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ORIGINAL ARTICLE

Analytical approach to Fokker–Planck equation with space- and time-fractional derivatives by means of the homotopy perturbation method

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Abstract In this study, we present numerical solutions for the space- and time-fractional Fokker–Planck equation using the homotopy perturbation method (HPM). The fractional derivatives are described in the Caputo sense. The methods give an analytic solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. Some examples are given and comparisons are made, the comparisons show that the homotopy perturbation method is very effective and convenient and overcome the difficulty of traditional methods. The numerical results show that the approaches are easy to implement and accurate when applied to space- and time-fractional Fokker–Planck equations. The methods introduce a promising tool for solving many space–time fractional partial differential equations.

1. Introduction

The Fokker–Planck equation arises in various fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker–Planck equation was first used by Fokker and Plank (for instance, see Risken (1989)) to describe the Brownian motion of particles. A FPE describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form (Risken, 1989)

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u(x,t), \tag{1}$$

with the initial condition given by

$$u(x,0) = f(x), \quad x \in R, \tag{2}$$

where $B(x) > 0$ is the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Eq. (1) is a linear second-order partial differential equation of parabolic type.

There is a more general form of FPE which is called nonlinear Fokker–Planck equation. Nonlinear FPE has important

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applications in various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing (see Frank, 2004 and reference therein). In one variable case, the nonlinear FPE is written in the following form

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x,t,u)+\frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t), \tag{3}$$

with the initial condition given by

$$u(x,0)=f(x), \quad x\in R, \tag{4}$$

In recent years there has been a great deal of interest in fractional diffusion equations. These equations arise in continuous time random walks, modelling of anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon, and simplification of the results (Agrawal, 2002).

Our concern in this work is to consider the numerical solution of the nonlinear FPE with space- and time-fractional derivatives of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left[-\frac{\partial^\beta}{\partial x^\beta}A(x,t,u)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B(x,t,u)\right]u(x,t), \quad t>0, \quad 0<\alpha, \quad \beta\leqslant 1, \tag{5}$$

where α and β are parameters describing the order of the fractional time- and space derivatives, respectively. The function $u(x, t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical nonlinear FPE (3).

The objective of this paper is to extend the application of the homotopy perturbation method (HPM) to obtain analytic solutions to the space- and time-fractional Fokker–Planck equations. The homotopy perturbation method is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduce the solution in the form of a convergent fractional series with elegantly computable terms.

The homotopy perturbation method was first proposed by the Chinese mathematician Ji-Huan He (He and Wu, 2006; He, 2004, 2005a,b, 2006c, 1999, 2000, 2003). The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and non-linear equations (Öziş and Yıldırım, 2007a,b,c,d; Yıldırım and Öziş, 2007; Yıldırım, 2008a, 2010, 2008b,c; Shakeri and Dehghan, 2007; Dehghan and Shakeri, 2007, 2008; Shakeri and Dehghan, 2008; Saadatmandi et al., 2009; Yusufoğlu 2007a,b; Chowdhury and Hashim, 2009a,b). The interested reader can see the Refs. He (2006a,b, 2008) for last development of HPM. This homotopy perturbation method will become a much more interesting method to solving nonlinear differential equations in science and engineering. We extend the method to solve the space- and time-fractional Fokker–Planck equations.

2. Fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu\in R$ if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu$, $m\in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha\geqslant 0$, of a function $f\in C_\mu, \mu\geqslant -1$, is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \qquad \alpha>0, \quad x>0,$$
$$J^0 f(x)=f(x).$$

Properties of the operator J^α can be found in Refs. Miller and Ross (1993), Samko et al. (1993) Oldham and Spanier (1974), we mention only the following. For $f\in C_\mu, \mu\geqslant -1$, $\alpha,\beta\geqslant 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta = J^{\alpha+\beta}f(x)$,
2. $J^\alpha J^\beta = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model realworld phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity (Luchko and Gorneflo, 1998).

Definition 2.3. The fractional derivative $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x)=J^{m-\alpha}D^m f(x) =\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt, \tag{6}$$

for $m-1<\alpha\leqslant m, \quad m\in N, \quad x>0, \quad f\in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If* $m-1<\alpha\leqslant m$, $m\in N$ *and* $f\in C_\mu^m, \mu\geqslant -1$, *then*

$$D^\alpha J^\alpha f(x)=f(x), \quad \textit{and},$$
$$J^\alpha D^\alpha f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{x^k}{k!}, \quad x>0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the fractional partial differential equations in fluid mechanics, and the fractional derivatives are taken in Caputo sense as follows.

