a bounded set in $\mathscr{C}([c, d], \mathscr{X})$, then $\mathscr{B}(\delta)$ is bounded in $\mathscr{X}$, and $\alpha(\mathscr{B}(\delta)) \leqslant \alpha(\mathscr{B})$.

Lemma 2.13 (*Deimling, 2010; Heinz, 1983*). If a bounded and countable set $\mathscr{B} = \{v_n\} \subset \mathscr{C}([c, d], \mathscr{X})(n = 1, 2, ..)$ then $\alpha(\mathscr{B}(\delta))$ is Lebesgue integrable on $[c, d]$ with

$$\alpha\left(\left\{\int_c^d v_n(\delta)d\delta\right\}_{n=1}^{\infty}\right) \leqslant 2\int_c^d \alpha(\mathscr{B}(\delta))d\delta.$$

Theorem 2.14 Mönch, 1980. Let $\mathscr{U}$ be a closed convex subset of a Banach space $\mathscr{Z}$ and $0 \in \mathscr{U}$. Assume that $X : \mathscr{U} \to \mathscr{Z}$ is a continuous map which satisfies Mönch's condition, that is, ($\mathscr{M} \subseteq \mathscr{U}$ is countable, $\mathscr{M} \subseteq \overline{co}(\{0\} \cup X(\mathscr{M})) \Rightarrow \overline{\mathscr{M}}$ is compact). Then X has a fixed point in $\mathscr{U}$.

3. Main Results

Now, let us look into the existence of (1). Suppose $\hat{\mathscr{U}} \in U^{\zeta}(\alpha_0, l_0)$ then $\|P_{\zeta}(\delta)\| \leqslant \mathscr{T}e^{l\delta}$ and $\|Q_{\zeta}(\delta)\| \leqslant \mathscr{C}e^{l\delta}(1 + \delta^{\zeta-1}), \forall\ \ \delta > 0, l > l_0$. Thus, $\hat{\mathscr{T}} = \sup_{\delta\geqslant 0}\|P_{\zeta}(\delta)\|, \hat{\mathscr{T}}_1 = \sup_{\delta\geqslant 0}\mathscr{C}e^{l\delta}(1 + \delta^{\zeta-1})$. So we get $\|P_{\zeta}(\delta)\| \leqslant \hat{\mathscr{T}}$ and $\|Q_{\zeta}(\delta)\| \leqslant \delta^{\zeta-1}\hat{\mathscr{T}}_1$. One can also refer (Shu et al., 2011).

Now we assume the following assumptions.

(H₁) $\mathscr{S}_2 : \mathscr{J} \times \mathscr{U} \times \mathscr{X} \to \mathscr{X}$ is a function that fits the following requirements
- **(i)** It satisfies Carathedory condition i.e. $\mathscr{S}_2(\cdot, \cdot, \cdot, w)$ is Lebesgue measurable and $\mathscr{S}_2(\delta, \cdot, \cdot, \cdot)$ is continuous.
- **(ii)** $\exists$ a non decreasing continuous function $\mathscr{T}_{\mathscr{S}_2} : [0, \infty) \to (0, \infty)$ and a function $\varphi \in L^{\frac{1}{\zeta_1}}(U, \mathbb{R}^+)$, where $\zeta_1 \in (0, \zeta) \ni$

$$\|\mathscr{S}_2(\delta, u_1, u_2)\| \leqslant \varphi(\delta)\mathscr{T}_{\mathscr{S}_2}(\|u_1\| + \|u_2\|),$$

and

$$\liminf_{n\to\infty}\frac{\mathscr{T}_{\mathscr{S}_2}(n)}{n} = \chi < \infty.$$

(**iii**) $\exists$ constants $\mathscr{L}_1 \in L^{\frac{1}{\zeta_1}}([0,\mathscr{P}];R^+)$ for any countable set $u_1, u_2 \subset \mathscr{X}$,

$$\alpha(\mathscr{S}_2(\delta, u_1, u_2)) \leqslant \mathscr{L}_1[\alpha(u_1) + \alpha(u_2)], \forall \delta \in [0,\mathscr{P}].$$

(**H$_2$**) For each $(\delta,\sigma) \in \Lambda, \mathscr{R} : \Lambda \times \mathscr{U} \to \mathscr{X}$ is a continuous function and it fits the following requirements

(**i**) there exist constants $\mathscr{T}_{\mathscr{R}}$, such that,

$$\|\mathscr{R}(\delta,\sigma,u_1)\| \leqslant \mathscr{T}_{\mathscr{R}}[1 + \|u_1\|],$$

for $u_1 \in \mathscr{X}, \delta, \sigma \in \mathscr{J}$.

