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ORIGINAL ARTICLE

Fractional variational iteration method via modified Riemann–Liouville derivative

Naeem Faraz [a,*], Yasir Khan [a], Hossein Jafari [b], Ahmet Yildirim [c], M. Madani [d]

[a] Modern Textile Institute, Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, China
[b] Department of Mathematics and Computer Science, University of Mazandaran, P.O. Box 47416-1467, Babolsar, Iran
[c] Ege University, Science Faculty, Department of Mathematics, 35100 Bornova, Turkey
[d] Chemical Engineering Department, Amirkabir University of Technology, No. 424, Hafez Ave., Tehran, Iran

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Abstract The aim of this paper is to present an efficient and reliable treatment of the variational iteration method (VIM) for partial differential equations with fractional time derivative. The fractional derivative is described in the Jumarie sense. The obtained results are in good agreement with the existing ones in open literature and it is shown that the technique introduced here is robust, efficient and easy to implement.

1. Introduction

Mathematics is the art of giving things misleading names. The beautiful and at first look mysterious name the fractional calculus is just one of those misnomers which are the essence of mathematics. Differential equations of fractional order appear more and more frequently in various research areas and engineering applications. For example nonlinear oscillation of earthquake can be modeled with fractional derivatives (He, 1998), the fluid-dynamic traffic model with fractional derivatives (He, 1999) can eliminate the deficiency arising from the assumptions of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in He (1998), and many differential equations and fractional differential equations have recently proved to be valuable tools to the modeling of many physical phenomena (Achouri and Omrani, 2009; Yildirim and Momani, 2010; Koçak and Yildirim, 2009; Yildirim and Gülkanat, 2010; Podlubny, 1999; Diethelm and Ford, 2002; Miller and Ross, 2003; Achouri et al., xxxx; Barari et al., 2008; Ghotbi et al., xxxx; Barari et al., xxxx; Fouladi et al., 2010; Faraz et al., xxxx; Khan, 2009; Khan et al., xxxx; Khan and Austin, 2010). Different fractional partial differential equations have been studied and solved including the space-time fractional diffusion-wave equation (Khan et al., xxxx; Wu and Lee, 2010; Das, 2009), the fractional telegraph equation (Momani, 2005), the fractional Kdv equation (Momani, 2005), the space and time fractional Burgers equations (Mustafa Inc., 2008) and the

* Corresponding author. Tel.: +86 2162378473.
E-mail addresses: nfaraz_math@yahoo.com (N. Faraz), yasirmath@yahoo.com (Y. Khan), jafari@umz.ac.ir (H. Jafari), ahmet.yildirim@ege.edu.tr (A. Yildirim).

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space-time-fractional Fokker–Planck equation (Yildirim, 2010). In recent times, Jumarie (1993) proposed a new modified Riemann–Liouville left derivative. Jumarie's modified derivative was successfully applied in the stochastic fractional models Jumarie, 2006, fractional Laplace problems (Jumarie, 2009) etc.

He's variational iteration method (He, 1999; He et al., 2010) based on the use of restricted variations, correction functional and Lagrange multiplier technique developed by Inokuti et al. (1978). This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. This technique is, in fact, a modifying of the general Lagrange multiplier method into an iteration method, which is called correction functional. The method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems, generally one or two iterations lead to high accurate solutions. In this technique, the equations are initially approximated with possible unknowns. A correction functional is established by the general Lagrange multiplier which can be identified optimally via the variational theory. The method provides rapidly the convergent successive approximations of the exact solution. Besides, the VIM has no restrictions or unrealistic assumptions such as linearization or small parameters that are used in the nonlinear operators.

The basic idea described in this paper is expected to be further employed to solve other linear as well as nonlinear problem in fractional calculus.