Definition 2.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}d\tau, & \text{for } m-1<\alpha<m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases} \tag{7}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. Basic ideas of HPM

To illustrate the basic idea of He's homotopy perturbation method, consider the following general nonlinear differential equation;

$$A(u) - f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{8}$$

with boundary conditions;

$$B(u, \partial u/\partial n) = 0, \quad \mathbf{r} \in \Gamma \tag{9}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided in to two parts L and N, where L is linear, and N is nonlinear, therefore Eq. (8) can be written as,

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{10}$$

By using homotopy technique, one can construct a homotopy $v(r,p){:}\Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \tag{11.a}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{11.b}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation of Eq. (8) which satisfies the boundary conditions. Clearly, we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{12}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{13}$$

the changing process of p from zero to unity is just that of $v(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If, the embedding parameter p; $(0 \leqslant p \leqslant 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can naturally assume that the solution of Eqs. (12) and (13) can be given as a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{14}$$

and setting $p = 1$ results in the approximate solution of Eq. (11) as;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{15}$$

The convergence of series (15) has been proved by He (2004). It is worth to note that the major advantage of He's homotopy perturbation method is that the perturbation equation can be freely constructed in many ways (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

4. Applications

In this section we shall illustrate the homotopy perturbation technique by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special case $\alpha = 1$ and $\beta = 1$, is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the space- and time-fractional derivatives on the behavior of the solution. All the results are calculated by using the symbolic calculus software Maple.

Example 1. Consider the linear space fractional FPE

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial^\beta}{\partial x^\beta}.x + \frac{\partial^{2\beta}}{\partial x^{2\beta}}.\frac{x^2}{2}\right]u(x,t), \quad t > 0, \quad x > 0, \tag{16}$$

where $0 < \beta \leqslant 1$, subject to the initial condition

$$u(x,0) = x. \tag{17}$$

To solve Eqs. (16) and (17) by homotopy perturbation method, we construct the following homotopy:

$$\left(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}\right) = p\left(-\frac{\partial^\beta (xu)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2 u}{2}\right)}{\partial x^{2\beta}} - \frac{\partial u_0}{\partial t}\right), \tag{18}$$

Assume the solution of Eq. (18) to be in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots\dots \tag{19}$$

Substituting Eq. (19) into Eq. (18) and collecting terms of the same power of p give

$$p^0 : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \tag{20}$$

$$p^1 : \frac{\partial u_1}{\partial t} = -\frac{\partial^\beta (xu_0)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2 u_0}{2}\right)}{\partial x^{2\beta}} - \frac{\partial u_0}{\partial t}, \tag{21}$$

$$p^2 : \frac{\partial u_2}{\partial t} = -\frac{\partial^\beta (xu_1)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2 u_1}{2}\right)}{\partial x^{2\beta}}, \tag{22}$$

$$p^3 : \frac{\partial u_3}{\partial t} = -\frac{\partial^\beta (xu_2)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2 u_2}{2}\right)}{\partial x^{2\beta}}, \tag{23}$$

$$\vdots$$

The given initial value admits the use of

$$u_0(x,t) = x, \tag{24}$$

The solution reads

$$u_1(x,t) = \left[\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)} - \frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right]t, \quad (25)$$

$$u_2(x,t) = \left[\begin{array}{l} \frac{2\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)} - \left(\frac{3\Gamma(5-2\beta)}{\Gamma(4-2\beta)} + \frac{\Gamma(5-\beta)}{\Gamma(3-\beta)}\right)\frac{x^{4-3\beta}}{\Gamma(5-3\beta)} + \\ \frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)} \end{array}\right]\frac{t^2}{2}, \quad (26)$$

⋮

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained.

