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Analysis of mixed type nonlinear Volterra–Fredholm integral equations involving the Erdélyi–Kober fractional operator

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ABSTRACT

This paper investigates the existence, uniqueness and stability of solutions to the nonlinear Volterra–Fredholm integral equations (NVFIE) involving the Erdélyi–Kober (E–K) fractional integral operator. We use the Leray–Schauder alternative and Banach's fixed point theorem to examine the existence and uniqueness of solutions, and we also explore Hyers–Ulam (H–U) and Hyers–Ulam–Rassias (H–U–R) stability in the space $C([0,\beta],\mathbb{R})$. Furthermore, three solution sets $U_{\sigma,\lambda}$, $U_{\theta,1}$ and $U_{1,1}$ are constructed for $\sigma > 0$, $\lambda > 0$, and $\theta \in (0,1)$, and then we obtain local stability of the solutions with some ideal conditions and by using Schauder fixed point theorem on these three sets, respectively. Also, to achieve the goal, we choose the parameters for the NVFIE as $\delta \in (\frac{1}{2},1)$, $\rho \in (0,1)$, $\gamma > 0$. Three examples are provided to clarify the results.

1. Introduction

Integral equations evolve spontaneously in various fields of basic sciences and engineering, like mathematical physics, solid state physics, astrophysics, microscopy, chemical reactions, plasma diagnostics, X-ray radiography, semiconductors, fluid flow, mathematical biology, scattering theory, etc. (Ganji, 2006; He, 2005; Liu and Gu, 2001; Rahman, 2007; Wazwaz, 2011; Marzban, 2023b; Marzban and Nezami, 2022; Marzban, 2023a; Marzban and Korooyeh, 2022; Marzban and Ashani, 2020; Rahimkhani and Ordokhani, 2023, 2022). Recent years have seen a major increase in interest in the theory of fractional integral equations, which is now a significant field of nonlinear analysis. Although integral equations containing the Erdélyi–Kober (E–K) fractional integral operator are typically used in kinetic theory of gases, to describe the medium with non-integer mass dimension, traffic theory, porous media, viscoelasticity, and electrochemistry (Alamo and Rodrıguez, 1994; Kilbas et al., 2006; Lakshmikantham et al., 2009; Hilfer, 2000; Kiryakova, 1994; Mainardi, 1997), it is essential to analyze such integral equations.

Analysis of the existence criteria for the solutions of different kinds of integral equations is an essential part of the study. One can use these requirements to identify the situation under which the problem's solution exists. The concepts of fixed-point approaches are significant in this sense.

In light of other viewpoints, stability is a crucial consideration for numerical solutions and might be necessary to compare the results and effectiveness of numerical methods. For instance, the papers in Refs. Nwaigwe (2022), Nwaigwe and Benedict (2023), Nwaigwe and Micula (2023), Nwaigwe et al. (2023) deal with numerical solutions of integral equations. Also, different forms of stability analysis have been performed on both differential and integral equations. In this regard, Lyapunov stability has been studied in a wide range of real-world problem settings. Further, exponential and Mittag-Leffler stabilities have been implemented for many topics. In recent times, researchers seem to be steadily more interested in H–U and H–U–R stability. In Refs. Akkouchi (2011), Ali et al. (2019), Amin et al. (2022b,a), Kumam et al. (2017), Morales and Rojas (2011), Subramanian et al. (2022),

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Paul et al. (2023), some current studies on H–U and H–U–R stabilities are included.

It ought to be noticed that Ma and Pečarić (2008) examined the following integral equation with the E–K fractional integral operator, i.e.,

$$\zeta(y) = g(y) + \frac{\lambda y^{-\rho s}}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^s \zeta(\mu)\, d\theta, \qquad y > 0. \tag{1.1}$$

An explicit bounds of the solution for Eq. (1.1) is achieved by constructing a generalized weakly singular integral inequality.

Wang et al. (2012) used the Schauder fixed point theorem to study the solvability of the following integral equation with the E–K fractional operator, i.e.,

$$\zeta(y) = g(y) + \frac{b(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H(\mu, \zeta(\mu))\, d\mu, \quad y \in [0,\beta], \quad \beta > 0. \tag{1.2}$$

In Wang et al. (2012), they also discussed the local stability result for the following nonlinear integral equation, i.e.,

$$\zeta(y) = g(y) + \frac{b(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H(y, \mu, \zeta(\mu))\, d\mu, \qquad y \in [0,\infty).$$

In 2022, Amin et al. (2022b,a) have examined the uniqueness and H–U stability to the solution of the mixed type Volterra–Fredholm fractional integral equations, those are,

$$\begin{aligned} \zeta(y) = g(y) &+ \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y - \mu)^{\delta-1} H_1(\mu, \zeta(\mu))\, d\mu \\ &+ \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta - \mu)^{\delta-1} H_2(\mu, \zeta(\mu))\, d\mu, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \zeta(y) = g(y) &+ \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y - \mu)^{\delta-1} \zeta(\mu)\, d\mu \\ &+ \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta - \mu)^{\delta-1} \zeta(\mu)\, d\mu, \quad y \in [0,\beta], \ \beta > 0. \end{aligned} \tag{1.4}$$

Motivated by the above literature's, this paper begins the investigation into the existence, uniqueness, H–U–R stability, H–U stability and local stability to the solutions of the nonlinear Volterra–Fredholm integral equation containing the Erdélyi–Kober fractional integral operator, i.e.,

$$\begin{aligned} \zeta(y) = g(y) &+ \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y, \mu, \zeta(\mu)) d\mu \\ &+ \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y, \mu, \zeta(\mu)) d\mu, \\ & y \in \mathbb{R}_+, \end{aligned} \tag{1.5}$$

where δ, ρ, and γ are positive parameters, $\mathbb{R}_+ = [0,\infty)$, $0 < \beta < \infty$, the functions $g, b_1, b_2 : \mathbb{R}_+ \to \mathbb{R}$ and $H_1, H_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. More assumptions will be discussed later.

To establish our proposed results, we use three useful theorems namely, Leray–Schauder alternative, Schauder fixed point theorem and Banach's fixed point theorem.

This paper is arranged as follows: Notations and supporting information are included in Section 2. In Section 3, theoretical analysis of existence and uniqueness of solutions have been discussed under some suitable conditions. Stability results of solutions have been given under some interesting conditions in Section 4. Three examples are discussed to interpret our established results in Section 5. Conclusions with notions for further research are discussed in Section 6.

2. Notations, definitions and auxiliary facts

Let $\mathcal{A} = [0,\beta]$, where $0 < \beta < \infty$. Let $V = C(\mathcal{A}, \mathbb{R})$ be the space of all continuous functions $v(y) : \mathcal{A} \to \mathbb{R}$. Let $D = C(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions $d(y) : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}$ is the set of real numbers. Then $(V, \|.\|)$ and $(D, \|.\|)$ are the Banach spaces with norm $\|v\| = \sup\{|v(y)| : y \in [0,\beta]\}$ and $\|d\| = \sup\{|d(y)| : y \in \mathbb{R}_+\}$, respectively.

Definition 2.1. (Pagnini, 2012; Kilbas et al., 2006). The Erdélyi–Kober fractional integral of a continuous function $\zeta : [0,\infty) \to \mathbb{R}$ is defined by

$$\mathcal{I}_\rho^{s,\delta} \zeta(y) = \frac{\rho y^{-\rho(\delta+s)}}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^{\rho s+\rho-1} \zeta(\mu)\, d\mu,$$

with $s \in \mathbb{R}$, and $\delta, \rho > 0$, provided the right side is point-wise defined on $[0,\infty)$.

