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A new numerical scheme for solving pantograph type nonlinear fractional integro-differential equations

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ABSTRACT

In this work, a general class of pantograph type nonlinear fractional integro-differential equations (PT-FIDEs) with non-singular and non-local kernel is considered. A numerical scheme based on the orthogonal basis functions including the shifted Legendre polynomials (SLPs) is proposed. First, we expand the unknown function and its derivatives in terms of the SLPs with unknown coefficients. Then, we present several theorems based on the SLPs for the help to achieve the approximate solution of the problem under study. Finally, by utilizing these theorems together with the collocation points, the main problem is transformed to a system of linear or nonlinear algebraic equations, which can be simply solved. An investigation for error estimate is discussed. The accuracy and efficiency of the proposed scheme are reported by four illustrative examples.

1. Introduction

Fractional calculus is an extension of the classical one which deal with derivatives and integrals of arbitrary real or complex order (Atangana and Hammouch, 2019; Baleanu et al., 2012; Podlubny, 1999; Srivastava et al., 2021; Yang et al., 2020). Fractional derivatives have been widely applied to describing various problems in different fields of applied science. These derivatives are useful in rheology as crucial features of cell rheological behavior (Djordjevic et al., 2003). Recently, the dynamics of coronavirus (2019-nCov) have modeled by with fractional derivative in Khan and Atangana (2020). Since in definition of the most important fractional operators such as Riemann–Liouville (RL) and Caputo exists a kernel of type local and sinqular, it is difficult or impossible to describe many non-local dynamics systems. Hence several definitions for fractional integral and derivative operators have been introduced such as Caputo–Fabrizio (CF) (Caputo and Fabrizio, 2015; Losada and Nieto, 2015), Atangana–Baleanu (AB) (Atangana and Baleanu, 2016) and Yang–Abdel–Aty–Cattani (YAAC) (Yang et al., 2019) operators. The most important advantage of these operators is the existence of the non-local and non-sinqular kernel which introduced to describe complex physical problems (Algahtani, 2016; Djida et al., 2017).

In this work, we consider a class of PT-FIDEs of the form

$$ {}^{ABC}D_t^{\alpha} z(t) = \lambda F\big(t, z(t), z(qt),\ I_t z(t), I_{qt} z(t)\big), \quad t \in [0,T], \quad 0 < q < 1, \quad 0 < \alpha \leqslant 1, \tag{1} $$

with

$$ \begin{aligned} I_t z(t) &= \int_0^t K_1(t,\tau)\,\phi_1(\tau, z(\tau))\,d\tau, \\ I_{qt} z(t) &= \int_0^{qt} K_2(t,\tau)\,\phi_2(\tau, z(\tau))\,d\tau, \end{aligned} \tag{2} $$

and the initial condition

$$ z(0) = z_0, \tag{3} $$

where λ and z_0 are real constants, K_1, K_2, ϕ_1 and ϕ_2 are given functions, $z(t)$ is a solution to be determined in $[0,T]$. ${}^{ABC}D_t^{\alpha}$ denotes the

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AB derivative in the Caputo sense. This new fractional derivative is introduced by Atangana and Baleanu which has non-singular kernel. In definition of this operator, there exists a kernel that is included a Mittag–Leffler (ML) function which is non-local and non-singular. Many properties of this operator are investigated in Atangana and Kocab (2016). The special cases of the Eq. (1) have been solved in Muroya et al. (2003), Rahimkhani et al. (2017), Zhao et al. (2017), Nemati et al. (2018).

Volterra nonlinear fractional integro-differential equations (V-NFIDEs) appear widely in many fields of science. The class of PV-IDEs is one of the most important classes of V-FIDEs. Many researchers have presented several numerical techniques for solving these equations (Muroya et al., 2003; Rahimkhani et al., 2017; Zhao et al., 2017).

Orthogonal basis functions have been generally used to achieve approximate solution for many problems in various fields of science. Approximation of the solution using these functions is known as a useful tool in solving many classes of equations, numerically, e.g., differential equations (Jafari et al., 2011; Mishra et al., 2016; Sabermahani et al., 2018; Sabermahani et al., 2020; Singh and Srivastava, 2019; Srivastava et al., 2019), partial differential equations (Ait Touchent et al., 2018; Deiveegan et al., 2019; Ganji et al., 2019; Yang et al., 2018; Yang and Tenreiro Machado, 2017; Ziane et al., 2019) and integro-differential equations (Ganji and Jafari, 2020; Ganji and Jafari, 2019; Nemati et al., 2018; Sedaghat et al., 2014; Nieto and Samet, 2017; Jothimani et al., 2019) of various orders (fixed, fractional or variable order).

The outline of this work is as follows. A brief review of definitions of RL and AB operators and their important properties are presented in Section 2. Section 3 the SLPs with their properties are reviewed. We proposed a numerical scheme for solving problem (1) under the initial condition given by (3) in section (4). In section (5), we discussed about error bound of the proposed scheme. Some illustrative examples are solved in Section 6. In the last section, we conclude the paper.

2. RL and AB operators and their properties

In this section, we first recall many special functions and then bring definitions of (RL and AB)- integral and derivative operators with their properties which will be used further.

Definition 1 (*See* Podlubny (1999)). The Beta and Mittag–Leffler functions are defined, respectively, by

(The Beta function) $B(\mu,\nu)=\int_0^1 \tau^{\mu-1}(1-\tau)^{\nu-1}\,d\tau$, $Re(\mu)\&Re(\nu)>0$,

(One parameter ML function) $E_\alpha(t)=\sum_{i=0}^{\infty}\frac{t^i}{\Gamma(\alpha i+1)}$,

(Two parameters ML function) $E_{\alpha,\beta}(t)=\sum_{i=0}^{\infty}\frac{t^i}{\Gamma(\alpha i+\beta)}$.