2. Basic definition

We give some basic definitions and properties of the fractional calculus theory which are used further in this letter

Definition 2.1. Jumarie is defined the fractional derivative (Jumarie, 2009) as the following limit form

$$f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^\alpha[f(x) - f(0)]}{h^\alpha} \tag{2.1}$$

This definition is close to the standard definition of derivatives, and as a direct result, the αth derivative of a constant, $0 < \alpha < 1$ is zero

Definition 2.2. Fractional integral operator of order $\alpha \geqslant 0$ is defined as

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^0 (x-\xi)^{\alpha-1} f(\xi)d\xi, \quad \alpha > 0, \tag{2.2}$$

Definition 2.3. The modified Riemann–Liouville derivative (Jumarie, 2009) is defined as

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^0 (x-\xi)^{n-\alpha}(f(\xi) - f(0))d\xi, \tag{2.3}$$

where $x \in [0,1]$, $n-1 \leqslant \alpha < n$ and $n \geqslant 1$.

Definition 2.4. Fractional derivative of compounded functions (Jumarie, 2009) is defined as

$$d^\alpha f \cong \Gamma(1+\alpha)df, \quad 0 < \alpha < 1 \tag{2.4}$$

Definition 2.5. The integral with respect to $(dx)^\alpha$ (Jumarie, 2009) is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^\alpha, \quad x \geqslant 0, \quad y(0) = 0, \quad 0 < \alpha < 1 \tag{2.5}$$

Lemma 2.1. Let $f(x)$ denote a continuous function (Jumarie, 2009) then the solution of the Eq. (2.5) is defined as

$$y = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi, \quad 0 < \alpha \leqslant 1 \tag{2.6}$$

For example $f(x) = x^\gamma$ in Eq. (2.6) one obtains

$$\int_0^x \xi^\gamma (d\xi)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad 0 < \alpha \leqslant 1 \tag{2.7}$$

Definition 2.6. Assume that the continuous function $f : R \to R$, $x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any $\alpha, 0 < \alpha \leqslant 1$; then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\alpha k!} f^{\alpha k}(x), \quad 0 < \alpha \leqslant 1. \quad 0 < \alpha \leqslant 1 \tag{2.8}$$

On making the substitution $h \to x$ and $x \to 0$ we obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\alpha k!} f^{\alpha k}(0), \quad 0 < \alpha \leqslant 1 \tag{2.9}$$

3. Fractional variation iteration method (FVIM)

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = K[x]u(x,t) + p(x,t), \quad t > 0, \quad x \in \mathbb{R},$$
$$u(x,0) = f(x), \quad \alpha > o \tag{3.1}$$

where $K[x]$ is the differential operator in x, $f(x)$ and $p(x,t)$ are continuous functions. According to the VIM, we can construct a correct functional for Eq. (3.1) as follows

$$u_{n+1}(x,t) = u_n(x) + I^\alpha\left[\lambda\left(\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - K[x]u(x,t) - p(x,t)\right)\right],$$
$$u_{n+1}(x,t) = u_n(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1}\lambda(\xi) \left(\frac{\partial^\alpha}{\partial \xi^\alpha}u(x,\xi) - K[x]u(x,\xi) - p(x,\xi)\right)d\xi, \tag{3.2}$$

Using Eq. (2.6), we obtain a new correction functional

$$u_{n+1}(x,t) = u_n(x) + \frac{1}{\Gamma(\alpha+1)} \times \int_0^t \lambda(\xi)\left(\frac{\partial^\alpha}{\partial \xi^\alpha}u(x,\xi) - K[x]u(x,\xi) - p(x,\xi)\right)(d\xi)^\alpha. \tag{3.3}$$

It is obvious that the successive approximations $u_j, j \geqslant 0$ can be established by determining λ, a general Lagrange's multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier

λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), n \geqslant 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations $u_j(x,t), j \geqslant 0$, follows immediately. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{3.4}$$

4. Applications

In this section we shall illustrate the FVIM by following examples.