Lemma 2.2. *(Prudnikov et al., 1981). For $0 < \mu < 1$ and $0 < p < q$, we have $|p^\mu - q^\mu| \le (p-q)^\mu$.*

Lemma 2.3. *(Prudnikov et al., 1981). Let ρ, σ, v and q be positive constants, then*

$$\int_0^y (y^\rho - \mu^\rho)^{v(\sigma-1)} \mu^{v(q-1)}\, d\mu = \frac{y^\alpha}{\rho} \mathbb{B}\left(\frac{v(q-1)+1}{\rho}, v(\sigma-1)+1\right), \quad y \in \mathbb{R}_+,$$

where $\alpha = v\big[\rho(\sigma-1) + q - 1\big] + 1$,
and $\mathbb{B}(r,x) = \int_0^1 \theta^{r-1}(1-\theta)^{x-1} d\theta$, $\big(Re(r) > 0, Re(x) > 0\big)$, is the well-known Beta function.

Theorem 2.4. *(Deimling, 1985). If $\mathcal{U}$ is a nonempty closed, bounded convex subset of a Banach space $\mathcal{X}$ and $f : \mathcal{U} \to \mathcal{U}$ is completely continuous, then f has a fixed point in $\mathcal{U}$.*

Theorem 2.5 (*Leray–Schauder alternative (Subramanian et al., 2022)*)**.** *Let $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ be a completely continuous operator. Let $\Omega = \{y \in \mathcal{E} : y = \eta\mathcal{M}(y)$, for some $0 < \eta < 1\}$. Then, either the set Ω is unbounded or $\mathcal{M}$ has at least one fixed point.*

Theorem 2.6 (*Arzelà-Ascoli theorem (Subramanian et al., 2022)*)**.** *A subset $\mathcal{H}$ in $\mathcal{E}([c,d], \mathbb{R})$ is relatively compact if it is uniformly bounded and equicontinuous on $[c,d]$.*

Theorem 2.7 (*Banach's fixed point theorem (Banach, 1922)*)**.** *Assume that V is a Banach space. Every contraction mapping $\mathcal{M}$ defined on V into itself has a unique fixed point in V.*

The following Definitions 2.8 and 2.9 are stated in the sense of the papers given in Refs. Akkouchi (2011), Amin et al. (2022b,a), Morales and Rojas (2011), Paul et al. (2023).

Definition 2.8. The Eq. (1.5) has the H–U–R stability, if for each $\zeta(y) \in V$ satisfying

$$\begin{aligned} \Big|\zeta(y) - g(y) &- \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y, \mu, \zeta(\mu)) d\mu \\ &- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y, \mu, \zeta(\mu)) d\mu\Big| \le w(y), \end{aligned} \tag{2.6}$$

where $w(y) : \mathcal{A} \to \mathbb{R}_+$, $\exists$ a solution $\zeta_s(y) \in V$ of Eq. (1.5) and a constant $\Lambda > 0$ independent of ζ and ζ_s with $|\zeta(y) - \zeta_s(y)| \le \Lambda w(y)$, for all $y \in \mathcal{A}$.

Definition 2.9. We say that the Eq. (1.5) has the H–U stability when w(y) is a constant function in Definition 2.8.

Furthermore, the following definition of the local stability is stated in the sense of the paper given in Ref. Wang et al. (2012).

Definition 2.10. If there exists a solution $\zeta(y)$ of Eq. (1.5) such that

$$\lim_{y \to +\infty} |\zeta(y)| = 0,$$

then the solution of Eq. (1.5) is said to be locally stable.

Now, to prove the main results, we introduce an operator $\mathcal{M}$ as

$$\mathcal{M}(\zeta(y)) = g(y) + \frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^\rho - \mu^\rho)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu + \frac{b_2(y)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu. \tag{2.7}$$

3. Existence and uniqueness of solutions

We consider the following assumptions for Eq. (1.5):

(A_1) $g, b_1, b_2, \int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu, \int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu \in V$.

(A_2) There exist constants $G>0$, $B_1>0$ and $B_2>0$, such that $|g(y)|\le G$, $|b_1(y)|\le B_1$ and $|b_2(y)|\le B_2$, for all $y\in[0,\beta]$.

(A_3) There exist constants $G_{H_1}>0$, $G_{H_2}>0$, such that $|H_1(y,\mu,\zeta_1) - H_1(y,\mu,\zeta_2)| \le G_{H_1}|\zeta_1-\zeta_2|$ and $|H_2(y,\mu,\zeta_1) - H_2(y,\mu,\zeta_2)|\le G_{H_2}|\zeta_1-\zeta_2|$, for all $\zeta_1,\zeta_2\in\mathbb{R}$ and $y,\mu\in[0,\beta]$ with $0\le\mu\le y$.

(A_4) There exist constants $N_1>0$ and $N_2>0$ such that $|H_1(y,\mu,\zeta)|\le N_1$ and $|H_2(y,\mu,\zeta)|\le N_2$, $\forall\ \zeta\in\mathbb{R}$, $y\in[0,\beta]$, $0\le\mu\le y$.

Theorem 3.1. *Assume that, assumptions (A_1)–(A_4) hold for Eq. (1.5), then with the parameters $\delta\in(\frac{1}{2},1)$, $\rho\in(0,1)$ and $\gamma>0$, Eq. (1.5) has at least one solution defined on $[0,\beta]$.*

Proof. Assumption (A_1) ensures that $\mathcal{M}: V\to V$.

Now we will establish this theorem in the following four steps:

Step 1. $\mathcal{M}$ is continuous.

Let $\zeta\in V$ and $\{\zeta_n\}$ be a sequence in V such that $\lim_{n\to\infty}\|\zeta_n-\zeta\|=0$. Then for $y\in\mathcal{A}$,

$$\begin{aligned}
\left|\mathcal{M}(\zeta_n(y)) - \mathcal{M}(\zeta(y))\right| &= \Big|\frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta_n(\mu))d\mu \\
&\quad + \frac{b_2(y)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta_n(\mu))d\mu \\
&\quad - \frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_2(y)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu\Big| \\
&\le \Big|\frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left[H_1(y,\mu,\zeta_n(\mu)) - H_1(y,\mu,\zeta(\mu))\right]d\mu\Big| \\
&\quad + \Big|\frac{b_2(y)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left[H_2(y,\mu,\zeta_n(\mu)) - H_2(y,\mu,\zeta(\mu))\right]d\mu\Big| \\
&\le \frac{B_1}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left|H_1(y,\mu,\zeta_n(\mu)) - H_1(y,\mu,\zeta(\mu))\right|d\mu \\
&\quad + \frac{B_2}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left|H_2(y,\mu,\zeta_n(\mu)) - H_2(y,\mu,\zeta(\mu))\right|d\mu \\
&\le \frac{B_1G_{H_1}}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left|\zeta_n(\mu)-\zeta(\mu)\right|d\mu \\
&\quad + \frac{B_2G_{H_2}}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left|\zeta_n(\mu)-\zeta(\mu)\right|d\mu \\
&\le \frac{B_1G_{H_1}}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma d\mu\|\zeta_n-\zeta\| \\
&\quad + \frac{B_2G_{H_2}}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma d\mu\|\zeta_n-\zeta\| \\
&\le \frac{B_1G_{H_1}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\|\zeta_n-\zeta\| \\
&\quad + \frac{B_2G_{H_2}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\|\zeta_n-\zeta\|.
\end{aligned}$$

This implies that, $\|\mathcal{M}(\zeta_n)-\mathcal{M}(\zeta)\|\to0$ as $n\to\infty$. So, the operator $\mathcal{M}$ is continuous.

Step 2. Bounded sets of V are mapped into bounded sets of V under the mapping $\mathcal{M}$.