Definition 2 (*See* Podlubny (1999)). The α order RL-integral is given by

$$^{RL}I_t^\alpha z(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}z(\tau)\,d\tau.$$

The RL-integral of order α satisfies the following property

$$^{RL}I_t^\alpha t^\zeta=\frac{\Gamma(1+\zeta)}{\Gamma(\alpha+\zeta+1)}t^{\alpha+\zeta},\quad \zeta\geqslant 0.$$

Definition 3 (*See* Atangana and Baleanu (2016), Yang (2019)). Let $0<\alpha\leqslant 1, z\in H^1(0,1)$ and $\aleph(\alpha)$ be a normalization function such that $\aleph(0)=\aleph(1)=1$ and for $0<\alpha<1,\ \aleph(\alpha)=1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$. Then

(1) The ABC-derivative is defined as follows

$$^{ABC}D_t^\alpha z(t)=\begin{cases}\frac{AB(\alpha)}{1-\alpha}\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)z\prime(\tau)\,d\tau, & \\ z\prime(t) & \alpha=1.\end{cases}\tag{4}$$

(2) The AB-integral is given by

$$^{AB}I_t^\alpha z(t)=\frac{1-\alpha}{\aleph(\alpha)}z(t)+\frac{\alpha}{\aleph(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}z(\tau)\,d\tau.\tag{5}$$

Let $\upsilon_\alpha=\frac{1-\alpha}{\aleph(\alpha)}$and $\omega_\alpha=\frac{1}{\aleph(\alpha)\Gamma(\alpha)}$, then we can rewrite (5) by

$$^{AB}I_t^\alpha z(t)=\upsilon_\alpha z(t)+\omega_\alpha\Gamma(\alpha+1)\,^{RL}I_t^\alpha z(t).\tag{6}$$

It is easy to report that the AB operators satisfy the following properties (Atangana and Baleanu, 2016; Ganji and Jafari, 2020; Ganji et al., 2020)

$$\begin{aligned}&^{ABC}D_t^\alpha C=0,\quad C\in\mathbb{R},\\&^{ABC}D_t^\alpha t^\beta=\frac{\aleph(\alpha)\beta! t^\beta}{1-\alpha}E_{\alpha,1+\beta}\left(-\frac{\alpha}{1-\alpha}t^\alpha\right),\quad \beta\geqslant 0,\\&^{AB}I_t^\alpha C=C(\upsilon_\alpha+\omega_\alpha t^\alpha),\quad C\in\mathbb{R},\\&^{AB}I_t^\alpha t^\beta=t^\beta(\upsilon_\alpha+\omega_\alpha(\alpha+1+\beta)B(1+\beta,1+\alpha)t^\alpha),\\&^{AB}I_t^\alpha\left(^{ABC}D_t^\alpha z(t)\right)=z(t)-z(0).\end{aligned}$$

Theorem 1 (*See* Tajadodi (2020)). *Let* $0<\alpha\leqslant 1$. *Then, we can rewrite the AB-derivative by*

$$^{ABC}D_t^\alpha z(t)=\frac{AB(\alpha)}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\,^{RL}I_t^{r\alpha+1}z\prime(t).$$

3. The SLPs and their properties

Now, firstly we express some basic properties of the SLPs. After that we explain to approximate a function with SLPs and obtaining OM based on SLPs.

3.1. The SLPs

The explanation of the SLPs on $[0,T]$ is

$$L_i^*(t)=L_i\left(\frac{2}{T}t-1\right),\quad i=0,1,2,\ldots,\tag{7}$$

where $L_i(t)$ is the well-known Legendre polynomial (LP) of degree i. The recursive formula of LP on $[-1,1]$ given by

$$L_{i+1}(t)=\frac{1+2i}{1+i}tL_i(t)-\frac{i}{1+i}L_{i-1}(t),\quad i=1,2,3,\ldots,$$

where $L_0(t)=1$ and $L_1(t)=t$.

The SLPs $L_i^*(t)$ given in (7), could be written the following analytic form

$$L_i^*(t)=\sum_{s=0}^{i}a_{i,s}t^s,\tag{8}$$

where

$$a_{i,s}=\frac{(-1)^{i+s}(i+s)!}{(i-s)!(s!)^2T^s}.\tag{9}$$

For the SLPs, the orthogonality condition is as follows

$$\int_0^T L_i^*(t)L_s^*(t)\,dt = \begin{cases} \frac{T}{1+2i}, & i=s,\\ 0, & i\neq s.\end{cases}$$

For two arbitrary functions z_1, z_2 in $L^2(0,T)$, the inner product and norm are defined, respectively, by

$$\langle z_1(t), z_2(t)\rangle = \int_0^T z_1(t)z_2(t)\,dt,$$
$$\|z_1(t)\|_{L^2(0,T)} = \left\langle z_1(t), z_1(t)\right\rangle^{\frac{1}{2}} = \left(\int_0^T |z_1(t)|^2\,dt\right)^{\frac{1}{2}}.$$

3.2. Approximation of a function

Assume that we can expand $z(t)\in L^2(0,T)$ in terms of the SLPs as

$$z(t) = \sum_{i=0}^{\infty} z_i\, L_i^*(t), \tag{10}$$

where

$$z_i = \frac{1+2i}{T}\int_0^T z(t)\, L_i^*(t)\,dt.$$

We can present z by using a truncated series as

$$z(t) \simeq z_M(t) = \sum_{i=0}^{M} z_i L_i^*(t) = Z^T \mathscr{L}(t), \tag{11}$$

where $Z = [z_0, z_1, \ldots, z_M]^T$ and

$$\mathscr{L}(t) = \left[L_0^*(t), L_1^*(t), \ldots, L_M^*(t)\right]^T. \tag{12}$$

Also, we can approximate the function $z(t,\tau)\in L^2((0,T)\times(0,T))$ in terms of the SLPs by

$$z(t,\tau) \simeq \mathscr{L}^T(t)\,\mathscr{Z}\,\mathscr{L}(\tau),$$

where $\mathscr{Z} = [z_{i,j}]$is an $(M+1)\times(M+1)$ matrix which $z_{i,j}, i,j = 0,1,\ldots,M$ are given by

$$z_{i,j} = \frac{\left\langle\left\langle z(t,\tau), L_i^*(t)\right\rangle, L_j^*(\tau)\right\rangle}{\|L_i^*(t)\|_2^2\,\|L_j^*(\tau)\|_2^2}, \quad i,j = 0,1,\ldots,M.$$