The solution of Eqs. (16) and (17) can be obtained by setting $p = 1$ in Eq. (19):

$$u = u_0 + u_1 + u_2 + u_3 + \ldots\ldots \quad (27)$$

Thus, we have

$$u(x,t) = x + \left[\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)} - \frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right]t + \left[\frac{2\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)} - \left(\frac{3\Gamma(5-2\beta)}{\Gamma(4-2\beta)} + \frac{\Gamma(5-\beta)}{\Gamma(3-\beta)}\right) \times \frac{x^{4-3\beta}}{\Gamma(5-3\beta)} + \frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^2}{2} + \ldots, \quad (28)$$

Setting $\beta = 1$ in (28), we reproduce the solution of problem as follows

$$u(x,t) = x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots..\right), \quad (29)$$

This solution is equivalent to the exact solution in a closed form

$$u(x,t) = xe^t \quad (30)$$

It is clear that no linearization or perturbation was used and a closed form solution is obtainable by adding more terms to the homotopy perturbation series.

The results for the exact solution (30) and the approximate solution (28) obtained using the homotopy perturbation method, for the special case $\beta = 1$, are shown in Fig. 1. It can be seen from Fig. 1 that the solution obtained by the present method is nearly identical with the exact solution. Fig. 2a and b show the approximate solutions when $\beta = 0.5$ and $\beta = 0.75$, respectively. It is to be noted that only the second-order term of the homotopy perturbation solution was used in evaluating the approximate solutions for Fig. 2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $u(x, t)$ when the homotopy perturbation method is used.

Example 2. Consider the nonlinear time-fractional FPE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}.\left(\frac{4u}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}.u\right]u(x,t), \quad t > 0, \quad x > 0, \quad (31)$$

where $0 < \alpha \leqslant 1$, subject to the initial condition

$$u(x,0) = x^2. \quad (32)$$

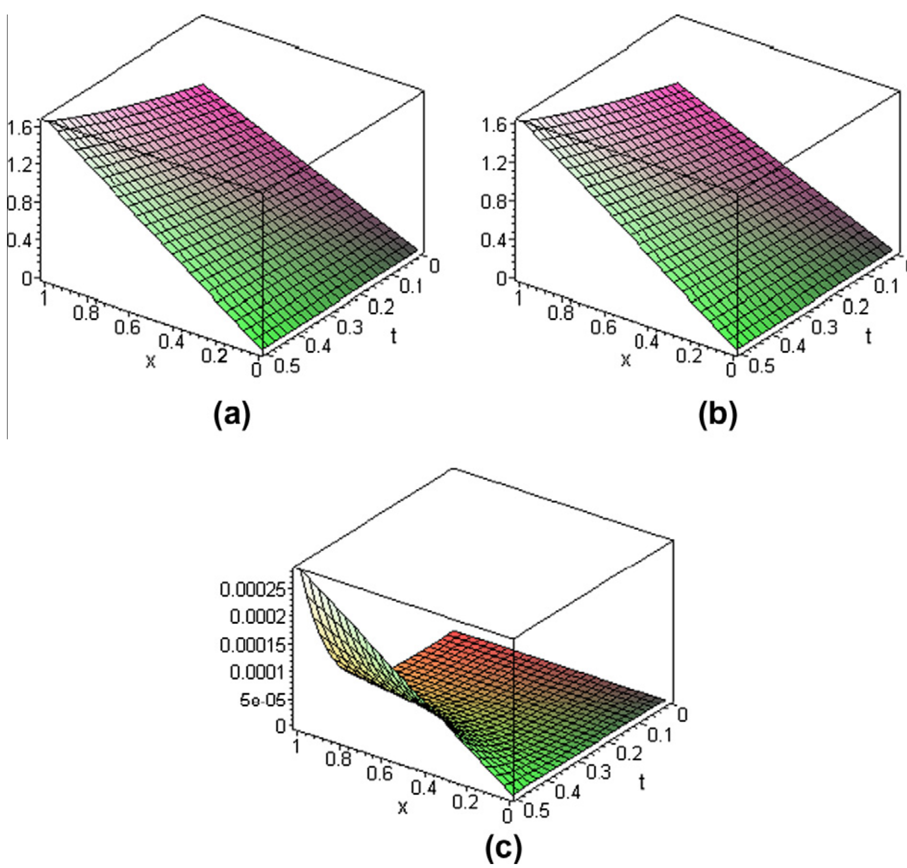

Figure 1 The surface shows the solution $u(x, t)$ for Eqs. (16) and (17) when $\beta = 1$: (a) exact solution (30) (b) approximate solution (29) (c) $|u_{ex} - u_{app}|$.