Now, for $\zeta\in B_\epsilon$ and for all $y\in\mathcal{A}$, we get

$$\begin{aligned}
|\mathcal{M}(\zeta(y))| &= \Big|g(y) + \frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu \\
&\quad + \frac{b_2(y)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu\Big| \\
&\le \left|g(y)\right| + \frac{|b_1(y)|}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma|H_1(y,\mu,\zeta(\mu))|d\mu \\
&\quad + \frac{|b_2(y)|}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma\left|H_2(y,\mu,\zeta(\mu))\right|d\mu \\
&\le G + \frac{B_1N_1}{\Gamma(\delta)}\int_0^y (y^\rho-\mu^\rho)^{\delta-1}\mu^\gamma d\mu \\
&\quad + \frac{B_2N_2}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma d\mu \\
&\le G + \frac{B_1N_1}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1} \\
&\quad + \frac{B_2N_2}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}.
\end{aligned}$$

Thus,

$$|\mathcal{M}(\zeta(y))| \le G + \left[\frac{B_1N_1}{\rho\Gamma(\delta)} + \frac{B_2N_2}{\rho\Gamma(\delta)}\right]\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1},$$

i.e.,

$$\|\mathcal{M}(\zeta)\| < \infty.$$

Step 3. $\mathcal{M}(B_\epsilon)$ is equi-continuous.

Let $\zeta\in B_\epsilon$ and $y_2, y_1\in\mathcal{A}$ with $y_2>y_1$. Then

$$\begin{aligned}
|\mathcal{M}(\zeta(y_2)) - \mathcal{M}(\zeta(y_1))| &= \Big|g(y_2) \\
&\quad + \frac{b_1(y_2)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_2,\mu,\zeta(\mu))d\mu \\
&\quad + \frac{b_2(y_2)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_2,\mu,\zeta(\mu))d\mu \\
&\quad - g(y_1) - \frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_1,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_2(y_1)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_1,\mu,\zeta(\mu))d\mu\Big| \\
&\le \left|g(y_2)-g(y_1)\right| + \Big|\frac{b_1(y_2)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_2,\mu,\zeta(\mu))d\mu \\
&\quad + \frac{b_2(y_2)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_2,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_1,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_2(y_1)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_1,\mu,\zeta(\mu))d\mu\Big| \\
&\le \left|g(y_2)-g(y_1)\right| \\
&\quad + \Big|\frac{b_1(y_2)-b_1(y_1)+b_1(y_1)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_2,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_1,\mu,\zeta(\mu))d\mu\Big| \\
&\quad + \Big|\frac{b_2(y_2)-b_2(y_1)+b_2(y_1)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_2,\mu,\zeta(\mu))d\mu \\
&\quad - \frac{b_2(y_1)}{\Gamma(\delta)}\int_0^\beta (\beta^\rho-\mu^\rho)^{\delta-1}\mu^\gamma H_2(y_1,\mu,\zeta(\mu))d\mu\Big| \\
&\le \left|g(y_2)-g(y_1)\right| \\
&\quad + \frac{|b_1(y_2)-b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma\left|H_1(y_2,\mu,\zeta(\mu))\right|d\mu \\
&\quad + \frac{|b_1(y_1)|}{\Gamma(\delta)}\Big|\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y_2,\mu,\zeta(\mu))d\mu
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{y_1} \left(y_1^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma H_1(y_1,\mu,\zeta(\mu))d\mu \Big| \\
&+ \frac{|b_2(y_2)-b_2(y_1)|}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{|b_2(y_1)|}{\Gamma(\delta)} \Big| \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y_2,\mu,\zeta(\mu)) d\mu \\
&- \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y_1,\mu,\zeta(\mu)) d\mu \Big| \\
\le{}& \left|g(y_2)-g(y_1)\right| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma d\mu \\
&+ \frac{B_1}{\Gamma(\delta)} \Big| \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma H_1(y_2,\mu,\zeta(\mu)) d\mu \\
&- \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma H_1(y_1,\mu,\zeta(\mu)) d\mu \Big| + \frac{B_1}{\Gamma(\delta)} \\
&\Big| \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma H_1(y_1,\mu,\zeta(\mu)) d\mu \\
&- \int_0^{y_1} \left(y_1^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma H_1(y_1,\mu,\zeta(\mu)) d\mu \Big| \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu \\
\le{}& \left|g(y_2)-g(y_1)\right| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_2,\mu,\zeta(\mu)) - H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_1} \left[\left(y_1^\rho - \mu^\rho\right)^{\delta-1} - \left(y_2^\rho - \mu^\rho\right)^{\delta-1}\right] \mu^\gamma \left|H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1}{\Gamma(\delta)} \int_{y_1}^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu \\
\le{}& \left|g(y_2)-g(y_1)\right| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_2,\mu,\zeta(\mu)) - H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1N_1}{\Gamma(\delta)} \int_0^{y_1} \left[\left(\frac{1}{y_1^\rho - \mu^\rho}\right)^{1-\delta} - \left(\frac{1}{y_2^\rho - \mu^\rho}\right)^{1-\delta}\right] \mu^\gamma d\mu \\
&+ \frac{B_1N_1}{\Gamma(\delta)} \int_{y_1}^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma d\mu \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu \\
\le{}& \left|g(y_2)-g(y_1)\right| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_2,\mu,\zeta(\mu)) - H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1N_1}{\Gamma(\delta)} \int_0^{y_1} \left[\frac{y_2^\rho - y_1^\rho}{\left(y_1^\rho - \mu^\rho\right)^2}\right]^{1-\delta} \mu^\gamma d\mu + \frac{B_1N_1\beta^{\gamma+1-\rho}}{\rho\delta\Gamma(\delta)} \left(y_2^\rho - y_1^\rho\right)^\delta \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu \\
\le{}& |g(y_2)-g(y_1)| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_2,\mu,\zeta(\mu)) - H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1N_1\left(y_2 - y_1\right)^{\rho(1-\delta)}}{\Gamma(\delta)} \int_0^{y_1} \left(y_1^\rho - \mu^\rho\right)^{2(\delta-1)} \mu^\gamma d\mu \\
&+ \frac{B_1N_1\beta^{\gamma+1-\rho}}{\rho\Gamma(\delta+1)} \left(y_2 - y_1\right)^{\rho\delta} \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu \\
\le{}& |g(y_2)-g(y_1)| + \frac{|b_1(y_2)-b_1(y_1)|N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1}{\Gamma(\delta)} \int_0^{y_2} \left(y_2^\rho - \mu^\rho\right)^{\delta-1} \mu^\gamma \left|H_1(y_2,\mu,\zeta(\mu)) - H_1(y_1,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{B_1N_1\left(y_2 - y_1\right)^{\rho(1-\delta)}}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},2\delta-1\right) \beta^{2\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_1N_1\beta^{\gamma+1-\rho}}{\rho\Gamma(\delta+1)} \left(y_2 - y_1\right)^{\rho\delta} \\
&+ \frac{|b_2(y_2)-b_2(y_1)|N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y_2,\mu,\zeta(\mu)) - H_2(y_1,\mu,\zeta(\mu))\right| d\mu.
\end{aligned}
$$

Thus, $\left|\mathcal{M}(\zeta(y_2)) - \mathcal{M}(\zeta(y_1))\right| \to 0$ as $y_2 \to y_1$.

So, $\mathcal{M}(B_\epsilon)$ is equi-continuous.

Hence, combining all the above steps, the operator $\mathcal{M}$ is completely continuous by the consequence of Arzelà-Ascoli theorem.

Step 4. Let $\Omega = \{\zeta \in V : \zeta = \eta\mathcal{M}(\zeta),$ for some $0 < \eta < 1\}$.

We need to show that the set Ω is bounded.

Let $\zeta(y) \in \Omega$, this indicates that $\zeta(y) = \eta\mathcal{M}(\zeta(y))$, for some $0 < \eta < 1$.

