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Soliton behavior of algae growth dynamics leading to the variation in nutrients concentration

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ABSTRACT

This paper provides the effects of nutrients concentrations on the algae growth via solitary wave analysis and classical existence theory. There is a well known Diffusive Nutrient-Algae model based on the Sanyang wetland, is under consideration. The new families of exact solutions which represent nutrient concentration and algae density in different form are observed using the ϕ^6-model expansion. Furthermore, the existence of these solutions are also discussed under different restrictive conditions and variables of nutrient concentration and algae density are represented in hyperbolic, trigonometric and rational forms. The graphics of these solutions are also sketched in 3D and 2D representations.

1. Introduction

The essential elements for the life processes in aquatic organisms are called nutrients. Major nutrients are carbon, nitrogen, phosphorus, silicon, and many others. Those elements which are required for plants and animals in small quantities are called micronutrient. They might be manganese, copper, zinc, and cobalt (Bowie et al., 1985). Nutrients are important in water purification but most often are associated with algae growth. The microorganisms that containing a nucleus enclosed within a well-defined membrane are called algae which apply to a diverse group of eucaryotic. They are unicellular and multicellular plants that occur in freshwater, marine water, and damp environment (Deas and Orlob, 1999).

Algae are found in rivers, lakes, and oceans. The photosynthesis of algae consumes inorganic carbon to produce organic matter and release oxygen by using light. Nutrient elements are important to the growth of algae and water eutrophication factors. Through photosynthesis, algae produce oxygen in the daytime and consume it in the nighttime. Thus, high algae density may lead to low dissolved oxygen concentration. Although carbon is also available in water. So, algae is required for the growth of cells using carbon. When carbon is dissolved in water it interfaces as carbon dioxide $CO_2(g) \rightarrow CO_2(aq)$. However, this process is insufficient to keep up with algae dependence. Under such conditions algae remove CO_2 from bicarbonate ion:

$$\mathrm{HCO_3^-} \rightarrow \mathrm{CO_2} + \mathrm{OH^-}. \tag{1}$$

This shows that OH^- is increasing the concentration.

The mathematical model have been constructed to study of growth of algae and its dependence on nutrients or light or both of them. There are three different situations first one is algae compete only for nutrients in oligotrophic aquatic ecosystems by using of light (Hsu et al., 2013; Nie et al., 2015). Second algae compete only for light in eutrophic aquatic ecosystems (Du and Hsu, 2010; Du et al., 2015; Du and Mei, 2011). Thirdly, algae compete

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both light and nutrients eutrophic aquatic ecosystems (see Du and Hsu, 2008; Du and Hsu, 2008).

The connection between algae and inorganic carbon is more complicated with a variety of biological mechanisms. An ODE model was constructed in Van de Waal et al. (2011) to describe competition for dissolved inorganic carbon in dense algal blooms in a completely well-mixed water column. After that, the authors also constructed PDE in the connection of algae and nutrients (Hsu et al., 2017; Nie et al., 2016) and some new and important developments for searching for analytical wave solutions for PDE (Çelik and Seadawy, 2021; Özkan et al., 2021; Yesim Glam Ozkan, 2020; Yildirim, 2017; Yasar, 2010; Iqbal et al., 2022; Raddadi et al., 2021; Seadawy et al., 2021; Alruwaili et al., 2021). So, the reaction–diffusion nutrients-algae model has been proposed in Wang et al. (2015) and Zhang et al. (2013),

$$\phi_t = d_1\Delta\phi + I - b\phi + \frac{\phi^2}{1+\phi^2} - a\phi\psi, \tag{2}$$

$$\psi_t = d_2\Delta\psi + r\psi\left(1-\frac{\psi}{k}\right) + ae\phi\psi - m\psi, \tag{3}$$

where $\phi(x,y,t)$ and $\psi(x,y,t)$ represent the nutrient concentration and the algae density respectively. In the wetland the phosphorus is limited and nitrogen is abundant. So, I be the input rate of nutrients flowing into the water, a is the maximum growth rate of the algae population, e represents the efficiency of conversion, r is the intrinsic growth rate of algae, k is the carrying capacity of the algae population, m is the death rate of the algae population, while d_1 and d_2 are the diffusion coefficients of nutrients and algae, respectively.