Example 4.1. Consider the following one-dimensional linear inhomogeneous fractional wave equation

$$\begin{aligned}&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{2}x^2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0<x<1, \quad t>0, \quad 1<\alpha\leqslant 2\\ &u(0,t)=0, \quad u(1,t)=1+\sinh t,\\ &u(x,0)=x. \quad u_t(x,0)=x^2.\end{aligned} \tag{4.1}$$

To solve Eq. (4.1) by means of FVIM, we construct a correctional functional which reads

$$u_{n+1}(x,t) = u_n(x) + \frac{1}{\Gamma(1+\alpha)} \times \int_0^t \lambda(\xi)\left\{\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} - \frac{x^2}{2}\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right\}(d\xi)^\alpha \tag{4.2}$$

Imposing the stationary condition ($\delta\tilde{u}_n = 0$) on the correctional functional,

$$\frac{\partial^\alpha \lambda(\xi)}{\partial \xi^\alpha} = 0, \text{ and } 1+\lambda(\xi)|_{\xi=t} = 0. \tag{4.3}$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(\xi) = -1. \tag{4.4}$$

Substituting Eq. (4.4) into the functional Eq. (4.2) yields the iteration formulation as follows

$$u_{n+1}(x,t) = u_n(x) - \frac{1}{\Gamma(1+\alpha)} \times \int_0^t \left\{\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} - \frac{x^2}{2}\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right\}(d\xi)^\alpha. \tag{4.5}$$

From the initial value, we can derive

$$\begin{aligned}&u_0(x,t) = x + x^2 t,\\ &u_1(x,t) = \frac{x^2\Gamma(2)}{\Gamma(2+\alpha)}t^{\alpha+1},\\ &u_2(x,t) = \frac{x^2\Gamma(2)}{\Gamma(2+2\alpha)}t^{1+2\alpha},\\ &u_3(x,t) = \frac{x^2\Gamma(2)}{\Gamma(2+3\alpha)}t^{1+3\alpha},\\ &\vdots\end{aligned} \tag{4.6}$$

Therefore the solution is

$$\begin{aligned}&u(x,t) = x + x^2 t + \frac{x^2\Gamma(2)}{\Gamma(2+\alpha)}t^{\alpha+1} + \frac{x^2\Gamma(2)}{\Gamma(2+2\alpha)}t^{1+2\alpha} + \frac{x^2\Gamma(2)}{\Gamma(2+3\alpha)}t^{1+3\alpha}\dots\\ &u(x,t) = x + x^2 t\left[1 + \frac{1}{\Gamma(2+\alpha)}t^\alpha + \frac{1}{\Gamma(2+2\alpha)}t^{2\alpha} + \frac{1}{\Gamma(2+3\alpha)}t^{3\alpha}\dots\right]\\ &u(x,t) = x + x^2 t E_{\alpha,2}(t^\alpha)\end{aligned} \tag{4.7}$$

where $E_{\alpha,2}(t^\alpha)$ denotes two-parameter Mittag–Leffler function. Result obtained in Eq. (4.7) is exactly the same result, obtained by Momani (2005).

Example 4.2. Consider the two-dimensional inhomogeneous wave equation

$$\begin{aligned}&\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}, \quad 0<\alpha\leqslant 1\\ &u(0,y,t)=0, u(2\pi,y,t)=0,\\ &u(x,0,t)=0, u(x,2\pi,t)=0,\\ &u(x,y,0)=\sin x \sin y,\end{aligned} \tag{4.8}$$

The correctional functional for Eq. (4.8) can be constructed as

$$u_{n+1}(x,y,t) = u_n(x,y) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda(\xi)\left\{\frac{\partial^\alpha u_n(x,y,\xi)}{\partial \xi^\alpha} - \frac{\partial^2 \tilde{u}_n(x,y,\xi)}{\partial x^2} - \frac{\partial^2 \tilde{u}_n(x,y,\xi)}{\partial y^2}\right\}(d\xi)^\alpha \tag{4.9}$$

Making the correction functional stationary, the Lagrange multiplier can be identified as

$$\lambda(\xi) = -1. \tag{4.10}$$

Substituting value of Lagrange multiplier in the Eq. (4.9), we get the following iteration formula

$$u_{n+1}(x,y,t) = u_n(x,y) - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left\{\frac{\partial^\alpha u_n(x,y,\xi)}{\partial \xi^\alpha} - \frac{\partial^2 \tilde{u}_n(x,y,\xi)}{\partial x^2} - \frac{\partial^2 \tilde{u}_n(x,y,\xi)}{\partial y^2}\right\}(d\xi)^\alpha. \tag{4.11}$$