Lemma 1. *Suppose* $0<q<1$*and* $\mathscr{L}(t)$*given by* (12). *Then*

$$\mathscr{L}(qt) \simeq \mathscr{H}\mathscr{L}(t),$$

where $\mathscr{H}$ *is given by*

$$\mathscr{H} = \begin{bmatrix} \sigma_{0,0,0} & \sigma_{0,1,0} & \cdots & \sigma_{0,M,0}\\ \sum_{s=0}^{1}\sigma_{1,0,s} & \sum_{s=0}^{1}\sigma_{1,1,s} & \cdots & \sum_{s=0}^{1}\sigma_{1,M,s}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{s=0}^{M}\sigma_{M,0,s} & \sum_{s=0}^{M}\sigma_{M,1,s} & \cdots & \sum_{s=0}^{M}\sigma_{M,M,s}\end{bmatrix},$$

with

$$\sigma_{i,k,s} = a_{i,s}h_{s,k}q^s.$$

Proof. By substituting $t = qt$ into (8), we get

$$L_i^*(qt) = \sum_{s=0}^{i} a_{i,s}q^s t^s \qquad i = 0,1,\ldots,M. \tag{13}$$

Now, we approximate the function t^s in terms of the SLPs by

$$t^s \simeq \sum_{k=0}^{M} h_{s,k}L_k^*(t). \tag{14}$$

Now for $i=0$ to $i=M$, By substituting (14) into (13), leads

$$L_i^*(qt) \simeq \sum_{s=0}^{i} a_{i,s}q^s\left(\sum_{k=0}^{M} h_{s,k}L_k^*(t)\right) = \sum_{k=0}^{M}\left(\sum_{s=0}^{i} a_{i,s}h_{s,k}q^s\right)L_k^*(t)$$
$$= \sum_{k=0}^{M}\left(\sum_{s=0}^{i}\sigma_{i,k,s}\right)L_k^*(t),$$

which completes the proof.

Lemma 2 (*See* Ganji et al. (2020)). *The operational matrix (OM) of the product and integration of the vector* $\mathscr{L}(t)$ *given by* (12) *can be approximated, respectively, as*

$$\mathscr{L}(t)\mathscr{L}^T(t)Z \simeq \widehat{Z}\mathscr{L}(t),$$
$$\int_0^t \mathscr{L}(\tau)\,d\tau \simeq \mathscr{P}\mathscr{L}(t),$$

where $\widehat{Z}$ *and* $\mathscr{P}$ *are given in* Ganji et al. (2020).

Theorem 2. *Suppose* $\mathscr{L}(t)$ *given by* (12). *Then*

$$\int_0^{qt}\mathscr{L}(\tau)\;d\tau \simeq \mathscr{P}^*\mathscr{L}(t),$$

where $\mathscr{P}^*$ *is given by*

$$\mathscr{P}^* = \begin{bmatrix} \varsigma_{0,0,0} & \varsigma_{0,1,0} & \cdots & \varsigma_{0,M,0}\\ \sum_{s=0}^{1}\varsigma_{1,0,s} & \sum_{s=0}^{1}\varsigma_{1,1,s} & \cdots & \sum_{s=0}^{1}\varsigma_{1,M,s}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{s=0}^{M}\varsigma_{M,0,s} & \sum_{s=0}^{M}\varsigma_{M,1,s} & \cdots & \sum_{s=0}^{M}\varsigma_{M,M,s}\end{bmatrix},$$

with

$$\varsigma_{i,k,s} = \frac{a_{i,s}d_{s,k}q^{s+1}}{s+1}.$$

Proof. By (12), for $i = 0,1,\ldots,M$, we have

$$\int_0^{qt} L_i^*(\tau)\;d\tau = \int_0^{qt}\left(\sum_{s=0}^{i} a_{i,s}\tau^s\right)\;d\tau = \sum_{s=0}^{i} a_{i,s}\left(\int_0^{qt}\tau^s\;d\tau\right)$$
$$= \sum_{s=0}^{i}\frac{a_{i,s}q^{s+1}}{s+1}t^{s+1}, \tag{15}$$

We expand t^{s+1} in the above equation by using the SLPs. It gives

$$t^{s+1} \simeq \sum_{k=0}^{M} d_{s,k}L_k^*(t). \tag{16}$$

By putting (16) into (15), we get

$$\int_0^{qt} L_i^*(\tau)\;d\tau \simeq \sum_{s=0}^{i}\frac{a_{i,s}q^{s+1}}{s+1}\left(\sum_{k=0}^{M} d_{s,k}L_k^*(t)\right)$$
$$= \sum_{k=0}^{M}\left(\sum_{s=0}^{i}\frac{a_{i,s}d_{s,k}q^{s+1}}{s+1}\right)L_k^*(t) = \sum_{k=0}^{M}\left(\sum_{s=0}^{i}\varsigma_{i,k,s}\right)L_k^*(t),$$

now the proof is completed.

Theorem 3. *Suppose $0 < \alpha \leqslant 1$. The α order AB-integral of a vector $\mathscr{L}(t)$ given in (12) might be approximated by*

$$ {}^{AB}I_t^{\alpha}\mathscr{L}(t) \simeq \mathscr{I}^{\alpha}\mathscr{L}(t), $$

where $\mathscr{I}^{\alpha} = \upsilon_{\alpha}I + \omega_{\alpha}\Gamma(\alpha+1)\mathscr{F}^{\alpha}$ is called the OM of the AB-integral based on the SLPs and I is an $(M+1)\times(M+1)$ identity matrix. Also, $\mathscr{F}^{\alpha}$ is called the OM of RL-integral based on the SLPs which is given by

$$ \mathscr{F}^{\alpha} = \begin{bmatrix} \rho_{0,0,0} & \rho_{0,1,0} & \cdots & \rho_{0,M,0} \\ \sum_{s=0}^{1}\rho_{1,0,s} & \sum_{s=0}^{1}\rho_{1,1,s} & \cdots & \sum_{s=0}^{1}\rho_{1,M,s} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{s=0}^{M}\rho_{M,0,s} & \sum_{s=0}^{M}\rho_{M,1,s} & \cdots & \sum_{s=0}^{M}\rho_{M,M,s} \end{bmatrix}, $$

with

$$ \rho_{i,k,s} = \frac{\Gamma(s+1)a_{i,s}e_{s,k}}{\Gamma(s+\alpha+1)}. $$

Proof. By applying the AB-integral operator on the vector $\mathscr{L}(t)$ yields

$$ {}^{AB}I_t^{\alpha}\mathscr{L}(t) = \upsilon_{\alpha}\mathscr{L}(t) + \omega_{\alpha}\Gamma(\alpha+1)\,{}^{RL}I_t^{\alpha}\mathscr{L}(t). \tag{17} $$