2. A-priori bounds for existence of concentration algae model

By using the heat kernels or simple 1st derivative inversion, we can invert the system (2–3). We want to adopt the later approach with some additional assumption of regularities of the solution with $(\phi,\psi)\in C^2(\Omega)$, where Ω is subset of R^2 and is a spatial domain of the problem.

Clearly, integrating w.r.t time t, the PDEs (2) and (3) are reduced to the following volterra type integral equations as,

$$\Phi = \phi(x,y,t) = \phi_0(x,y) + \int_0^t \left[d_1\Delta_2\phi(x,y,s) + I - b\phi(x,y,s) + \frac{\phi^2(x,y,s)}{1+\phi^2(x,y,t)} - (a\phi\psi)(x,y,s)\right] ds, \tag{4}$$

$$\Psi = \phi(x,y,t) = \psi_0(x,y) + \int_0^t \left[d_2\Delta_2\psi(x,y,s) + (r\psi - \tfrac{r\psi^2}{k})(x,y,s) + (ae\phi\psi)(x,y,t) - m\psi(x,y,t)\right] ds, \tag{5}$$

for existence and unique solution, we, consider the space of continuous functions $C[0,\rho]$ and a closed, convex and bounded set in $C[0,\rho]$, defined by

$$B_R(\Theta) = \{f, \quad f\in C[0,\rho], \qquad ||f||\leqslant R\},$$

where $||\cdot||$ is usual supremum norm defined by $||\cdot|| = \sup_{[0,\rho]}|\cdot|$. We can show the unique existence by Schauder fixed point theorem,

- $(\Phi,\Psi): B_R(\Theta)\to B_R(\Theta)$
- $(\Phi,\Psi)(B_R(\Theta))$: is relatively compact.

For former, we take the norm of Eq. (2) and assume that, $\Delta_2\psi\in C(\Omega)\Rightarrow\Delta_2\Phi$ is bounded and $\frac{1}{1+\phi^2}$ is globally bounded by 1.

$$||\Phi|| \leqslant k_1 + \rho_\phi(d_1k_2 + |I| + |b|R + R^2 + |a|R^2) \leqslant R, \tag{7}$$

$$\Rightarrow$$

$$\rho_\phi \leqslant \frac{R-k_1}{k_\phi(R)}, \tag{8}$$

where $k_\phi(R)$ is bound for R.H.S of Eq. (2) in ball (6), and also where

$$k(R) = d_1k_2 + |I| + |b|R + R^2 + |a|R^2, \qquad ||\phi_0||_{C[\Omega]}\leqslant k_1,$$

similarly,

$$\rho_\psi \leqslant \frac{R-k_3}{k_\psi(R)}, \tag{9}$$

where $k_\phi(R)$ is bound for R.H.S of Eq. (3) in ball (6). Conditions (8) & (9) are self mapping a priori estimate and let $\rho = \min\left(\frac{R-k_1}{k_\phi(R)}, \frac{R-k_3}{k_\psi(R)}\right)$, which serves as the length of continuity as $[0,\rho)$ provided the relative compactness conditions satisfies. For relative compactness we consider the families Φ_i for pre images ϕ_i and Ψ_i against ψ_i.