Beginning with an initial approximation $u_0(x,y,t) = u(x,y,0) = f(x) = \sin x \sin y$, we obtain the following first order approximation solution

$$\begin{aligned}u_1(x,y,t) &= \sin x\sin y - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left\{\frac{\partial^\alpha u_0(x,y,\xi)}{\partial \xi^\alpha} - \frac{\partial^2 \tilde{u}_0(x,y,\xi)}{\partial x^2} - \frac{\partial^2 \tilde{u}_0(x,y,\xi)}{\partial y^2}\right\}(d\xi)^\alpha\\ &= \sin x\sin y - \frac{1}{\Gamma(1+\alpha)} \times \int_0^t \left(\sin x\sin y\frac{\Gamma(1)}{\Gamma(1-\alpha)}\xi^{-\alpha} + 2\sin x\sin y\right)(d\xi)^\alpha\\ &= -\frac{2t^\alpha \sin x\sin y}{\Gamma(1+\alpha)}.\end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}u_2(x,y,t) &= u_1(x,y,t) - \frac{1}{\Gamma(1+\alpha)} \times \int_0^t \left(-2\sin x\sin y - \frac{4\sin x\sin y}{\Gamma(\alpha+1)}\xi^\alpha\right)(d\xi)^\alpha\\ &= \frac{4t^{2\alpha}\sin x\sin y}{\Gamma(1+2\alpha)},\\ u_3(x,y,t) &= -8\sin x\sin y\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\ &\vdots\end{aligned} \tag{4.13}$$

The solution in a series form is given by

$$u(x,y,t) = \sin x \sin y - 2\sin x \sin y \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\sin x \sin y \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 8\sin x \sin y \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \ldots,$$

$$u(x,y,t) = \sin x \sin y \sum_{m=0}^{\infty} \frac{(-2t^{\alpha})^m}{\Gamma(m\alpha+1)} = \sin x \sin y E_{\alpha}(-2t^{\alpha}) \qquad (4.14)$$

One can see that obtained result is good agreement of existing one in the literature (Jumarie, 2009).

5. Conclusion

Variational iteration method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations, integral equations and fractional differential equations. In this article, we have presented an new form of variational iteration method having integral w.r.t. $(d\xi)^{\alpha}$. The present work shows the validity and great potential of the variational iteration method for solving linear and nonlinear fractional partial differential equations. Both examples show that the results of the variational iteration method having integral w.r.t. $(d\xi)^{\alpha}$ are in excellent agreement with those obtained by classical VIM or other methods present in open literature.

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References

Achouri, T., Omrani, K., 2009. Numerical solutions for the damped generalized regularized long-wave equation with a variable coefficient by Adomian decomposition method. Commun. Nonl. Sci. Num. Sim. 14, 2025–2033.

Achouri, T., Omrani, K., xxxx. Application of the homotopy perturbation method to the modified regularized long-wave equation, Wiley. Int. Sci. doi:10.1002/num.20441.

Barari, A., Omidvar, M., Ghotbi, A.R., Ganji, D.D., 2008. Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations. Acta. Appl. Math. 104, 161–171.

Barari, A., Omidvar, M., Ganji, D.D., Poor, A.T., xxxx. An approximate solution for boundary value problems in structural engineering and fluid mechanics. Math. Prob. Engg. doi:10.1155/2008/394103.

Das, S., 2009. Analytical solution of a fractional diffusion equation by variational iteration method. Comput. Math. Appl. 57, 483–487.

Diethelm, K., Ford, N.J., 2002. Analysis of fractional differential equations. J. Math. Anal. Appl. 265, 229–248.

Faraz, N., Khan, Y., Yildirim, A., xxxx. Analytical approach to two-dimensional viscous flow with a shrinking sheet via variational iteration algorithm-II. J. King. Saud. Univ. Sci. doi:10.1016/j.jksus.2010.06.010.

Fouladi, F., Hosseinzadeh, E., Barari, A., Domairry, G., 2010. Highly nonlinear temperature-dependent fin analysis by variational iteration method. Heat. Transf. Res. 41, 155–165.