Now, we must obtain the OM of RL-integral of order α. To do this, we apply the LR-integral operator, ${}^{RL}I_t^{\alpha}$, on $L_i^*(t), i = 0,1,\ldots,M$as

$$ {}^{RL}I_t^{\alpha}L_i^*(t) = {}^{RL}I_t^{\alpha}\left(\sum_{s=0}^{i}a_{i,s}t^s\right) = \sum_{s=0}^{i}a_{i,s}\left({}^{RL}I_t^{\alpha}t^s\right) = \sum_{s=0}^{i}\frac{\Gamma(s+1)a_{i,s}}{\Gamma(s+\alpha+1)}t^{s+\alpha}. $$

By approximating the function $t^{s+\alpha}$ in terms of the SLPs, we get

$$ t^{s+\alpha} \simeq \sum_{k=0}^{M}e_{s,k}L_k^*(t). \tag{18} $$

In view of (18) and for $i = 0,1,\ldots,M$, we get

$$ \begin{aligned} {}^{RL}I_t^{\alpha}L_i^*(t) &\simeq \sum_{s=0}^{i}\frac{\Gamma(s+1)a_{i,s}}{\Gamma(s+\alpha+1)}\left(\sum_{k=0}^{M}e_{s,k}L_k^*(t)\right) = \sum_{k=0}^{M}\left(\sum_{s=0}^{i}\frac{\Gamma(s+1)a_{i,s}e_{s,k}}{\Gamma(s+\alpha+1)}\right)L_k^*(t) \\ &= \sum_{k=0}^{M}\left(\sum_{s=0}^{i}\rho_{i,k,s}\right)L_k^*(t). \end{aligned} $$

Therefore, for $i = 0,1,\ldots,M$, we can write

$$ {}^{RL}I_t^{\alpha}\mathscr{L}(t) = \mathscr{F}^{\alpha}\mathscr{L}(t), \tag{19} $$

where

$$ \mathscr{F}^{\alpha} = \begin{bmatrix} \rho_{0,0,0} & \rho_{0,1,0} & \cdots & \rho_{0,M,0} \\ \sum_{s=0}^{1}\rho_{1,0,s} & \sum_{s=0}^{1}\rho_{1,1,s} & \cdots & \sum_{s=0}^{1}\rho_{1,M,s} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{s=0}^{M}\rho_{M,0,s} & \sum_{s=0}^{M}\rho_{M,1,s} & \cdots & \sum_{s=0}^{M}\rho_{M,M,s} \end{bmatrix}, $$

with

$$ \rho_{i,k,s} = \frac{\Gamma(s+1)a_{i,s}e_{s,k}}{\Gamma(s+\alpha+1)}. $$

By substituting (19) into (17), the proof completes.

4. Numerical scheme

The purpose of this section is to present a numerical scheme for solving Eq. (1) under the initial condition (3). To this aim, we first approximate the function ${}^{ABC}D_t^{\alpha}z(t)$ in terms of the SLPs as

$$ {}^{ABC}D_t^{\alpha}z(t) \simeq Z^T\mathscr{L}(t). \tag{20} $$

First we apply the α order AB-integral on the both sides of (20) and use the initial condition, we have

$$ z(t) \simeq Z^T\mathscr{I}^{\alpha}\mathscr{L}(t) + z_0. \tag{21} $$

By approximating $z_0 \simeq \mathscr{B}^T\mathscr{L}(t)$, (21) is rewritten as

$$ z(t) \simeq \mathscr{Y}\mathscr{L}(t), \tag{22} $$

where $\mathscr{Y} = Z^T\mathscr{I}^{\alpha} + \mathscr{B}^T$. By putting $t = qt$ in (22) yields

$$ z(qt) \simeq \mathscr{Y}\mathscr{L}(qt). \tag{23} $$

By employing Lemma 1, (23) is approximated as

$$ z(qt) \simeq \mathscr{Y}\mathscr{H}\mathscr{L}(t). \tag{24} $$

For approximating the Volterra parts of Eq. (1), we expand K_1, K_2, ϕ_1 and ϕ_2 using the SLPs as

$$ \begin{aligned} &K_1(t,\tau) \simeq \mathscr{L}^T(t)\mathscr{K}_1\mathscr{L}(\tau), \\ &K_2(t,\tau) \simeq \mathscr{L}^T(t)\mathscr{K}_2\mathscr{L}(\tau), \\ &\phi_1(t,z(t)) \simeq \mathscr{C}^T\mathscr{L}(\tau), \\ &\phi_2(t,z(t)) \simeq \mathscr{D}^T\mathscr{L}(\tau). \end{aligned} \tag{25} $$

By utilizing (25), Lemma 2, Theorem 2, and in a similar way (Ganji et al., 2020), we obtain

$$ \begin{aligned} I_tz(t) &\simeq \int_0^t\left(\mathscr{L}^T(t)\mathscr{K}_1\mathscr{L}(\tau)\right)\left(\mathscr{L}^T(\tau)\mathscr{C}\right)d\tau = \mathscr{L}^T(t)\mathscr{K}_1\widehat{\mathscr{C}}\int_0^t\mathscr{L}(\tau)\,d\tau = \mathscr{L}^T(t)\mathscr{K}_1\widehat{\mathscr{C}}\mathscr{P}\mathscr{L}(t), \\ I_{qt}z(t) &\simeq \int_0^{qt}\left(\mathscr{L}^T(t)\mathscr{K}_2\mathscr{L}(\tau)\right)\left(\mathscr{L}^T(\tau)\mathscr{D}\right)d\tau = \mathscr{L}^T(t)\mathscr{K}_2\widehat{\mathscr{D}}\int_0^{qt}\mathscr{L}(\tau)\,d\tau = \mathscr{L}^T(t)\mathscr{K}_2\widehat{\mathscr{D}}\mathscr{P}^*\mathscr{L}(t). \end{aligned} \tag{26} $$