Then for $y \in \mathcal{A}$, we obtain

$$
\begin{aligned}
\left|\zeta(y)\right| \le{}& \left|g(y)\right| + \frac{|b_1(y)|}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_1(y,\mu,\zeta(\mu))\right| d\mu \\
&+ \frac{|b_2(y)|}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \left|H_2(y,\mu,\zeta(\mu))\right| d\mu \\
\le{}& G + \frac{B_1N_1}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu + \frac{B_2N_2}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \\
\le{}& G + \frac{B_1N_1}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} \\
&+ \frac{B_2N_2}{\rho\Gamma(\delta)} \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1},
\end{aligned}
$$

i.e.,

$$
\left|\zeta(y)\right| \le G + \left[\frac{B_1N_1}{\rho\Gamma(\delta)} + \frac{B_2N_2}{\rho\Gamma(\delta)}\right] \mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right) \beta^{\rho(\delta-1)+\gamma+1} < \infty,
$$

which shows that the set Ω is bounded. Hence, by the Leray–Schauder alternative, $\mathcal{M}$ has at least one fixed point, which is a solution of Eq. (1.5). □

Theorem 3.2. *Assume that, conditions (A_1)–(A_3) hold for Eq. (1.5), and $\mathcal{G} = \max\{G_{H_1}, G_{H_2}\}$ satisfies the relation $0 < \mathcal{G} < \frac{\rho\,\Gamma(\delta)}{(B_1+B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}$, then with the parameters $\delta \in (\frac{1}{2},1)$, $\rho \in (0,1)$ and $\gamma > 0$, Eq. (1.5) has a unique solution defined on $[0,\beta]$.*

Proof. Assumption (A_1) ensures that $\mathcal{M} : V \to V$. Now we need to show that $\mathcal{M}$ is a contraction.

Let $\zeta_1, \zeta_2 \in V$, then $\forall y \in \mathcal{A}$, we get

$$
\begin{aligned}
\left|\mathcal{M}(\zeta_1(y)) - \mathcal{M}(\zeta_2(y))\right| ={}& \Big| \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta_1(\mu)) d\mu \\
&+ \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta_1(\mu)) d\mu \\
&- \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta_2(\mu)) d\mu
\end{aligned}
$$

$$- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta_2(\mu)) d\mu \Big|$$
$$\le \frac{|b_1(y)|}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_1(y,\mu,\zeta_1(\mu)) - H_1(y,\mu,\zeta_2(\mu)) \Big| d\mu$$
$$+ \frac{|b_2(y)|}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_2(y,\mu,\zeta_1(\mu)) - H_2(y,\mu,\zeta_2(\mu)) \Big| d\mu$$
$$\le \frac{B_1 G_{H_1}}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \|\zeta_1 - \zeta_2\|$$
$$+ \frac{B_2 G_{H_2}}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \|\zeta_1 - \zeta_2\|$$
$$\le \frac{\mathbb{B}\left(\frac{\gamma+1}{\rho}, \delta\right) \beta^{\rho(\delta-1)+\gamma+1}}{\rho} \left(\frac{B_1 G_{H_1}}{\Gamma(\delta)} + \frac{B_2 G_{H_2}}{\Gamma(\delta)} \right) \|\zeta_1 - \zeta_2\|$$
$$\le \frac{\mathcal{G}(B_1 + B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho}, \delta\right) \beta^{\rho(\delta-1)+\gamma+1}}{\rho\, \Gamma(\delta)} \|\zeta_1 - \zeta_2\|.$$

Thus,

$$\|\mathcal{M}(\zeta_1(y)) - \mathcal{M}(\zeta_2(y))\| \le \kappa \, \|\zeta_1 - \zeta_2\|.$$

As by the condition of $\mathcal{G}$, we obtain $\kappa = \frac{\mathcal{G}(B_1+B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\,\Gamma(\delta)} < 1$. So, $\mathcal{M}$ is a contraction. Hence, by the Banach's fixed point theorem, $\mathcal{M}$ has a unique fixed point, i.e., Eq. (1.5) has a unique solution. □

4. Stability of solutions

We consider the following assumptions for Eq. (1.5):

(S_1) $g, b_1, b_2, \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta(\mu)) d\mu, \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta(\mu)) d\mu \in D$. Furthermore, the functions $g, b_1, b_2 : \mathbb{R}_+ \to \mathbb{R}$ are bounded.

(S_2) There exist two continuous functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Big|H_1(y,\mu,\zeta_1) - H_1(y,\mu,\zeta_2)\Big| \le h_1(y)|\zeta_1 - \zeta_2|$ and $\Big|H_2(y,\mu,\zeta_1) - H_2(y,\mu,\zeta_2)| \le h_2(y)\Big|\zeta_1 - \zeta_2|$,
for all $\zeta_1, \zeta_2 \in \mathbb{R}$ and $y, \mu \in \mathbb{R}_+$ with $\mu \le y$.

(S_3) There exist two continuous functions $N_2, N_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\max\{\Big|H_2(y,\mu,0) : 0 \le \mu \le y\} = N_2(y)$ and $\max\{\Big|H_1(y,\mu,0)\Big| : 0 \le \mu \le y\} = N_1(y)$.

Theorem 4.1. *Assume that Eq. (1.5) meets all of the requirements of Theorem 3.2. Suppose $\zeta(y) \in V$ is such that it satisfies (2.6). Then the Eq. (1.5) has the H–U–R stability.*

Proof. According to Theorem 3.2, $\exists$ a unique solution $\zeta_s \in V$ of Eq. (1.5).

As stated in Definition 2.8, we need to show that $\exists$ a constant $\Lambda > 0$ such that, $\Big|\zeta(y) - \zeta_s(y)\Big| \le \Lambda w(y)$. Now,

$$\Big|\zeta(y) - \zeta_s(y)\Big| = \Big|\zeta(y) - g(y) - \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta_s(\mu)) d\mu$$
$$- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta_s(\mu)) d\mu \Big|$$
$$\le \Big|\zeta(y) - g(y) - \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta(\mu)) d\mu$$
$$- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta(\mu)) d\mu \Big|$$
$$+ \Big| \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta(\mu)) d\mu$$
$$- \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta_s(\mu)) d\mu \Big|$$
$$+ \Big| \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta(\mu)) d\mu$$
$$- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta_s(\mu))\, d\mu \Big|$$
$$\le w(y) + \frac{|b_1(y)|}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_1(y,\mu,\zeta(\mu)) - H_1(y,\mu,\zeta_s(\mu)) \Big| d\mu$$
$$+ \frac{|b_2(y)|}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_2(y,\mu,\zeta(\mu)) - H_2(y,\mu,\zeta_s(\mu)) \Big| d\mu$$
$$\le w(y) + \frac{B_1 G_{H_1}}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \|\zeta - \zeta_s\|$$
$$+ \frac{B_2 G_{H_2}}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma d\mu \|\zeta - \zeta_s\|$$
$$\le w(y) + \frac{\mathbb{B}\left(\frac{\gamma+1}{\rho}, \delta\right) \beta^{\rho(\delta-1)+\gamma+1}}{\rho} \left(\frac{B_1 G_{H_1}}{\Gamma(\delta)} + \frac{B_2 G_{H_2}}{\Gamma(\delta)} \right) \|\zeta - \zeta_s\|$$
$$\le w(y) + \frac{\mathcal{G}(B_1 + B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho}, \delta\right) \beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)} \|\zeta - \zeta_s\|.$$

Thus, $\Big|\zeta(y) - \zeta_s(y)\Big| \le \|\zeta - \zeta_s\| \le w(y) + \frac{\mathcal{G}(B_1+B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)} \|\zeta - \zeta_s\|$, which implies that, $\Big|\zeta(y) - \zeta_s(y)\Big| \le \|\zeta - \zeta_s\| \le \Lambda w(y)$, where $\Lambda = \frac{1}{1-\kappa} > 0$, as $\kappa = \frac{\mathcal{G}(B_1+B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)} < 1$, by the condition of $\mathcal{G}$.