(**ii**) $\exists\ \mathscr{L}_2 \in L^{\frac{1}{\zeta_1}}\left(U^2, \mathbb{R}^+\right)$, for any bounded subset $u_2 \subset \mathscr{X} \to \mathscr{X}$

$$\alpha(\mathscr{T}_{\mathscr{R}}(\delta,\sigma,u_2)) \leqslant \mathscr{L}_2(\delta,\sigma)[\alpha(u_2)] \text{ for a.e. } \delta \in U,$$

with $\mathscr{L}_2^* = \int_0^\sigma \mathscr{L}_2(\delta,\varrho) < \infty$.

(**H$_3$**) For a function $\mathscr{S}_1 : \mathscr{J} \times \mathscr{U} \to \mathscr{X}$ is continuous then it should fulfill the following

(**i**) $\exists$ a constant $\mathscr{T}_{\mathscr{S}_1}, \hat{\mathscr{T}}_{\mathscr{S}_1} \ni$

$$\|\mathscr{S}_1(\delta,u_1)\| \leqslant \mathscr{T}_{\mathscr{S}_1}(1 + \|u_1\|) \text{ for } \delta \in U, \vartheta \in \mathscr{X},$$

$$\|\mathscr{S}_1(\delta,u_1) - \mathscr{S}_1(\delta,u_2)\| \leqslant \hat{\mathscr{T}}_{\mathscr{S}_1}\|u_1 - u_2\| \forall u_1, u_2 \in \mathscr{X}.$$

(**ii**) $\exists$ constants $\mathscr{L}_3 \ni$ for any countable set $u_3 \subset \mathscr{X}$,

$$\alpha(\mathscr{S}_1(\delta,u_3)) \leqslant \mathscr{L}_3\alpha(u_3), \forall \delta \in U.$$

(**H$_4$**) For a bounded linear operators $\mathscr{G}$ and $\mathscr{K}$ from $\mathscr{X}$ $\exists$ positive constants υ and μ fulfilling

$$\|\mathscr{G}\| \leqslant \upsilon \text{ and } \|\mathscr{K}\| \leqslant \mu.$$

Theorem 3.1. If (**H$_1$**)-(**H$_4$**) are fulfilled, the system (1) has at least one mild solution, assuming that,

$$2\left[\xi^*\mathscr{L}_1\left(1 + 2\mathscr{L}_2^*\right) + \xi^{**}\mathscr{L}_3\right] < 1. \tag{3.1}$$

Proof. To show that, the operator $\mathscr{E} : \mathscr{U}' \to \mathscr{U}'$ defined by

$$\mathscr{E}w(\delta) = \begin{cases} \Theta(\delta), \quad \delta = [-q, 0], \\ \mathscr{G}P_\zeta(\delta)[\Theta(0) - \mathscr{S}_1(0,w_0)] \\ + \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\mathscr{S}_2\left(\imath, w_\imath, \int_0^\imath \mathscr{R}(\imath,\theta,w_\theta)d\theta\right)d\imath \\ + \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\sigma^*\mathscr{S}_1(\imath, w_\imath)d\imath \\ + \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\mathscr{S}_2\left(\imath, w_\imath, \int_0^\imath \mathscr{R}(\imath,\theta,w_\theta)d\theta\right)d\imath \\ + \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\hat{\mathscr{U}}\mathscr{S}_1(\imath, w_\imath)d\imath, \quad \delta \in [0,\mathscr{P}], \end{cases} \tag{3.2}$$

has fixed point, which is a mild solution of (1). Rewriting the problem (1) as follows. For $\Theta \in \mathscr{U}$, we define $\hat{\Theta}$ by

$$\hat{\Theta}(\delta) = \begin{cases} \Theta(\delta), & \delta \in [-q,0], \\ \mathscr{G}P_\zeta(\delta)\Theta(0), & \delta \in \mathscr{J}. \end{cases}$$