Substituting (20), (22), (24) and (26) into Eq. (1), leads

$$ Z^T\mathscr{L}(t) = \lambda F\left(t, \mathscr{Y}\mathscr{L}(t), \mathscr{Y}\mathscr{H}\mathscr{L}(t), \mathscr{L}^T(t)\mathscr{K}_1\widehat{\mathscr{C}}\mathscr{P}\mathscr{L}(t), \mathscr{L}^T(t)\mathscr{K}_2\widehat{\mathscr{D}}\mathscr{P}^*\mathscr{L}(t)\right). \tag{27} $$

Also, by substituting (20) into (25) yields

$$ \begin{aligned} &\phi_1(t,\mathscr{Y}\mathscr{L}(t)) \simeq \mathscr{C}^T\mathscr{L}(t), \\ &\phi_2(t,\mathscr{Y}\mathscr{L}(t)) \simeq \mathscr{D}^T\mathscr{L}(t). \end{aligned} \tag{28} $$

Finally, by substituting the collocation points $\frac{k}{M+2}T, k = 1,\ldots,M+1$ into Eqs. (27) and (28), a system of $3(M+1)$ nonlinear equations of the vectors of $Z, \mathscr{C}$ and $\mathscr{D}$ is formed. By solving this system, the unknown parameters of the vectors of $Z, \mathscr{C}$ and $\mathscr{D}$ are obtained. Finally the approximate solution can be computed by (22).

5. Error estimation

This section deals an estimate for the error of the numerical solution of Eq. (1) with initial condition (3) obtained by the proposed scheme in Section 4.

It is well known in the interval (a,b), the Sobolev norm of integer order $\mu \geqslant 0$, is defined by

$$ \|z\|_{H^{\mu}(a,b)} = \left(\sum_{k=0}^{\mu}\|z^{(k)}\|_{L^2(a,b)}\right)^{\frac{1}{2}}, $$

where $z^{(k)}$ denotes the kth derivative of z and $H^{\mu}(a,b)$ is a Sobolev space.

Lemma 3 (*See Canuto et al. (2006)*). *Let $\mu \geqslant 0$ and $z \in H^m(-1,1)$. Suppose $P_M(z) = \sum_{i=0}^{M} z_i L_i(t)$ be the truncated Legendre series of z. Then,*

$$\|z - P_M(z)\|_{L^2(-1,1)} \leqslant CM^{-\mu}|z|_{H^{\mu;M}(-1,1)},$$

where

$$|z|_{H^{\mu;M}(-1,1)} = \left(\sum_{k=\min\{1+M,\mu\}}^{\mu} \|z^{(k)}\|^2_{L^2(-1,1)}\right)^{\frac{1}{2}},$$

and C is a positive constant and does not depend to z and integer M.

Lemma 4 (*See Ganji et al. (2020)*). *Let $z:(0,T)\longrightarrow\mathbb{R}$ be a function in $H^\mu(0,T)$. Suppose that function $\bar{z}:(-1,1)\longrightarrow\mathbb{R}$ is given by $\bar{z}(t) = z(\frac{T}{2}(t+1))$ for all $t\in(-1,1)$. Then, for $0\leqslant k\leqslant\mu$*

$$\|\bar{z}^{(k)}\|_{L^2(-1,1)} = \left(\frac{2}{T}\right)^{\frac{1}{2}-k} \|z^{(k)}\|_{L^2(0,T)}.$$

Theorem 4. *Suppose $\mu \geqslant 0$ and $z \in H^\mu(0,T)$. Let $z_M(t) = \sum_{i=0}^{M} z_i L_i^*(t)$ is the obtained approximate solution by the given scheme in Section 4. Then,*

$$\|z - z_M\|_{L^2(0,T)} \leqslant CM^{-\mu}|z|_{H^{\mu;M;0}(0,T)},$$

and

$$\|z^{(i)} - z_M^{(i)}\|_{L^2(0,T)} \leqslant CM^{-\mu}|z|_{H^{\mu;M;i}(0,T)},$$

where

$$|z|_{H^{\mu;M;r}(0,T)} = \left(\sum_{k=\min\{1+M,\mu\}}^{\mu} \left(\frac{T}{2}\right)^{2k} \|z^{(k+r)}\|^2_{L^2(0,T)}\right)^{\frac{1}{2}}, \qquad r \geqslant 0.$$

Proof. With the help Lemma 4, we obtain

$$\begin{aligned}\|z - z_M\|^2_{L^2(0,T)} &= \tfrac{T}{2}\|\bar{z} - P_M(\bar{z})\|^2_{L^2(-1,1)}\\ &\leqslant \tfrac{T}{2}C\prime M^{-2\mu}\sum_{k=\min\{1+M,\mu\}}^{\mu}\|\bar{z}^{(k)}\|^2_{L^2(-1,1)}\\ &= C\prime M^{-2\mu}\sum_{k=\min\{1+M,\mu\}}^{\mu}\left(\tfrac{T}{2}\right)^{2k}\|z^{(k)}\|^2_{L^2(0,T)}.\end{aligned}$$

By definition

$$|z|_{H^{\mu;M;0}(0,T)} = \left(\sum_{k=\min\{1+M,\mu\}}^{\mu} \left(\frac{T}{2}\right)^{2k} \|z^{(k)}\|^2_{L^2(0,T)}\right)^{\frac{1}{2}},$$

the proof completes. By similar way, we obtain

$$\|z^{(i)} - z_M^{(i)}\|_{L^2(0,T)} \leqslant CM^{-\mu}|z|_{H^{\mu;M;i}(0,T)},$$

where

$$|z|_{H^{\mu;M;i}(0,T)} = \left(\sum_{k=\min\{\mu,M+1\}}^{\mu} \left(\frac{T}{2}\right)^{2k} \|z^{(k+i)}\|^2_{L^2(0,T)}\right)^{\frac{1}{2}}.$$