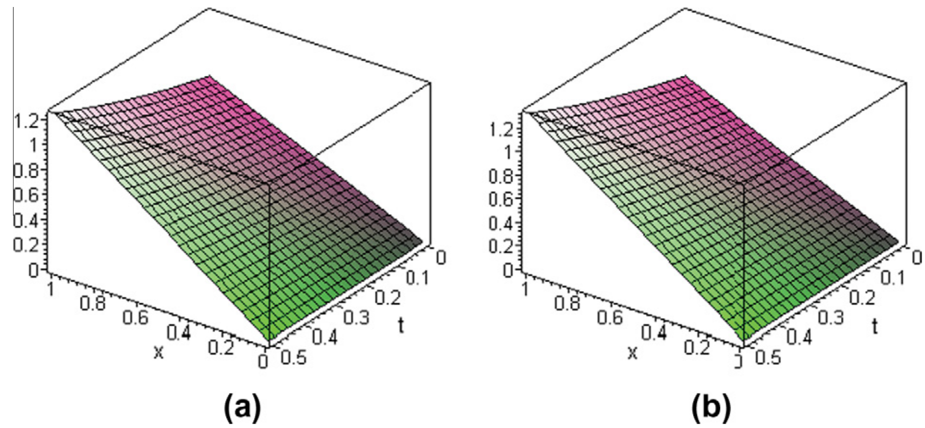

Figure 2 The surface shows the solution $u(x,\ t)$ for Eqs. (16) and (17): (a) $\beta = 0.5$, (b) $\beta = 0.75$.

To solve Eqs. (31) and (32) by homotopy perturbation method, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) = p\left(-\frac{\partial\left(\frac{4u^2}{x} - \frac{xu}{3}\right)}{\partial x} + \frac{\partial^2(u^2)}{\partial x^2} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right), \tag{33}$$

Substituting Eq. (19) into Eq. (33) and collecting terms of the same power of p give

$$p^0 : \frac{\partial^\alpha u_0}{\partial t} - \frac{\partial^\alpha u_0}{\partial t} = 0, \tag{34}$$

$$p^1 : \frac{\partial^\alpha u_1}{\partial t} = -\frac{\partial\left(\frac{4u_0^2}{x} - \frac{xu_0}{3}\right)}{\partial x} + \frac{\partial^2(u_0^2)}{\partial x^2} - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{35}$$

$$p^2 : \frac{\partial^\alpha u_2}{\partial t} = -\frac{\partial\left(\frac{8u_0u_1}{x} - \frac{xu_1}{3}\right)}{\partial x} + \frac{\partial^2(2u_0u_1)}{\partial x^2}, \tag{36}$$

$$p^3 : \frac{\partial^\alpha u_3}{\partial t} = -\frac{\partial\left(\frac{8u_0u_2+4u_1^2}{x} - \frac{xu_2}{3}\right)}{\partial x} + \frac{\partial^2\left(2u_0u_2 + u_1^2\right)}{\partial x^2}, \tag{37}$$

$\vdots$

The given initial value admits the use of

$$u_0(x,t) = x^2, \tag{38}$$

The solution reads

$$u_1(x,t) = x^2\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{39}$$

$$u_2(x,t) = x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{40}$$

$$u_3(x,t) = x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{41}$$

$\vdots$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained.

Thus, we have

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right), \tag{42}$$

Setting $\alpha = 1$ in (42), we reproduce the solution of problem as follows

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots..\right), \tag{43}$$

This solution is equivalent to the exact solution in a closed form

$$u(x,t) = x^2e^t \tag{44}$$

The results for the exact solution (44) and the approximate solution (42) obtained using the homotopy perturbation method, for the special case $\alpha = 1$, are shown in Fig. 3. It can be seen from Fig. 3 that the solution obtained by the present method is nearly identical with the exact solution. Fig. 4a and b show the approximate solutions when $\alpha = 0.5$ and $\alpha = 0.75$, respectively. It is to be noted that only the third-order term of the homotopy perturbation solution was used in evaluating the approximate solutions for Fig. 4.