Then $\hat{\Theta} \in \mathscr{U}'$. Let $w(\delta) = \mathscr{M}(\delta) + \hat{\Theta}(\delta), -q < \delta \leqslant \mathscr{P}$. Hence $\mathscr{M}$ satisfies $\mathscr{M}_0 = 0$ and

$$\begin{aligned} \mathscr{M}(\delta) &= \mathscr{G}P_\zeta(\delta)[-\mathscr{S}_1(0,w_0)] \\ &+ \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ &+ \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\sigma^*\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath \\ &+ \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ &+ \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\hat{\mathscr{U}}\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath \end{aligned}$$

if and only if w satisfies

$$\begin{aligned} w(\delta) &= \mathscr{G}P_\zeta(\delta)[\Theta(0) - \mathscr{S}_1(0,w_0)] \\ &+ \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ &+ \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\sigma^*\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath \\ &+ \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ &+ \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\hat{\mathscr{U}}\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath, \quad \delta \in [0,\mathscr{P}], \end{aligned}$$

and $w(\delta) = \Theta(\delta), \quad \delta \in [-q,0]$.

We define an operator $\mathscr{U}'' = \{\mathscr{M} \in \mathscr{U}' : \mathscr{M}_0 \in \mathscr{U}\}$. Let $\mathscr{B}_\xi = \left\{\mathscr{M} \in \mathscr{U}'' : \|\mathscr{M}\|_{\mathscr{U}'} \leqslant \xi\right\}$ for some $\xi > 0$, then $\mathscr{B}_\xi \subseteq \mathscr{U}''$ is uniformly bounded, we have.

$$\|\mathscr{M}_\delta + \hat{\Theta}_\delta\|_{\mathscr{U}} \leqslant \|\mathscr{M}_\delta\| + \|\hat{\Theta}_\delta\| \leqslant \xi + \left|\left|\hat{\Theta}_\delta\right|\right|_{\mathscr{U}'} = \xi'.$$

Define an operator $\hat{\mathscr{E}} : \mathscr{U}'' \to \mathscr{U}''$ by

$$\hat{\mathscr{E}}w(\delta) = \begin{cases} 0, \delta \in [-q,0], \\ \mathscr{G}P_\zeta(\delta)[-\mathscr{S}_1(0,w_0)] \\ + \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ + \frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\sigma^*\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath \\ + \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath \\ + \frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\hat{\mathscr{U}}\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath, \quad \delta \in [0,\mathscr{P}]. \end{cases}$$

Visibly, the operator $\mathscr{E}$ has a fixed point that is identical to $\hat{\mathscr{E}}$ has one. Therefore, it is enough to prove $\hat{\mathscr{E}}$ has fixed point.

Step 1: For a positive number $\xi > 0,\ \hat{\mathscr{E}}(\mathscr{B}_\xi) \subseteq \mathscr{B}_\xi$.

We assume the contrary, i.e, $\forall \xi, \exists\ \mathscr{M}^\xi \in \mathscr{B}_\xi$ but $\hat{\mathscr{E}}\left(\mathscr{M}^\xi\right) \notin \mathscr{B}_\xi$, i.e, $\|\hat{\mathscr{E}}\left(\mathscr{M}^\xi\right)(\delta)\| > \xi$ for some $\delta \in \mathscr{J}$.