Theorem 5. *Suppose $0 < \alpha \leqslant 1$ and $z \in H^\mu(0,T)$ satisfies in Theorem 4. Then*

$$\|{}^{ABC}D_t^\alpha z - {}^{ABC}D_t^\alpha z_M\|_{L^2(0,T)} \leqslant \frac{AB(\alpha)T}{1-\alpha}E_{\alpha,2}\left(-\frac{\alpha}{1-\alpha}T^\alpha\right)CM^{-\mu}|z|_{H^{\mu,M,1}(0,T)}.$$

Proof. By employing Theorems 1 and 4, we get

$$\begin{aligned}\|{}^{ABC}D_t^\alpha z - {}^{ABC}D_t^\alpha z_M\|_{L^2(0,T)} &= \left\|\tfrac{AB(\alpha)}{1-\alpha}\sum_{r=0}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^r {}^{RL}I_t^{r\alpha+1}(z\prime - z_M\prime)\right\|_{L^2(0,T)}\\ &\leqslant \tfrac{AB(\alpha)}{1-\alpha}\sum_{r=0}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^r \tfrac{T^{r\alpha+1}}{\Gamma(r\alpha+2)}\|z\prime - z_M\prime\|_{L^2(0,T)}\\ &\leqslant \tfrac{AB(\alpha)T}{1-\alpha}E_{\alpha,2}\left(-\tfrac{\alpha}{1-\alpha}T^\alpha\right)CM^{-\mu}|z|_{H^{\mu,M,1}(0,T)}.\end{aligned}$$

Lemma 5. *Suppose $k_1 = \max_{0\leqslant t,\tau\leqslant T}|K_1(t,\tau)|, k_2 = \max_{0\leqslant t,\tau\leqslant T}|K_2(t,\tau)|$, and ϕ_1 and ϕ_2 satisfy the Lipschitz conditions with the constants L_1 and L_2, respectively. Let $z \in H^\mu(0,1)$ satisfies in Theorem 4. Then*

$$\begin{aligned}&\|I_t z - I_t z_M\|_{L^2(0,T)} \leqslant k_1L_1TCM^{-\mu}|z|_{H^{\mu,M,0}(0,T)},\\ &\|I_t z - I_{qt} z_M\|_{L^2(0,T)} \leqslant k_2L_2qTCM^{-\mu}|z|_{H^{\mu,M,0},(0,T)}.\end{aligned}$$

Proof. According to (2) and using Theorem 4, the proof completes.

Theorem 6. *Suppose $\mu \geqslant 0$and $z \in H^\mu(0,T)$satisfies in Theorems 4, 5 and Lemma 5. Let F satisfies the Lipschitz conditions with the constant L. Then E_M, the error bound of the proposed scheme, is bounded as follows*

$$\|E_M\|_{L^2(0,T)} \leqslant CM^{-\mu}T\left(\frac{AB(\alpha)}{1-\alpha}E_{\alpha,2}\left(-\frac{\alpha}{1-\alpha}T^\alpha\right)|z|_{H^{\mu,M,1}(0,T)} + |\lambda|L\left(\frac{2}{T} + k_1L_1 + k_2L_2q\right)|z|_{H^{\mu,M,0}(0,T)}\right).$$

Proof. In view of Eq. (1), we get

$$\begin{aligned}\|E_M\|_{L^2(0,T)} \leqslant\; &\|{}^{ABC}D_t^\alpha z - {}^{ABC}D_t^\alpha z_M - \lambda F(t,z(t),z(qt),I_t z(t),I_{qt}z(t))\\ &+\lambda F(t,z_M(t),z_M(qt),I_t z_M(t),I_{qt}z_M(t))\|_{L^2(0,T)}\\ &\leqslant \|{}^{ABC}D_t^\alpha z - {}^{ABC}D_t^\alpha z_M\|_{L^2(0,T)} + |\lambda|L\Big(2\|z - z_M\|_{L^2(0,T)}\\ &+\|I_t z - I_t z_M\|_{L^2(0,T)} + \|I_{qt}z - I_{qt}z_M\|_{L^2(0,T)}\Big).\end{aligned}$$

By employing Theorems 4, 5 and Lemma 5, the proof completes.

6. Numerical results

Now, we solve some illustrative examples to show the accuracy and efficiency of the proposed scheme. The codes were written in Mathematica software. For the difference between the value of the exact and approximate solutions at some selected points, we use the following notations

$$\begin{aligned}&\text{Absolute error} = |z(t_k) - z_M(t_k)|, \quad 0\leqslant k\leqslant M,\\ &MAE = \max_{0\leqslant k\leqslant M}|z(t_k) - z_M(t_k)|.\end{aligned}$$

Example 1. Consider the following PT-FIDE

$$\begin{aligned}{}^{ABC}D_t^\alpha z(t) = \;& z(t) - \tfrac{1}{2}\ln\left(1+\tfrac{t}{2}\right)z\left(\tfrac{t}{2}\right) + \tfrac{1}{1+t} - \ln(1+t)\left(\tfrac{t}{2}\ln(1+t)+1\right)\\ &+\int_0^t \tfrac{t}{1+\tau}z(\tau)\,d\tau + \int_0^{\frac{t}{2}}\tfrac{1}{1+\tau}z(\tau)\,d\tau, \qquad t\in[0,1],\end{aligned}$$

under the initial condition

$$z(0) = 0.$$

By applying the proposed scheme, the approximate solution for this problem is computed. By considering $M = 5$, the approximate solu-