So, the Eq. (1.5) has the H–U–R stability. □

Theorem 4.2. *Assume that Eq. (1.5) meets all of the requirements of Theorem 3.2. Let $\zeta(y) \in V$ and $\epsilon > 0$ such that satisfies*

$$\Big|\zeta(y) - g(y) - \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta(\mu)) d\mu$$
$$- \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta(\mu)) d\mu \Big| \le \epsilon,$$

then the Eq. (1.5) has the H–U stability.

Proof. By using $w(y) = \epsilon$, in Theorem 4.1, we can establish this theorem similarly as well. □

Theorem 4.3. *Assume that, there are two constants $\sigma > 0$ and $\lambda > 0$ such that*

$$\Big|g(y) + \frac{b_1(y)}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_1(y,\mu,\zeta(\mu)) d\mu$$
$$+ \frac{b_2(y)}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma H_2(y,\mu,\zeta(\mu)) d\mu \Big| \le \lambda y^{-\sigma}$$

holds for Eq. (1.5). Then with the conditions (S_1)–(S_3), and with the parameters $\delta \in (\frac{1}{2}, 1)$, $\rho \in (0,1)$ and $\gamma > 0$, Eq. (1.5) has a solution which is locally stable.

Proof. Define a set $U_{\sigma,\lambda} = \{\zeta(y) : \zeta(y) \in D$ and $\Big|\zeta(y)\Big| \le \lambda y^{-\sigma}$, for $y \ge \hat{y} > 0\}$.

It is easy to observe that the set $U_{\sigma,\lambda}$ is closed, bounded, and convex subset of D.

Firstly, we show that $\mathcal{M}$ maps $U_{\sigma,\lambda}$ in $U_{\sigma,\lambda}$.

As by the assumption, we have $\Big|\mathcal{M}(\zeta(y))\Big| \le \lambda y^{-\sigma}$, for $y > 0$, then $\mathcal{M}(U_{\sigma,\lambda}) \subset U_{\sigma,\lambda}$.

Now we will establish this theorem in the following two steps:

Step 1. $\mathcal{M}$ is continuous.

Let $\{\zeta_n\}$ be a sequence in $U_{\sigma,\lambda}$ and $\zeta \in U_{\sigma,\lambda}$ such that $\lim_{n\to\infty} \|\zeta_n - \zeta\| = 0$. Let $\epsilon > 0$ be given, $\exists$ $\beta > 0$ such that $\lambda y^{-\sigma} < \frac{\epsilon}{2}$, for $y \ge \beta$. Now for $0 < y \le \beta$,

$$\Big|\mathcal{M}(\zeta_n(y)) - \mathcal{M}(\zeta(y))\Big| \le \frac{\Big|b_1(y)\Big|}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_1(y,\mu,\zeta_n(\mu))$$
$$- H_1(y,\mu,\zeta(\mu)) \Big| d\mu$$
$$+ \frac{\Big|b_2(y)\Big|}{\Gamma(\delta)} \int_0^\beta (\beta^\rho - \mu^\rho)^{\delta-1} \mu^\gamma \Big| H_2(y,\mu,\zeta_n(\mu)) - H_2(y,\mu,\zeta(\mu)) \Big| d\mu$$
$$\le \frac{\Big|b_1(y)\Big|}{\Gamma(\delta)} \int_0^y (y^\rho - \mu^\rho)^{\delta-1} \mu^\gamma h_1(y) \Big|\zeta_n(\mu) - \zeta(\mu)\Big| d\mu$$

$$\begin{aligned}
&\quad+\frac{\left|b_2(y)\right|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}h_2(y)\left|\zeta_n(\mu)-\zeta(\mu)\right|d\mu\\
&\le\frac{\sup_{y\in\mathcal{A}}\left(\left|b_1(y)\right|h_1(y)\right)}{\Gamma(\delta)}\int_0^{y}(y^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}d\mu\|\zeta_n-\zeta\|\\
&\quad+\frac{\sup_{y\in\mathcal{A}}\left(\left|b_2(y)\right|h_2(y)\right)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}d\mu\|\zeta_n-\zeta\|\\
&\le\frac{\sup_{y\in\mathcal{A}}\left(\left|b_1(y)\right|h_1(y)\right)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\|\zeta_n-\zeta\|\\
&\quad+\frac{\sup_{y\in\mathcal{A}}\left(\left|b_2(y)\right|h_2(y)\right)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\|\zeta_n-\zeta\|.
\end{aligned}$$

This implies that, $\|\mathcal{M}(\zeta_n)-\mathcal{M}(\zeta)\|\to 0$ as $n\to\infty$.

For $y\ge\beta$, we get

$$\begin{aligned}
&\left|\mathcal{M}(\zeta_n(y))-\mathcal{M}(\zeta(y))\right|\le\Big|g(y)+\frac{b_1(y)}{\Gamma(\delta)}\int_0^{y}(y^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_1(y,\mu,\zeta_n(\mu))d\mu\\
&\quad+\frac{b_2(y)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y,\mu,\zeta_n(\mu))d\mu\\
&\quad-\Big(g(y)+\frac{b_1(y)}{\Gamma(\delta)}\int_0^{y}(y^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_1(y,\mu,\zeta(\mu))d\mu\\
&\quad+\frac{b_2(y)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y,\mu,\zeta(\mu))d\mu\Big)\Big|\\
&\le 2\lambda y^{-\sigma}<\epsilon.
\end{aligned}$$

Thus, for $y>0$, $\|\mathcal{M}(\zeta_n)-\mathcal{M}(\zeta)\|\to 0$ as $n\to\infty$. So, the operator $\mathcal{M}$ is continuous.

Step 2. To prove that $\mathcal{M}(U_{\sigma,\lambda})$ is equi-continuous.

Let $y_1,y_2>0$ with $y_1<y_2$. Let $\epsilon>0$ be given, $\exists\ \beta>0$ such that $\lambda y^{-\sigma}<\frac{\epsilon}{2}$, for $y>\beta$.