Applying (**H$_1$**) – (**H$_4$**), we have

$$\begin{aligned} \xi &\leqslant \|\hat{\mathscr{E}}\left(\mathscr{M}^\xi\right)(\delta)\| \\ &\leqslant \|\mathscr{G}P_\zeta(\delta)[-\mathscr{S}_1(0,w_0)] \\ &+ \|\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath\| \\ &+ \|\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\sigma^*\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath\| \\ &+ \|\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\mathscr{S}_2\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath, \int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\right)d\imath\| \\ &+ \|\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^\delta Q_\zeta(\delta - \imath)\hat{\mathscr{U}}\mathscr{S}_1\left(\imath, \mathscr{M}_\imath + \hat{\Theta}_\imath\right)d\imath\| \\ &\leqslant \upsilon\hat{\mathscr{T}}\mathscr{T}_{\mathscr{S}_1}(1 + \|w_0\|) \\ &\quad + \frac{\mu\upsilon(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^\delta(\delta - \imath)^{\zeta-1}\varphi(\imath)\mathscr{T}_{\mathscr{S}_2}\Big[\|\mathscr{M}_\imath + \hat{\Theta}_\imath\| \\ &\quad + \|\int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\|\Big]d\imath \\ &\quad + \frac{\mu\upsilon}{\Gamma(\zeta+1)}P^\zeta\mathscr{T}_{\mathscr{S}_1}\left[1 + \|\mathscr{M}_\imath + \hat{\Theta}_\imath\|\right] \\ &\quad + \frac{\upsilon^2}{B(\zeta)}\hat{\mathscr{T}}_1\int_0^\delta(\delta - \imath)^{\zeta-1}\varphi(\imath)\mathscr{T}_{\mathscr{S}_2}\Big[\|\mathscr{M}_\imath + \hat{\Theta}_\imath\| \\ &\quad + \|\int_0^\imath \mathscr{R}\left(\imath,\theta,\mathscr{M}_\theta + \hat{\Theta}_\theta\right)d\theta\|\Big]d\imath \\ &\quad + \mu\upsilon\hat{\mathscr{T}}_1\frac{P^\zeta}{\zeta}\mathscr{T}_{\mathscr{S}_1}\left[1 + \|\mathscr{M}_\imath + \hat{\Theta}_\imath\|\right] \\ &\leqslant \upsilon\hat{\mathscr{T}}\mathscr{T}_{\mathscr{S}_1}(1 + \|w_0\|) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu\upsilon(1-\zeta)}{B(\zeta)\Gamma(\zeta+1)}P^{\zeta}\|\varphi\|\{\mathscr{T}_{\mathscr{S}_2}(\xi'+\mathscr{T}_{\mathscr{R}}(1+\xi')\mathscr{P})\}\\
&+\frac{\mu\upsilon}{\Gamma(\zeta+1)}P^{\zeta}\mathscr{T}_{\mathscr{S}_1}[1+\xi']\\
&+\frac{\upsilon^2}{B(\zeta)}\hat{\mathscr{T}}_1P^{\zeta}\|\varphi\|\{\mathscr{T}_{\mathscr{S}_2}(\xi'+\mathscr{T}_{\mathscr{R}}(1+\xi')\mathscr{P})\}\\
&+\mu\upsilon\hat{\mathscr{T}}_1\frac{P^{\zeta}}{\zeta}\mathscr{T}_{\mathscr{S}_1}[1+\xi']\\
&\leqslant \upsilon\hat{\mathscr{T}}\mathscr{T}_{\mathscr{S}_1}(1+\|w_0\|)\\
&+\left[\frac{\mu\upsilon(1-\zeta)}{B(\zeta)\Gamma(\zeta+1)}P^{\zeta}+\frac{\upsilon^2\hat{\mathscr{T}}_1P^{\zeta}}{B(\zeta)}\right]\|\varphi\|\{\mathscr{T}_{\mathscr{S}_2}(\xi'+\mathscr{T}_{\mathscr{R}}(1+\xi')\mathscr{P})\}\\
&+\left[\frac{\mu\upsilon P^{\zeta}}{\Gamma(\zeta+1)}+\frac{\mu\upsilon\hat{\mathscr{T}}_1P^{\zeta}}{\zeta}\right]\mathscr{T}_{\mathscr{S}_1}[1+\xi']\\
&\leqslant \upsilon\hat{\mathscr{T}}\mathscr{T}_{\mathscr{S}_1}(1+\|w_0\|)+\xi^*\|\varphi\|\{\mathscr{T}_{\mathscr{S}_2}(\xi'+\mathscr{T}_{\mathscr{R}}(1+\xi')\mathscr{P})\}+\xi^{**}\mathscr{T}_{\mathscr{S}_1}[1+\xi'].
\end{aligned}
$$

Let $v=\xi'+\mathscr{T}_{\mathscr{R}}(1+\xi')\mathscr{P}$. At the moment, $v\to\infty$ as $\xi\to\infty$. Now dividing (3.3) by ξ and allowing $\xi\to\infty$, one can obtain

$$1\leqslant\frac{\upsilon\hat{\mathscr{T}}\mathscr{T}_{\mathscr{S}_1}(1+\|w_0\|)}{\xi}+\xi^*\|\varphi\|\frac{\mathscr{T}_{\mathscr{S}_2}(v)}{v}\cdot\frac{v}{\xi}+\frac{\xi^{**}\mathscr{T}_{\mathscr{S}_1}[1+\xi']}{\xi}$$

then by $(\mathbf{H_1})$, we obtain $1\leqslant 0$.

This is a contraction. Hence, for some positive integer $\xi,\hat{\mathscr{E}}(\mathscr{B}_{\xi})\subseteq\mathscr{B}_{\xi}$.