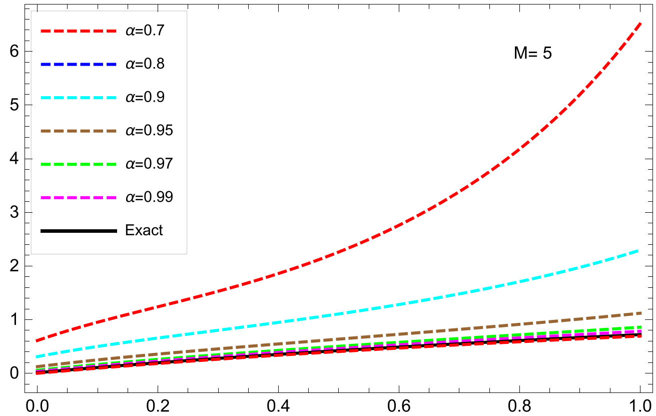

Fig. 1. (Example 1) Approximate solutions for different values of α.

tion together with the exact solution ($z(t) = \ln(1+t)$when $\alpha = 1$) for various values of α are illustrated in Fig. 1. Zhao et al. (2017) have solved this problem using the Sinc collocation method (SCM) for getting its approximate solution. Hence, in Table 1, the MAE of $z(t)$obtained by the proposed scheme with those obtained in Zhao et al. (2017) at different choices of M is compared. As seen from Fig. 1 and Table 1, by increasing the number of basis functions the numerical solution converges to the exact one. Also, Table 1 shows the proposed scheme only with a small number of basis functions gives more favorable results than the method given by Zhao et al. (2017).

Example 2. Consider the following PV-FIDE

$$ {}^{ABC}D_t^{\alpha} z(t) = z\left(\tfrac{t}{2}\right) - 1 + \tfrac{t^2}{4} - \tfrac{t^4}{64} + \tfrac{t^5}{80} - \tfrac{t^6}{384} + t\left(3 - \tfrac{1}{2} e^{(-1+t)t} - \frac{\sqrt{\pi} Erfi\left[\frac{1}{2}\right]}{4e^{\frac{1}{4}}}\right) + \frac{\sqrt{\pi} t Erfi\left[\frac{1}{2} - t\right]}{4e^{\frac{1}{4}}} + \int_0^t t\tau e^{z(\tau)}\, d\tau + \int_0^{\frac{t}{2}} \tau z^2(\tau)\, d\tau, \qquad t \in [0,1], $$

under the initial condition

$$ z(0) = 0, $$

where $Erfi(\cdot)$ is the imaginary error function. The exact solution is given by $z(t) = t^2 - t$ when $\alpha = 1$. For different values of α, in Fig. 2, by setting $M = 5, 7$ and $T = 1, 2$, we have reported the obtained numerical results by the proposed scheme at some selected points. Also, by considering $T = 1$, comparison of the absolute error at those selected points with different values M and α is shown in Tables 2 and 3.

Table 1
(Example 1) Comparison of the absolute error at some selected points for $\alpha = 1$.

Method of Zhao et al. (2017)		Present method		
M	MAE	M	MAE	CPU time
5	1.70e−3	3	9.96e−4	0.016
10	1.11e−4	5	3.62e−5	0.063
20	1.96e−6	7	1.57e−6	0.156
30	8.70e−8	10	5.17e−8	0.516
40	6.26e−9	12	2.50e−9	1.047

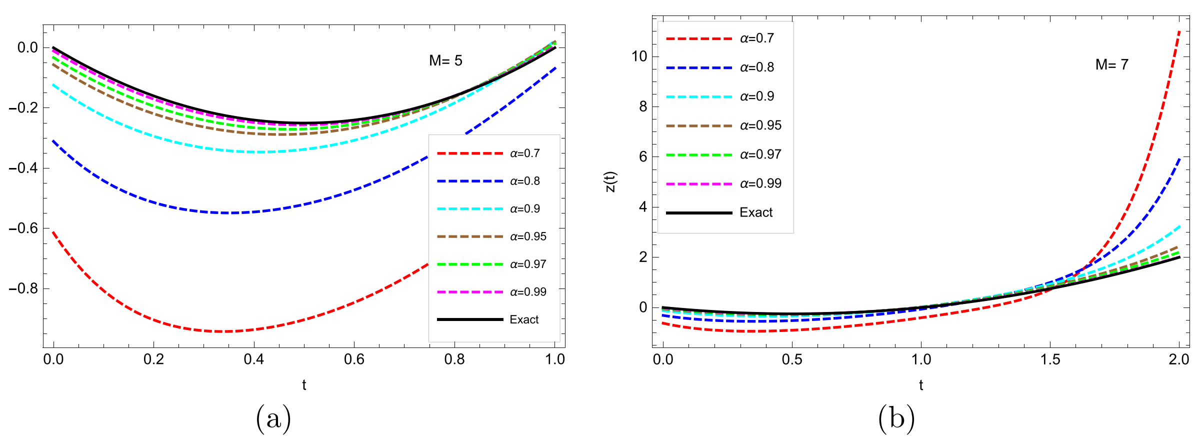

Fig. 2. (Example 2) The exact and approximate solutions given by different values of α (a) $M = 5$and $t \in [0,1]$ (b) $M = 7$ and $t \in [0,2]$.

Table 2
(Example 2) Comparison of the absolute error at some selected points with different values M.

		$\alpha = 1$	$T = 1$		
t	$M = 3$	$M = 5$	$M = 7$	$M = 9$	$M = 11$
0.1	1.94e−5	1.20e−6	1.13e−8	2.83e−11	1.62e−12
0.3	2.68e−5	2.73e−7	1.73e−8	1.54e−10	1.44e−12
0.5	8.38e−5	2.25e−6	4.61e−9	2.82e−10	1.19e−12
0.7	1.33e−4	1.41e−6	2.20e−8	6.32e−11	2.57e−12
0.9	5.66e−5	2.06e−6	1.91e−8	3.43e−10	5.13e−13

Table 3
(Example 2) Comparison of the absolute error at some selected points with different values α.

		$M = 7$	$T = 1$		
t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 1$
0.1	7.20e−1	3.53e−1	1.37e−2	1.13e−3	1.13e−8
0.3	7.28e−1	3.34e−1	1.22e−2	9.58e−3	1.73e−8
0.5	6.53e−1	2.71e−1	8.78e−2	6.14e−3	4.61e−9
0.7	5.51e−1	1.91e−1	4.54e−2	1.90e−3	2.20e−8
0.9	4.51e−1	1.08e−1	8.75e−5	2.66e−3	1.91e−8

Table 4
(Example 3) Comparison of the absolute error at some selected points for $\alpha = 1$.