For $y_1,y_2\in[0,\beta]$, let $K=\max\{|\zeta(y)|:y\in\mathcal{A}\}$. Then,

$$\begin{aligned}
&\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))\right|\\
&=\Big|g(y_2)+\frac{b_1(y_2)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_2,\mu,\zeta(\mu))d\mu\\
&\quad+\frac{b_2(y_2)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-g(y_1)-\frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\\
&\quad-\frac{b_2(y_1)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\le\left|g(y_2)-g(y_1)\right|+\Big|\frac{b_1(y_2)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_2,\mu,\zeta(\mu))d\mu\\
&\quad+\frac{b_2(y_2)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))\,d\mu\\
&\quad-\frac{b_2(y_1)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\le\left|g(y_2)-g(y_1)\right|\\
&\quad+\Big|\frac{b_1(y_2)-b_1(y_1)+b_1(y_1)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\quad+\Big|\frac{b_2(y_2)-b_2(y_1)+b_2(y_1)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\frac{b_2(y_1)}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\le\left|g(y_2)-g(y_1)\right|\\
&\quad+\frac{|b_1(y_2)-b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left|H_1(y_2,\mu,\zeta(\mu))\right|d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\Big|\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\int_0^{y_1}\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\quad+\frac{|b_2(y_2)-b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))\right|d\mu\\
&\quad+\frac{|b_2(y_1)|}{\Gamma(\delta)}\Big|\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}H_2(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\le\left|g(y_2)-g(y_1)\right|+\frac{|b_1(y_2)-b_1(y_1)|}{\Gamma(\delta)}\\
&\quad\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_2,\mu,0)+H_1(y_2,\mu,0)\right|d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\Big|\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_2,\mu,\zeta(\mu))d\mu\\
&\quad-\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\Big|\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\Big|\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\\
&\quad-\int_0^{y_1}\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}H_1(y_1,\mu,\zeta(\mu))d\mu\Big|+\frac{|b_2(y_2)-b_2(y_1)|}{\Gamma(\delta)}\\
&\quad\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_2,\mu,0)+H_2(y_2,\mu,0)\right|d\mu\\
&\quad+\frac{|b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\le|g(y_2)-g(y_1)|+\frac{|b_1(y_2)-b_1(y_1)|}{\Gamma(\delta)}\\
&\quad\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left(\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_2,\mu,0)\right|+\left|H_1(y_2,\mu,0)\right|\right)d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_1}\left[\left(y_1^{\rho}-\mu^{\rho}\right)^{\delta-1}-\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\right]\mu^{\gamma}\Big(\left|H_1(y_1,\mu,\zeta(\mu))-H_1(y_1,\mu,0)\right|\\
&\quad+\left|H_1(y_1,\mu,0)\right|\Big)d\mu+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_{y_1}^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\Big(\left|H_1(y_1,\mu,\zeta(\mu))-H_1(y_1,\mu,0)\right|\\
&\quad+\left|H_1(y_1,\mu,0)\right|\Big)d\mu+\frac{|b_2(y_2)-b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\Big(\Big|H_2(y_2,\mu,\zeta(\mu))\\
&\quad-H_2(y_2,\mu,0)\Big|+\left|H_2(y_2,\mu,0)\right|\Big)d\mu\\
&\quad+\frac{|b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\le\left|g(y_2)-g(y_1)\right|+\frac{|b_1(y_2)-b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left(h_1(y_2)|\zeta(\mu)|+N_1(y_2)\right)d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_1}\left[\left(\frac{1}{y_1^{\rho}-\mu^{\rho}}\right)^{1-\delta}-\left(\frac{1}{y_2^{\rho}-\mu^{\rho}}\right)^{1-\delta}\right]\mu^{\gamma}\left(h_1(y_1)|\zeta(\mu)|+N_1(y_1)\right)d\mu\\
&\quad+\frac{|b_1(y_1)|}{\Gamma(\delta)}\int_{y_1}^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left(h_1(y_1)|\zeta(\mu)|+N_1(y_1)\right)d\mu\\
&\quad+\frac{|b_2(y_2)-b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left(h_2(y_2)|\zeta(\mu)|+N_2(y_2)\right)d\mu\\
&\quad+\frac{|b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\le\left|g(y_2)-g(y_1)\right|\\
&\quad+\frac{\sup_{y_2\in\mathcal{A}}\left(h_1(y_2)K+N_1(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_1(y_2)-b_1(y_1)|\\
&\quad+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\quad+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)}{\Gamma(\delta)}\int_0^{y_1}\left[\frac{y_2^{\rho}-y_1^{\rho}}{\left(y_1^{\rho}-\mu^{\rho}\right)^2}\right]^{1-\delta}\mu^{\gamma}d\mu\\
&\quad+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)}{\Gamma(\delta)}\int_{y_1}^{y_2}\left(y_2^{\rho}-\mu^{\rho}\right)^{\delta-1}\mu^{\gamma}d\mu\\
&\quad+\frac{\sup_{y_2\in\mathcal{A}}\left(h_2(y_2)K+N_2(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_2(y_2)-b_2(y_1)|\\
&\quad+\frac{\sup_{y_1\in\mathcal{A}}|b_2(y_1)|}{\Gamma(\delta)}\int_0^{\beta}(\beta^{\rho}-\mu^{\rho})^{\delta-1}\mu^{\gamma}\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu\\
&\le\left|g(y_2)-g(y_1)\right|\\
&\quad+\frac{\sup_{y_2\in\mathcal{A}}\left(h_1(y_2)K+N_1(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_1(y_2)-b_1(y_1)|
\end{aligned}$$

$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_1,\mu,\zeta(\mu))\right|d\mu$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)}{\Gamma(\delta)}\int_0^{y_1}\left(y_2-y_1\right)^{\rho(1-\delta)}\left(y_1^\rho-\mu^\rho\right)^{2(\delta-1)}\mu^\gamma d\mu$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)\beta^{\gamma+1-\rho}}{\rho\delta\Gamma(\delta)}\left(y_2^\rho-y_1^\rho\right)^\delta$$
$$+\frac{\sup_{y_2\in\mathcal{A}}\left(h_2(y_2)K+N_2(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_2(y_2)-b_2(y_1)|$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_2(y_1)|}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu,$$

i.e.,

$$|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))|\le|g(y_2)-g(y_1)|$$
$$+\frac{\sup_{y_2\in\mathcal{A}}\left(h_1(y_2)K+N_1(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_1(y_2)-b_1(y_1)|$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma\left|H_1(y_2,\mu,\zeta(\mu))-H_1(y_1,\mu,\zeta(\mu))\right|d\mu$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)\beta^{2\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}$$
$$\times\mathbb{B}\left(\frac{\gamma+1}{\rho},2\delta-1\right)\left(y_2-y_1\right)^{\rho(1-\delta)}$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_1(y_1)|\left(h_1(y_1)K+N_1(y_1)\right)\beta^{\gamma+1-\rho}}{\rho\Gamma(\delta+1)}\left(y_2-y_1\right)^{\rho\delta}$$
$$+\frac{\sup_{y_2\in\mathcal{A}}\left(h_2(y_2)K+N_2(y_2)\right)\beta^{\rho(\delta-1)+\gamma+1}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)|b_2(y_2)-b_2(y_1)|$$
$$+\frac{\sup_{y_1\in\mathcal{A}}|b_2(y_1)|}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma\left|H_2(y_2,\mu,\zeta(\mu))-H_2(y_1,\mu,\zeta(\mu))\right|d\mu.$$

Thus, $\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))\right|\to0$ as $y_2\to y_1$.

For $y_1,y_2>\beta$, we have

$$\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))\right|\le\left|g(y_2)\right.$$
$$+\frac{b_1(y_2)}{\Gamma(\delta)}\int_0^{y_2}\left(y_2^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu$$
$$+\frac{b_2(y_2)}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu$$
$$-g(y_1)-\frac{b_1(y_1)}{\Gamma(\delta)}\int_0^{y_1}\left(y_1^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_1(y,\mu,\zeta(\mu))d\mu$$
$$\left.-\frac{b_2(y_1)}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma H_2(y,\mu,\zeta(\mu))d\mu\right|\le\lambda y_2^{-\sigma}+\lambda y_1^{-\sigma}<\epsilon.$$

For $0<y_1<\beta<y_2$, observe that $y_2\to y_1$ implies $y_2\to\beta$ and $\beta\to y_1$, then
$\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))\right|\le\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(\beta))\right|+\left|\mathcal{M}(\zeta(\beta))-\mathcal{M}(\zeta(y_1))\right|\to0$ as $y_2\to y_1$.

Thus, $\left|\mathcal{M}(\zeta(y_2))-\mathcal{M}(\zeta(y_1))\right|\to0$ as $y_2\to y_1$ for $y_2,y_1\in[0,\infty)$. Therefore, $\mathcal{M}(U_{\sigma,\lambda})$ is equi-continuous. Subsequently, $\mathcal{M}(U_{\sigma,\lambda})$ is relatively compact as $\mathcal{M}(U_{\sigma,\lambda})\subset U_{\sigma,\lambda}$ is uniformly bounded. Hence, $\mathcal{M}$ is completely continuous on $U_{\sigma,\lambda}$. By Schauder fixed point theorem, $\mathcal{M}$ has a fixed point in $U_{\sigma,\lambda}$ which is the solution of the Eq. (1.5), and it is easy to see that the solution tends to zero as $y\to\infty$. Thus, the solution of Eq. (1.5) is locally stable. □

In the next two theorems, we provide another easy checked sufficient conditions for the local stability of the solutions of Eq. (1.5).