Example 3. Consider the linear space- and time-fractional FPE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta}\cdot\left(\frac{x}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\cdot\left(\frac{x^2}{12}\right)\right]u(x,t), \quad t > 0, \quad x > 0, \tag{45}$$

where $0 < \alpha, \beta \leqslant 1$, subject to the initial condition

$$u(x,0) = x^2. \tag{46}$$

To solve Eqs. (45) and (46) by homotopy perturbation method, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) = p\left(-\frac{\partial^\beta\left(\frac{xu}{6}\right)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2u}{12}\right)}{\partial x^{2\beta}} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right), \tag{47}$$

Substituting Eq. (19) into Eq. (47) and collecting terms of the same power of p give

$$p^0 : \frac{\partial^\alpha u_0}{\partial t} - \frac{\partial^\alpha u_0}{\partial t} = 0, \tag{48}$$

$$p^1 : \frac{\partial^\alpha u_1}{\partial t} = -\frac{\partial^\beta\left(\frac{xu_0}{6}\right)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2u_0}{12}\right)}{\partial x^{2\beta}} - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{49}$$

$$p^2 : \frac{\partial^\alpha u_2}{\partial t} = -\frac{\partial^\beta\left(\frac{xu_1}{6}\right)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2u_1}{12}\right)}{\partial x^{2\beta}}, \tag{50}$$

$$p^3 : \frac{\partial^\alpha u_3}{\partial t} = -\frac{\partial^\beta\left(\frac{xu_2}{6}\right)}{\partial x^\beta} + \frac{\partial^{2\beta}\left(\frac{x^2u_2}{12}\right)}{\partial x^{2\beta}}, \tag{51}$$

$\vdots$

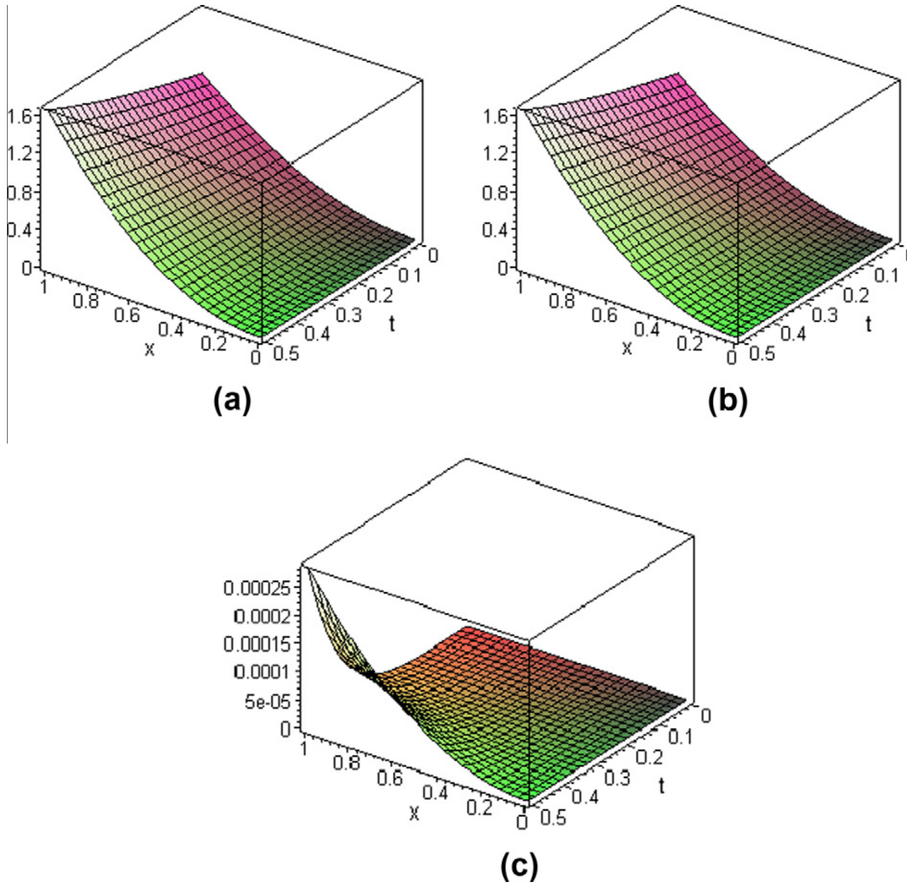

Figure 3 The surface shows the solution $u(x, t)$ for Eqs. (31) and (32) when $\alpha = 1$: (a) exact solution (44) (b) approximate solution (43) (c) $|u_{ex} - u_{app}|$.