Step 2: $\hat{\mathscr{E}}$ is continuous on $\mathscr{B}_{\xi}$.

$$
\begin{aligned}
&\|\mathscr{E}_2w_n(\delta)-\mathscr{E}_2w(\delta)\|\leqslant\|\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\\
&-\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)\Big]d\iota\|\\
&+\|\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\sigma^*\Big[\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)\\
&-\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|+\|\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\iota)\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\\
&-\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)\Big]d\iota\|\\
&+\|\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\iota)\hat{\mathscr{U}}\Big[\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)-\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|\\
&\leqslant\frac{\mu\upsilon(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\\
&\|\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\\
&-\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)\|\\
&+\frac{\mu\upsilon}{\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\|\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)\\
&-\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\|+\frac{\zeta\upsilon^2\hat{\mathscr{T}}_1}{B(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\\
&\|\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\\
&-\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)\|\\
&+\mu\upsilon\int_0^{\delta}Q_{\zeta}(\delta-\iota)\|\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)-\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\|.
\end{aligned}
$$

We acquire $\lim_{n\to\infty}\mathscr{E}\Big(\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)=\mathscr{E}\Big(\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big))$ in $\mathscr{B}_{\xi}$, since the functions $\mathscr{S}_1,\mathscr{S}_2$ are continuous.

Hence $\mathscr{E}$ is continuous on $\mathscr{B}_{\xi}$.

Step 3: $\hat{\mathscr{E}}(\mathscr{B}_{\xi})$ is equicontinuous family of function on $\mathscr{J}$.

For $w\in\hat{\mathscr{E}}(\mathscr{B}_{\xi})$ and $0<\delta_1<\delta_2\leqslant\mathscr{P}$ then $\exists\ \mathscr{M}\in\mathscr{B}_{\xi}\ni$

$$
\begin{aligned}
&\|(\mathscr{E})(\delta_2)-(\mathscr{E})(\delta_1)\|\leqslant\|\mathscr{G}[P_{\zeta}(\delta_2)-P_{\zeta}(\delta_1)][\Theta(0)-\mathscr{S}_1(0,w_0)]\|\\
&+\|\tfrac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta_1}\Big[(\delta_2-\iota)^{\zeta-1}-(\delta_1-\iota)^{\zeta-1}\Big]\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}_2\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)+\sigma^*\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|\\
&+\|\tfrac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta_1}[Q_{\zeta}(\delta_2-\iota)-Q_{\zeta}(\delta_1-\iota)]\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}_2\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)+\hat{\mathscr{U}}\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|\\
&+\|\tfrac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_{\delta_1}^{\delta_2}(\delta_2-\iota)^{\zeta-1}\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}_2\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)+\sigma^*\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|\\
&+\|\tfrac{\zeta\mathscr{G}^2}{B(\zeta)}\int_{\delta_1}^{\delta_2}Q_{\zeta}(\delta_2-\iota)\\
&\Big[\mathscr{S}_2\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota},\int_0^{\iota}\mathscr{R}_2\Big(\iota,\theta,\mathscr{M}_{\theta}+\hat{\Theta}_{\theta}\Big)d\theta\Big)+\hat{\mathscr{U}}\mathscr{S}_1\Big(\iota,\mathscr{M}_{\iota}+\hat{\Theta}_{\iota}\Big)\Big]d\iota\|.
\end{aligned}
$$

When $\delta_1\to\delta_2\Rightarrow$ *RHS* tends to 0, and the compactness of strongly continuous operators $P_{\zeta}(\delta)$ and $Q_{\zeta}(\delta)$ for $\delta>0$ implicit the continuity in the uniform operators topology.

$\Rightarrow\hat{\mathscr{E}}(B_{\xi})$ is equicontinuous.

Step 4: To prove: Mönch's condition holds.