Theorem 4.4. *Assume that,* $|g(y)|\le\frac{1}{3}y^{-\theta}$, $y\in[\hat{y},+\infty)$, $\hat{y}>0$, $\theta\in(0,1)$ *and there are four constants* $L_{b_1},L_{b_2},L_{H_1},L_{H_2}>0$ *such that* $|b_1(y)|\le L_{b_1}y^{\rho(1-\delta)-\gamma-1}$, $|H_1(y,\mu,\zeta(\mu))|\le L_{H_1}|\zeta|$, $|b_2(y)|\le L_{b_2}y^{-\theta}$, $|H_2(y,\mu,\zeta(\mu))|\le L_{H_2}|\zeta|$, *where* $L_{H_1}L_{b_1}\le\frac{\rho\Gamma(\delta)}{3\mathbb{B}\left(\frac{\gamma-\theta+1}{\rho},\delta\right)}$ *and* $L_{H_2}L_{b_2}\le\frac{\rho\Gamma(\delta)}{3\mathbb{B}\left(\frac{\gamma-\theta+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma-\theta+1}}$.

Then with the conditions (S_1)–(S_3), *and with the parameters* $\delta\in(\frac{1}{2},1)$, $\rho\in(0,1)$ *and* $\gamma>0$, *Eq.* (1.5) *has a locally stable solution on* $[\hat{y},+\infty)$.

Proof. We define a set $U_{\theta,1}=\{\zeta(y):\zeta(y)\in D$ and $\left|\zeta(y)\right|\le y^{-\theta}$, for $y\ge\hat{y}>0\}$.

It is easy to observe that the set $U_{\theta,1}$ is closed, bounded, and convex subset of D.

Now, we need to show that $\mathcal{M}$ maps $U_{\theta,1}$ in $U_{\theta,1}$.

For $y>0$, we get

$$|\mathcal{M}(\zeta(y))|\le|g(y)|+\frac{|b_1(y)|}{\Gamma(\delta)}\int_0^y\left(y^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma|H_1(y,\mu,\zeta(\mu))|d\mu$$
$$+\frac{|b_2(y)|}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma|H_2(y,\mu,\zeta(\mu))|d\mu$$
$$\le\frac{1}{3}y^{-\theta}+\frac{L_{H_1}L_{b_1}y^{\rho(1-\delta)-\gamma-1}}{\Gamma(\delta)}\int_0^y\left(y^\rho-\mu^\rho\right)^{\delta-1}\mu^{\gamma-\theta}d\mu$$
$$+\frac{L_{H_2}L_{b_2}y^{-\theta}}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^{\gamma-\theta}d\mu$$
$$\le\frac{1}{3}y^{-\theta}+\frac{L_{H_1}L_{b_1}y^{-\theta}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma-\theta+1}{\rho},\delta\right)$$
$$+\frac{L_{H_2}L_{b_2}y^{-\theta}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{\gamma-\theta+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma-\theta+1}$$
$$\le y^{-\theta}.$$

Then $\mathcal{M}(U_{\theta,1})\subset U_{\theta,1}$.

The remaining part of the proof is the same as the proof of Theorem 4.3, it is simple to complete. □

Theorem 4.5. *Assume that,* $|g(y)|\le\frac{1}{3}y^{-1}$, $y\in[\hat{y},+\infty)$, $\hat{y}>0$, *and there are two constants* $N_{b_1},N_{b_2}>0$ *such that* $|b_1(y)|\le N_{b_1}y^{-\theta-1}$, $|H_1(y,\mu,\zeta(\mu))|\le e^{-\delta\mu}|\zeta|$, $|b_2(y)|\le N_{b_2}y^{-1}$, $|H_2(y,\mu,\zeta(\mu))|\le e^{-\delta\mu}|\zeta|$, *where* $N_{b_1}=\frac{\rho\Gamma(\delta)(\delta q)^{\frac{1}{q}}}{3\mathbb{B}\left(\frac{p(\gamma-1)+1}{\rho},p(\delta-1)+1\right)}$ *and* $N_{b_2}=\frac{\rho\Gamma(\delta)(\delta q)^{\frac{1}{q}}}{3\mathbb{B}\left(\frac{p(\gamma-1)+1}{\rho},p(\delta-1)+1\right)\beta^\theta}$,
$\theta=p\left[\rho(\delta-1)+\gamma-1\right]+1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Then with the conditions (S_1)–(S_3), *and with the parameters* $\delta\in(\frac{1}{2},1)$, $\rho\in(0,1)$ *and* $\gamma>0$, *Eq.* (1.5) *has a locally stable solution on* $[\hat{y},+\infty)$.

Proof. We define a set $U_{1,1}=\{\zeta(y):\zeta(y)\in D$ and $\left|\zeta(y)\right|\le y^{-1}$, for $y\ge\hat{y}>0\}$.

It is easy to observe that the set $U_{1,1}$ is closed, bounded, and convex subset of D.

Now, we need to show that $\mathcal{M}$ maps $U_{1,1}$ in $U_{1,1}$.

For $y>0$, we obtain

$$|\mathcal{M}(\zeta(y))|\le|g(y)|+\frac{|b_1(y)|}{\Gamma(\delta)}\int_0^y\left(y^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma|H_1(y,\mu,\zeta(\mu))|d\mu$$
$$+\frac{|b_2(y)|}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^\gamma|H_2(y,\mu,\zeta(\mu))|d\mu$$
$$\le\frac{1}{3}y^{-1}+\frac{N_{b_1}y^{-\theta-1}}{\Gamma(\delta)}\int_0^y\left(y^\rho-\mu^\rho\right)^{\delta-1}\mu^{\gamma-1}e^{-\delta\mu}d\mu$$
$$+\frac{N_{b_2}y^{-1}}{\Gamma(\delta)}\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{\delta-1}\mu^{\gamma-1}e^{-\delta\mu}d\mu$$
$$\le\frac{1}{3}y^{-1}+\frac{N_{b_1}y^{-\theta-1}}{\Gamma(\delta)}\left(\int_0^y\left(y^\rho-\mu^\rho\right)^{p(\delta-1)}\mu^{p(\gamma-1)}d\mu\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^y e^{-\delta q\mu}d\mu\right)^{\frac{1}{q}}$$
$$+\frac{N_{b_2}y^{-1}}{\Gamma(\delta)}\left(\int_0^\beta\left(\beta^\rho-\mu^\rho\right)^{p(\delta-1)}\mu^{p(\gamma-1)}d\mu\right)^{\frac{1}{p}}\left(\int_0^\beta e^{-\delta q\mu}d\mu\right)^{\frac{1}{q}},$$

i.e.,

$$|\mathcal{M}(\zeta(y))| \le \frac{1}{3}y^{-1} + \frac{N_{b_1}y^{-\theta-1}y^{\theta}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{p(\gamma-1)+1}{\rho}, p(\delta-1)+1\right)\frac{1}{(\delta q)^{\frac{1}{q}}} + \frac{N_{b_2}y^{-1}\beta^{\theta}}{\rho\Gamma(\delta)}\mathbb{B}\left(\frac{p(\gamma-1)+1}{\rho}, p(\delta-1)+1\right)\frac{1}{(\delta q)^{\frac{1}{q}}} \le y^{-1}.$$

Then $\mathcal{M}(U_{1,1}) \subset U_{1,1}$.

The remaining part of the proof is the same as the proof of Theorem 4.3, it is simple to complete. □

5. Examples

Three examples are given in this section to interpret our established results.

Example 5.1. Consider the NVFIE with the E–K fractional integral operator as

$$\zeta(y) = y^2 + y + \frac{\sin(y)}{\Gamma(\frac{4}{5})}\int_0^y \left(y^{\frac{1}{2}} - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_1(y,\mu,\zeta(\mu))d\mu + \frac{1}{\Gamma(\frac{4}{5})}\int_0^1 \left(1 - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_2(y,\mu,\zeta(\mu))d\mu, \tag{5.8}$$

with $H_1(y,\mu,\zeta(\mu)) = \sin(y\mu)\sin(\zeta(\mu))$, $H_2(y,\mu,\zeta(\mu)) = \sin(\mu)\cos(\zeta(\mu))$, $y \in [0,1]$.