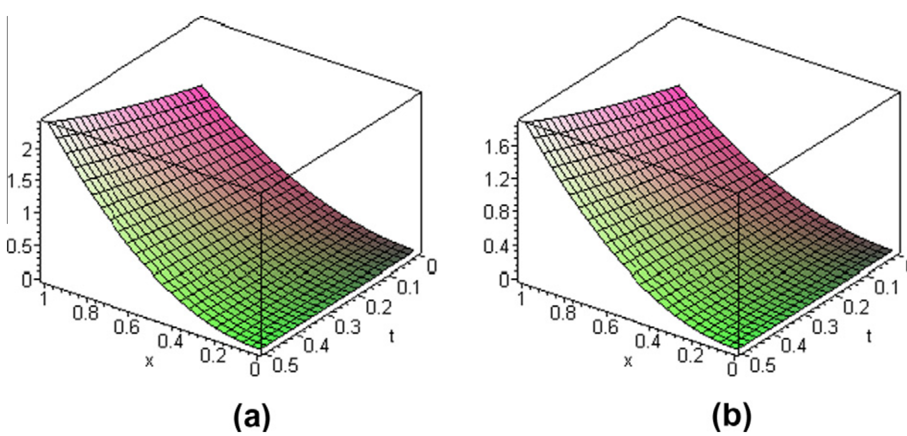

Figure 4 The surface shows the solution $u(x, t)$ for Eqs. (31) and (32): (a) $\alpha = 0.5$, (b) $\alpha = 0.75$.

The given initial value admits the use of

$$u_0(x,t) = x^2, \tag{52}$$

The solution reads

$$u_1(x,t) = \left[\frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{53}$$

$$u_2(x,t) = \left[\begin{array}{l}\frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} - \left(\frac{\Gamma(6-2\beta)}{3\Gamma(5-2\beta)} + \frac{\Gamma(6-\beta)}{12\Gamma(4-\beta)}\right)\frac{x^{5-3\beta}}{\Gamma(6-3\beta)} \\ + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)}\end{array}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{54}$$

$\vdots$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained. Thus, we have

$$\begin{aligned} u(x,t) = x^2 &+ \left[\frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \left[\frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} - \left(\frac{\Gamma(6-2\beta)}{3\Gamma(5-2\beta)} + \frac{\Gamma(6-\beta)}{12\Gamma(4-\beta)}\right)\frac{x^{5-3\beta}}{\Gamma(6-3\beta)} \right. \\ &\left. + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \end{aligned} \tag{55}$$

Setting $\alpha = 1$ and $\beta = 1$ in (55), we reproduce the solution of problem as follows