Let $\Xi\subseteq\mathscr{B}_{\xi}$ is countable and $\Xi\subseteq\overline{conv}(\{0\}\cup\hat{\mathscr{E}}(\Xi))$. Now, we show that $\alpha(\Xi)=0$ where α Hausdorff MNC. Without loss of generality we consider $\Xi=\{\mathscr{M}^n\}_{n=1}^{\infty}$. Now we need to show that $\hat{\mathscr{E}}(\Xi)(\delta)$ is relatively compact in $\mathscr{X}\ \forall\delta\in\mathscr{J}$. By referring lemma (2.12) we have,

$$
\begin{aligned}
&\alpha(\{\hat{\mathscr{E}}(\sigma)_{n=1}^{\infty}\})\\
&\leqslant\alpha\Big\{\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\\
&\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)d\iota\\
&+\frac{\mathscr{K}\mathscr{G}(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\sigma^*\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\iota\\
&+\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\iota)\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)d\iota\\
&+\frac{\zeta\mathscr{G}^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\iota)\hat{\mathscr{U}}\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\iota\Big\}_{n=1}^{\infty}\\
&\leqslant 2\Big\{\frac{\mu\upsilon(1-\zeta)}{B(\zeta)\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\\
&\Big[\alpha\Big(\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\Big)\Big]d\iota\\
&+\frac{\mu\upsilon}{\Gamma(\zeta)}\int_0^{\delta}(\delta-\iota)^{\zeta-1}\Big[\alpha\Big(\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\iota}+\hat{\Theta}_{n\iota}\Big)\Big)\Big]d\iota\\
&+\frac{\zeta\upsilon^2}{B(\zeta)}\int_0^{\delta}Q_{\zeta}(\delta-\iota)\\
&\Big[\alpha\Big(\mathscr{S}_2\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta},\int_0^{\iota}\mathscr{R}\Big(\iota,\theta,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)d\theta\Big)\Big)\Big]d\iota\\
&+\upsilon\mu\int_0^{\delta}Q_{\zeta}(\delta-\iota)\hat{\mathscr{U}}\Big[\alpha\Big(\mathscr{S}_1\Big(\iota,\mathscr{M}_{n\theta}+\hat{\Theta}_{n\theta}\Big)\Big)\Big]d\iota\Big\}\\
&\leqslant 2\Big\{\Big[\frac{\mu\upsilon(1-\zeta)P^{\zeta}}{B(\zeta)\Gamma(\zeta+1)}+\frac{\upsilon^2\hat{\mathscr{T}}_1P^{\zeta}}{B(\zeta)}\Big]\mathscr{L}_1(1+2\mathscr{L}_2^*)\alpha(w(\iota))\\
&+\Big[\frac{\upsilon\mu P^{\zeta}}{\Gamma(\zeta+1)}+\frac{\upsilon\mu\hat{\mathscr{T}}_1P^{\zeta}}{\zeta}\mathscr{L}_3\Big]\alpha(w(\iota))\Big\}\\
&\leqslant 2[\xi^*\mathscr{L}_1(1+2\mathscr{L}_2^*)+\xi^{**}\mathscr{L}_3]\alpha(w(\iota)),
\end{aligned}
$$

$\Rightarrow$ from lemma (2.10),

$\alpha(\hat{\mathscr{E}}(\Xi)) \leqslant \tilde{\mathscr{L}}\alpha(\Xi)$.

Through Mönch's condition, we get.

$\alpha(\Xi) \leqslant \alpha(\overline{conv}(\{0\} \cup \hat{\mathscr{E}}(\Xi))) = \alpha(\hat{\mathscr{E}}(\Xi)) \leqslant \tilde{\mathscr{L}}\alpha(\Xi)$, which gives $\alpha(\Xi) = 0$. Thus, from Theorem (2.13) $\hat{\mathscr{E}}$ has a fixed point $\mathscr{M} \in \mathscr{B}_\xi$, then $w = \mathscr{M} + \hat{\Theta}$ is the mild solution of the system (1). This completes the proof.

4. Example

This part focus mostly on the application of our theoretical findings.

$$
\begin{aligned}
&{}^{ABC}D^{\zeta}_{\delta}\left[w(\delta,\theta) - \frac{e^{-\delta}}{25+e^{\delta}}\left(\frac{|w(\delta-q,\theta)|}{1+|w(\delta-q,\theta)|}\right)\right] = \frac{\partial^2}{\partial\theta^2}w(\delta,\theta)\\
&+\frac{e^{-\delta}}{49+e^{\delta}}\left(\frac{|w(\delta-q,\theta)|}{1+|w(\delta-q,\theta)|}\right) + \int_0^{\delta}\left(\frac{e^{-\iota}}{50}\right)\frac{|w(\iota-q,\theta)|}{1+|w(\iota-q,\theta)|}d\iota,\\
&\quad \delta\in[0,1], \delta \neq \frac{1}{2},\\
&w(\delta,0) = w(\delta,\pi) = 0, \quad \delta\in[0,1],\\
&w(\delta,\theta) = \Theta(\delta,\theta), \quad \delta\in[-q,0], \theta\in[0,\pi].
\end{aligned} \tag{4.1}
$$