Comparing Eq. (5.8) with Eq. (1.5), we get $\delta = \frac{4}{5}$, $\rho = \frac{1}{2}$, $\gamma = \frac{5}{2}$, $b_2(y) = 1$, $b_1(y) = \sin(y)$, $g(y) = y^2 + y$, $\beta = 1$.

Then, $|g(y)| = |y^2 + y| \le 2$, $|b_1(y)| = |\sin(y)| \le 1$, $|b_2(y)| = 1$.

As $\left|H_2(y,\mu,\zeta_1) - H_2(y,\mu,\zeta_2)\right| = |\sin(\mu)\cos(\zeta_1) - \sin(\mu)\cos(\zeta_2)| \le |\zeta_1 - \zeta_2|$,
and $\left|H_1(y,\mu,\zeta_1) - H_1(y,\mu,\zeta_2)\right| = |\sin(y\mu)\sin(\zeta_1) - \sin(y\mu)\sin(\zeta_2)| \le |\zeta_1 - \zeta_2|$.

Also, $\left|H_2(y,\mu,\zeta(\mu))\right| = |\sin(\mu)\cos(\zeta(\mu))| \le 1$, and $\left|H_1(y,\mu,\zeta(\mu))\right| = |\sin(y\mu)\sin(\zeta(\mu))| \le 1$.

Therefore, assumptions $(A_1)-(A_4)$ are satisfied with $G = 2$, $B_1 = 1$, $B_2 = 1$, $G_{H_1} = 1$, $G_{H_2} = 1$, $N_1 = 1$, $N_2 = 1$.

As a result, all the requirements of Theorem 3.1 are satisfied for Eq. (5.8). Hence, we can say that Eq. (5.8) has at least one solution defined on $[0,1]$.

Example 5.2. Consider the NVFIE with the E–K fractional integral operator as

$$\zeta(y) = \frac{y^2}{2} + \frac{y}{\Gamma(\frac{4}{5})}\int_0^y \left(y^{\frac{1}{2}} - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_1(y,\mu,\zeta(\mu))d\mu + \frac{\cos(y)}{\Gamma(\frac{4}{5})}\int_0^1 \left(1 - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_2(y,\mu,\zeta(\mu))d\mu, \tag{5.9}$$

with $H_1(y,\mu,\zeta(\mu)) = \sin(y\mu)\sin(\zeta(\mu))$, $H_2(y,\mu,\zeta(\mu)) = \frac{|\zeta(\mu)|}{(\mu+8)^2(1+|\zeta(\mu)|)}$, $y \in [0,1]$.

Comparing Eq. (5.9) with Eq. (1.5), we get $\delta = \frac{4}{5}$, $\rho = \frac{1}{2}$, $\gamma = \frac{5}{2}$, $g(y) = \frac{y^2}{2}$, $b_2(y) = \cos(y)$, $b_1(y) = y$, $\beta = 1$.

Then, $|g(y)| = |\frac{y^2}{2}| \le \frac{1}{2}$, $|b_2(y)| = |\cos(y)| \le 1$, $|b_1(y)| = |y| \le 1$.

As $\left|H_1(y,\mu,\zeta_1) - H_1(y,\mu,\zeta_2)\right| = |\sin(y\mu)\sin(\zeta_1) - \sin(y\mu)\sin(\zeta_2)| \le |\zeta_1 - \zeta_2|$,
and $\left|H_2(y,\mu,\zeta_1) - H_2(y,\mu,\zeta_2)\right| \le \frac{1}{64}|\zeta_1 - \zeta_2|$.

Therefore, assumptions $(A_1)-(A_3)$ are satisfied with $G = \frac{1}{2}$, $B_1 = 1$, $B_2 = 1$, $G_{H_1} = 1$ and $G_{H_2} = \frac{1}{64}$. Then $\mathcal{G} = \max\{G_{H_1}, G_{H_2}\} = 1$ and $0 < \mathcal{G} < \frac{\rho\Gamma(\delta)}{(B_1+B_2)\mathbb{B}\left(\frac{\gamma+1}{\rho},\delta\right)\beta^{\rho(\delta-1)+\gamma+1}} = 1.1725$.

As a result, all the requirements of Theorem 3.2 are satisfied for Eq. (5.9). Hence, we can say that Eq. (5.9) has a unique solution on $[0,1]$.

Also for this equation, we can apply Theorems 4.1 and 4.2 to analyze the corresponding H–U–R and H–U stability.

Example 5.3. Consider the NVFIE with the E–K fractional integral operator as

$$\zeta(y) = \frac{y^{-5}}{4\Gamma(\frac{4}{5})}\int_0^y \left(y^{\frac{1}{2}} - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_1(y,\mu,\zeta(\mu))d\mu + \frac{y^{-\frac{8}{5}}}{4\Gamma(\frac{4}{5})}\int_0^1 \left(1 - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} H_2(y,\mu,\zeta(\mu))d\mu, \tag{5.10}$$

where $H_1(y,\mu,\zeta(\mu)) = \sin(\mu + \zeta(\mu))$, $H_2(y,\mu,\zeta(\mu)) = \cos(\zeta(\mu))$, $y \ge \hat{y} > 0$.

Comparing Eq. (5.10) with Eq. (1.5), we get $\delta = \frac{4}{5}$, $\rho = \frac{1}{2}$, $\gamma = \frac{5}{2}$, $b_2(y) = \frac{y^{-\frac{8}{5}}}{4}$, $b_1(y) = \frac{y^{-5}}{4}$, $g(y) = 0$.

Then,

$$\left|g(y) + \frac{b_1(y)}{\Gamma(\delta)}\int_0^y (y^{\rho} - \mu^{\rho})^{\delta-1}\mu^{\gamma} H_1(y,\mu,\zeta(\mu))d\mu + \frac{b_2(y)}{\Gamma(\delta)}\int_0^{\beta} (\beta^{\rho} - \mu^{\rho})^{\delta-1}\mu^{\gamma} H_2(y,\mu,\zeta(\mu))d\mu\right|$$

$$\le \frac{y^{-5}}{4\Gamma(\frac{4}{5})}\int_0^y \left(y^{\frac{1}{2}} - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} d\mu + \frac{y^{-\frac{8}{5}}}{4\Gamma(\frac{4}{5})}\int_0^1 \left(1 - \mu^{\frac{1}{2}}\right)^{-\frac{1}{5}} \mu^{\frac{5}{2}} d\mu$$

$$= \frac{y^{-\frac{8}{5}}}{2\Gamma(\frac{4}{5})}\mathbb{B}\left(7,\frac{4}{5}\right) + \frac{y^{-\frac{8}{5}}}{2\Gamma(\frac{4}{5})}\mathbb{B}\left(7,\frac{4}{5}\right) = \frac{y^{-\frac{8}{5}}}{\Gamma(\frac{4}{5})}\mathbb{B}\left(7,\frac{4}{5}\right) = \frac{\Gamma(7)}{\Gamma(\frac{39}{5})}y^{-\frac{8}{5}}.$$

As a result, all the requirements of Theorem 4.3 are satisfied for Eq. (5.10). Hence, we can say that Eq. (5.10) has a solution which is locally stable.

6. Conclusions and future work

In this study, we investigated existence and uniqueness of solutions for the NVFIE given in Eq. (1.5), by using Leray–Schauder alternative and Banach's fixed point theorem, we also analyzed H–U and H–U–R stability in the space $C([0,\beta],\mathbb{R})$. Moreover, three different solutions sets has been constructed and under some ideal conditions, local stability of solutions has been obtained. Also, to achieve our aim, we have been chosen the parameters as $\delta \in (\frac{1}{2},1)$, $\rho \in (0,1)$ and $\gamma > 0$, for Eq. (1.5).

In the future, more results can be investigated, such as local stability results on some different solutions sets by assuming different conditions. Moreover, one can investigate the above results for the quadratic Volterra–Fredholm integral equations involving more generalized integral operator.

Declaration of competing interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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