Ghotbi, A.R., Barari, A., Ganji, D.D., xxxx. Solving ratio-dependent predator-prey system with constant effort harvesting using homotopy perturbation method. Math. Prob. Engg. doi:10.1155/2008/945420.

He, J.H., 1998. Nonlinear oscillation with fractional derivative and its applications. International Conference on Vibrating Engineering'98, Dalian, China, 288–291.

He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Meth. Appl. Mech. Eng. 167, 57–68.

He, J.H., 1999. Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Tech. 15, 86–90.

He, J.H., 1999. Variational iteration method – a kind of non-linear analytical technique: some examples. Int. J. Nonl. Mech. 34, 699–708.

He, J.H., Wu, G.C., Austin, F., 2010. The variational iteration method which should be followed. Nonl. Sci. Lett. A 1, 1–30.

Inokuti, M., Sekine, H., Mura, T., 1978. General use of the Lagrange multiplier in nonlinear mathematical physics. In: Nemat-Nasser, S. (Ed.), Variational Methods in the Mechanics of Solids. Pergamon Press, New York, pp. 156–162.

Jumarie, G., 1993. Stochastic differential equations with fractional Brownian motion input. Int. J. Syst. Sci. 6, 1113–1132.

Jumarie, G., 2006. New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations. Math. Comput. Model. 44, 231–254.

Jumarie, G., 2009. Laplace's transform of fractional order via the Mittag–Leffler function and modified Riemann–Liouville derivative. Appl. Math. Lett. 22.

Jumarie, G., 2009. Table of some basic fractional calculus formulae derived from a modified Riemann–Liouvillie derivative for non-differentiable functions. Appl. Math. Lett. 22.

Khan, Y., 2009. An effective modification of the laplace decomposition method for nonlinear equations. Int. J. Nonl. Sci. Num. Sim. 10, 1373–1376.

Khan, Y., Austin, F., 2010. Application of the Laplace decomposition method to nonlinear homogeneous and non-homogenous advection equations. Z. Naturforsch. 65a, 1–5.

Khan, Y., Faraz, N., xxxx. Application of modified laplace decomposition method for solving boundary layer equation. J. King. Saud. Univ. Sci. doi:10.1016/j.jksus.2010.06.018.

Khan, Y., Faraz, N., xxxx. Modified fractional decomposition method having integral $(d\xi)^{\alpha}$. J. King. Saud. Univ. Sci. doi:10.1016/j.jksus.2010.06.024.

Koçak, H., Yildirim, A., 2009. Numerical solution of 3D Green's function for the dynamic system of anisotropic elasticity. Phys. Lett. A 373, 3145–3150.

Miller, K.S., Ross, B., 2003. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons, Inc., New York.

Momani, S., 2005. Analytic and approximate solutions of the space- and time-fractional telegraph equations. Appl. Math. Comput. 170, 1126–1134.

Momani, S., 2005. Explicit and numerical solutions of the fractional KdV equation. Math. Comput. Sim. 70, 110–118.

Momani, S., 2005. Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficient using the decomposition method. Appl. Math. Comput. 165, 459–472.

Mustafa Inc., 2008. The approximate and exact solutions of the space and time-fractional Burgers equations with initial conditions by variational iteration method. J. Math. Anal. Appl. 345, 476–484.

Podlubny, I., 1999. Fractional Differential Equations. Academic Press, San Diego, CA.

Wu, G.C., Lee, E.W.M., 2010. Fractional variational iteration method and its application. Phys. Lett. A 374, 2506–2509.

Yildirim, Ahmet, 2010. Analytical approach to Fokker–Planck equation with space- and time-fractional derivatives by means of the homotopy perturbation method. J. King Saud. Univ. Sci.. doi:10.1016/j.jksus.2010.05.008.

Yildirim, A., Gülkanat, Y., 2010. Analytical approach to fractional Zakharov–Kuznetsov equations by He's homotopy perturbation method. Commun. Theor. Phys. 53, 1005–1010.

Yildirim, A., Momani, S., 2010. Series solution of a fractional oscillator by means of the homotopy perturbation method. Int. J. Comp. Math. 87, 1072–1082.