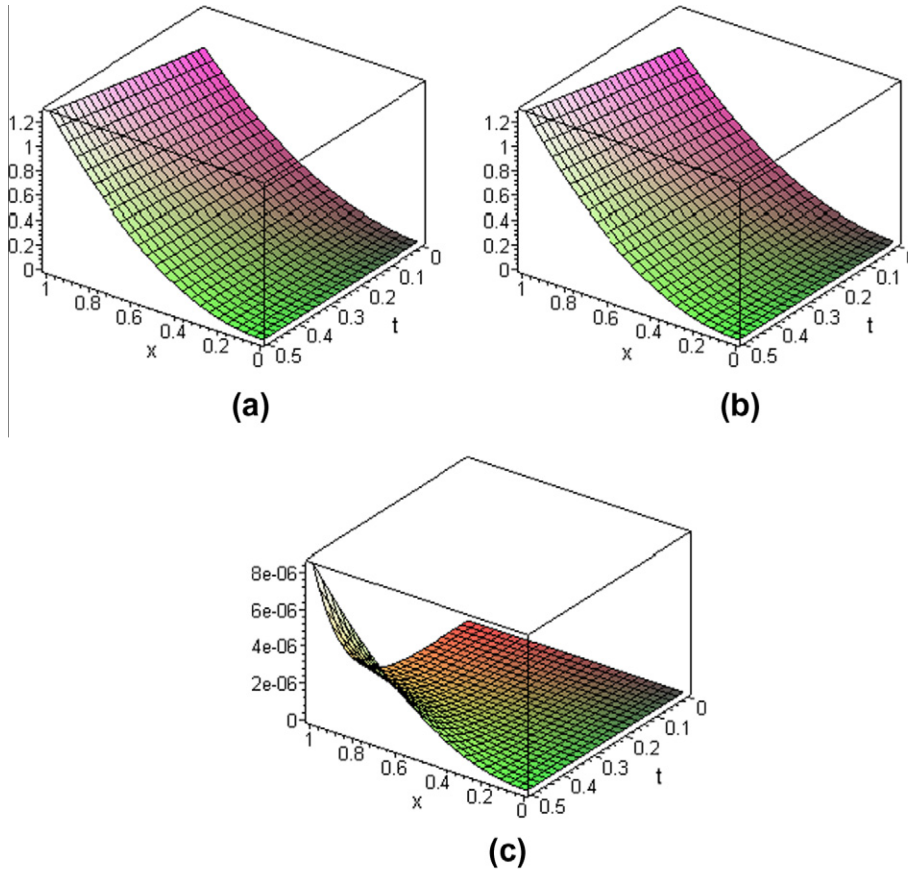

Figure 5 The surface shows the solution $u(x, t)$ for Eqs. (45) and (46) when $\alpha = 1$, $\beta = 1$: (a) exact solution (57) (b) approximate solution (56) (c) $|u_{ex} - u_{app}|$.

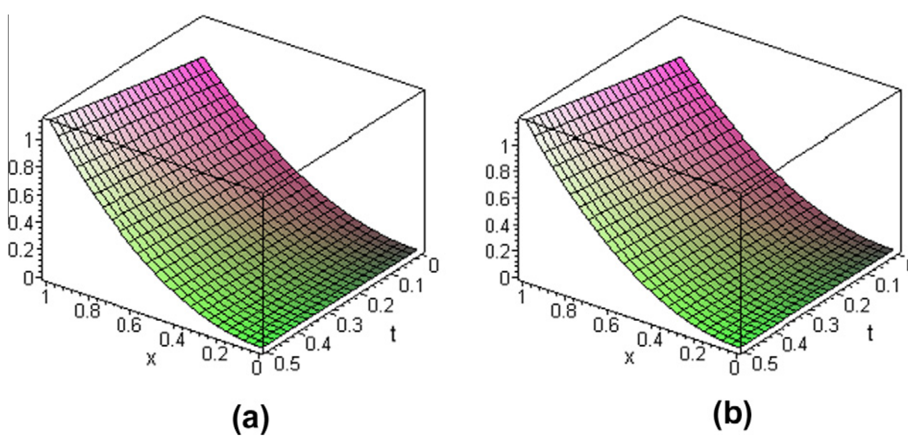

Figure 6 The surface shows the solution $u(x, t)$ for Eqs. (45) and (46): (a) $\alpha,\beta = 0.5$, (b) $\alpha,\beta = 0.75$.

$$u(x,t) = x^2\left(1 + \left(\frac{t}{2}\right) + \frac{\left(\frac{t}{2}\right)^2}{2!} + \frac{\left(\frac{t}{2}\right)^3}{3!} + \frac{\left(\frac{t}{2}\right)^4}{4!} + \dots\right), \tag{56}$$

This solution is equivalent to the exact solution in a closed form

$$u(x,t) = x^2 e^{t/2} \tag{57}$$

The results for the exact solution (57) and the approximate solution (55) obtained using the homotopy perturbation method, for the special case $\alpha = 1$ and $\beta = 1$, are shown in Fig. 5. It can be seen from Fig. 5 that the solution obtained by the present method is nearly identical with the exact solution. Fig. 6a and b show the approximate solutions when $\alpha,\beta = 0.5$ and $\alpha,\beta = 0.75$, respectively. It is to be noted that only the second-order term of the homotopy perturbation solution was used in evaluating the approximate solutions for Fig. 6.

5. Conclusion

In this study, the homotopy perturbation method is implemented to solve the space- and time-fractional Fokker–Planck equation. It may be concluded that the method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of space–time fractional partial differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The study shows that the technique requires less computational work than existing approaches while supplying quantitatively reliable results. Finally, the homotopy perturbation method is more effective and overcome the difficulty of traditional methods.

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