	Method of Zhao et al. (2017)		Present method	
M	MAE	M	MAE	CPU time
5	3.60e−3	3	2.30e−3	0.031
10	2.23e−4	5	2.83e−5	0.078
20	5.72e−6	7	1.20e−7	0.406
30	2.89e−7	10	4.50e−10	1.016
40	2.21e−8	12	1.42e−10	2.203

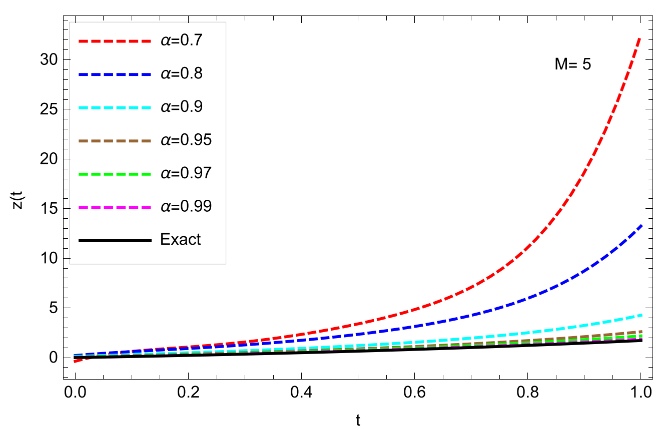

Fig. 3. (Example 3) The exact and approximate solutions given by different values of α.

Example 3. Consider the following PT-FIDE

$$ {}^{ABC}D_t^{\alpha}z(t) = \tfrac{1}{2}z(t) + z(\tfrac{t}{4}) + \tfrac{1}{2} - \tfrac{t}{4}e^{\frac{t}{4}} + \tfrac{t^2}{32} - \tfrac{1}{2}e^{3t} + e^{2t} + \int_0^t e^{t+\tau}z(\tau)\,d\tau + \int_0^{\frac{t}{4}} \tau z(\tau)\,d\tau, \qquad t \in [0,1], $$

under the initial condition

$z(0) = 0.$

Zhao et al. (2017) have considered this example and solved it by the SCM to achieve its approximate solution. Hence, in Table 4, the MAE of $z(t)$ obtained by the proposed scheme with those obtained in Zhao et al. (2017) at various values of M is compared. Also, by taking $M = 5$, the approximate solution together with the exact solution ($z(t) = e^t - 1$ when $\alpha = 1$) with different choices of α are shown in Fig. 3.

Example 4. Consider the fractional pantograph differential equation

$$ {}^{ABC}D_t^{\alpha}z(t) = -z(t) + 0.1z(0.2t) - 0.1e^{-0.2t}, \qquad t \in [0,1], $$

under the initial condition

$z(0) = 1.$

By employing the proposed scheme, we have achieve the approximate solution by setting $M = 5$ and plotted the approximate solution along with the exact solution ($z(t) = e^{-t}$ when $\alpha = 1$) at various values of α. This problem is solved with different methods given in Muroya et al. (2003), Rahimkhani et al. (2017), Nemati et al. (2018) which include collocation method, operational matrix based on Bernoulli wavelets and hat functions, respectively. By setting $M = 10, \alpha = 1$ and $T = 1$, the results obtained are compared with methods given in Muroya et al. (2003), Rahimkhani et al. (2017), Nemati et al. (2018) at some selected points in Table 5. Table 5 shows the proposed scheme gives more favorable results than the method given by Muroya et al. (2003), Rahimkhani et al. (2017), Nemati et al. (2018). Also, comparison of the absolute error at some selected points with different values $\alpha$is shown in Table 6.

Table 5
(Example 4) Comparison of the absolute errors for $\alpha = 1$.

	Muroya et al. (2003)	Rahimkhani et al. (2017)	Nemati et al. (2018)	Presented method
t		$M = 32$	$k = 2,\ M = 6$	$M = 10$
2^{-2}	$1.08e-5$	$8.79e-9$	$1.05e-8$	$2.22e-15$
2^{-3}	$3.81e-5$	$1.89e-8$	$5.79e-9$	$2.66e-15$
2^{-4}	$1.26e-5$	$8.92e-9$	$2.00e-8$	$3.11e-15$
2^{-5}	$4.09e-5$	$3.55e-8$	$3.70e-9$	$7.77e-15$
2^{-6}	$1.20e-5$	$1.83e-6$	$2.03e-8$	$4.33e-15$

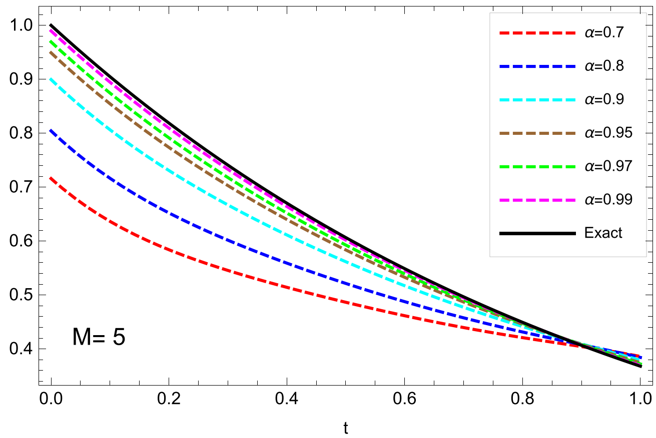

Fig. 4. (Example 4) Approximate solutions given by different values of α.

7. Conclusion

In this article, an efficient method has been proposed to obtain numerical solution of pantograph Volterra nonlinear fractional integro-differential equations which is described in the Atangana-Baleanu sense. For solving the considered equations, the properties of the shifted Legendre polynomials together with the collocation points have been used. By this way, the problem under study is reduced to a system of algebraic equations which greatly simplifies the problem. Then, an error estimate is proved for the proposed scheme. Finally, some examples have been presented to show the accuracy and efficiency of the proposed scheme. The numerical results confirm the superiority of this method compared to the other existing state of the art methods. see Fig. 4.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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