Set $\mathscr{X} = L^2[0,\pi]$, and $\hat{\mathscr{U}} : D\left(\hat{\mathscr{U}}\right) \subset \mathscr{X} \to \mathscr{X}$ an operator defined as $\hat{\mathscr{U}}\mathscr{Z} = \mathscr{Z}'', \mathscr{Z} \in D\left(\hat{\mathscr{U}}\right)$, whereas the domain $D\left(\hat{\mathscr{U}}\right) = \{\mathscr{Z} \in \mathscr{X}; \mathscr{Z}, \mathscr{Z}'$ are absolutely continuous, $\mathscr{Z}'' \in \mathscr{X}, \mathscr{Z}(0) = \mathscr{Z}(1) = 0\}$. Then

$$\hat{\mathscr{U}}\mathscr{Z} = \sum_{n=1}^{\infty} n^2(\mathscr{Z},\mathscr{Z}_n)\mathscr{Z}_n, \mathscr{Z} \in D\left(\hat{\mathscr{U}}\right).$$

At this moment $\mathscr{Z}_n(\iota) = \sqrt{\frac{2}{\pi}}\sin(n\iota), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of $\hat{\mathscr{U}}$. It is obvious that $\hat{\mathscr{U}}$ is a generator of an analytic semigroup $(P(\delta))_{\delta\leqslant 0}$ in $\mathscr{X}$ defined as

$$P(\delta)\mathscr{Z} = \sum_{n=1}^{\infty} e^{-n^2\delta}(\mathscr{Z},\mathscr{Z}_n)\mathscr{Z}_n, \mathscr{Z} \in \mathscr{X}, \delta > 0.$$

Hence $(P(\delta))_{t\geqslant 0}$ is a uniformly bounded compact semigroup, in order that $R\left(\lambda, \hat{\mathscr{U}}\right) = \left(\lambda - \hat{\mathscr{U}}\right)^{-1}$ is a compact operator $\forall$ $\lambda \in \mu\left(\hat{\mathscr{U}}\right) \Rightarrow \hat{\mathscr{U}} \in \hat{\mathscr{U}}^{\zeta}(\alpha_0, \mathscr{Z}_0)$. Futhermore, the subordination principle of solution operator $(P_\zeta(\delta))_\delta \geqslant 0 \ni \|P_\zeta(\delta)\| \leqslant \hat{\mathscr{M}}$ for $\delta \in [0,1]$.

Thus, for $(\delta,\theta) \in [0,1]\times[0,\pi], \alpha \in [-q,0]$ and $\phi \in \mathscr{C}([0,1],[-q,1])$, where

$$
\begin{aligned}
w(\delta)(\theta) = w(\delta,\theta), \mathscr{S}_2&\left(\delta, w_\delta, \int_0^{\delta}\mathscr{R}(\delta,\sigma,w_\sigma)d\sigma\right)(\theta)\\
&= \frac{e^{-\delta}}{49+e^{\delta}}\left(\frac{|w(\delta-q,\theta)||}{1+|w(\delta-q,\theta)|}\right) + \int_0^{\delta}\left(\frac{e^{-\iota}}{50}\right)\frac{|w(\iota-q,\theta)|}{1+|w(\iota-q,\theta)|}d\iota,\\
\mathscr{S}_1(\delta, w_\delta)(\theta) &= \frac{e^{-\delta}}{25+e^{\delta}}\left(\frac{|w(\delta-q,\theta)||}{1+|w(\delta-q,\theta)|}\right).
\end{aligned}
$$

The system (4.1) is the theoretical form of (1). Additionally, the conditions $(\mathbf{H_1})$-$(\mathbf{H_4})$ are fulfilled. Hence there exist at least one mild solution in the system (4.1).

5. Conclusion

As a result, we studied Atangana-Baeanu fractional neutral delay integro-differential systems in Banach spaces. We proved our major conclusions by applying the abstract notions associated with fractional calculus and the fixed point approach. Through Mönch fixed point theorem, the system existence is proved. The theoretical outcomes are demonstrated through an application provided. In the future, we will extend our text to study the controllability of Atangana-Baeanu fractional neutral delay integro-differential systems.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Authors' contributions

All the authors have contributed equally to this paper.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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