$$\Phi_i = \phi_{i0} + \int_0^t \left[d_1\Delta_2\phi_i(x,y,s) + I - b\phi_i(x,y,s) + \frac{\phi_i^2(x,y,s)}{1+\phi_i^2(x,y,t)} - (a\phi_i\psi_i)(x,y,s)\right] ds,$$

$$\Phi_i(t) - \Phi_i(t^*) = \int_t^{t^*} \left[d_1\Delta_2\phi_i(x,y,s) + I - b\phi_i(x,y,s) + \frac{\phi_i^2(x,y,s)}{1+\phi_i^2(x,y,t)} - (a\phi_i\psi_i)(x,y,s)\right] ds,$$

$$||\Phi_i(t) - \Phi_i(t^*)|| \leqslant \int_t^{t^*} \left|\left|\left[d_1\Delta_2\phi_i(x,y,s) + I - b\phi_i(x,y,s) + \frac{\phi_i^2(x,y,s)}{1+\phi_i^2(x,y,t)} - (a\phi_i\psi_i)(x,y,s)\right]\right|\right| ds,$$

let

$$G(x,y,s) = \left[d_1\Delta_2\phi_i(x,y,s) + I - b\phi_i(x,y,s) + \frac{\phi_i^2(x,y,s)}{1+\phi_i^2(x,y,t)} - (a\phi_i\psi_i)(x,y,s)\right],$$

$$||\Phi_i - \Phi_i(t^*)|| \leqslant \kappa_i|t-t^*| \quad ||G(x,y,t)||\leqslant\kappa_i,$$

when $t\to t^*$ implies $\Phi_i(t)\to\Phi_i(t^*)$,

i.e., Φ_i are equi-continuous. Similarly Ψ_i are also equi-continuous. Hence by Arzela-Assoli theorem, there exist uniformly convergent subsequence Φ_{ij} of ϕ_i and Ψ_{ij} of ψ_i. So, both operators Φ & Ψ are relatively compact by Schauder fixed point theorem (Iqbal, 2011) is applicable and therefore under conditions (8) & (9) both operators Φ & Ψ has fixed points.

Theorem: Suppose the function $\phi(x,y,t)$ and $\psi(x,y,t)$ are twice continuously differentiable in space and continuously differentiable in time and if the conditions (8) & (9) hold. Thus the problem for nutrient concentration and algae growth has the least one classical solution.

3. Analytical Study

To find the exact solution of Eq. (1), we take the transformation $\phi(x,y,t) = u(\rho)$ and $\psi(x,y,t) = v(\rho)$ where $\rho = x+y-ct$ (Younis et al., 2021; Bashir et al., 2021; Bilal et al., 2021). Using this transformation we convert Eqs. (2) & (3) into an ordinary differential equation (ODE) as, (Seadawy et al. (2021), Seadawy et al. (2021), Nasreen et al. (2019), Rizvi et al. (2020), Younas et al. (2021), Dianchen et al. (2018), Aly (2019), Ijaz Ali et al. (2020)).

$$cu' + 2d_1u'' + I - bu + \frac{u^2}{1+u^2} - auv = 0,$$

$$cv' + 2d_2v'' + rv\left(1 - \frac{v}{k}\right) + aeuv - mv = 0.$$

Let the solutions of eqs. (10) & (11) which can be expressed in the following polynomial $Z^i(\rho)$.

$$u(\rho) = \sum_{i=0}^{2m} a_i Z^i(\rho), \quad a_i \neq 0, \tag{12}$$

$$v(\rho) = \sum_{i=0}^{2m} b_i Z^i(\rho), \quad b_i \neq 0, \tag{13}$$

where $a_i, b_i (0 \leqslant i \leqslant 2m)$ are constants and $Z(\rho)$ satisfies the eqs. (10) & (11). Here we take

$$(Z')^2(\rho) = h_0 + h_2Z^2(\rho) + h_4Z^4(\rho) + h_6Z^6(\rho), \tag{14}$$

$$Z''(\rho) = h_2Z(\rho) + 2h_4Z^3(\rho) + 3h_6Z^5(\rho), \tag{15}$$

where $h_i (i = 0, 2, 4, 6)$ are real constants. By applying the homogenous balancing principle (Jin-Liang et al., 2003), we obtain $m = 1$. So, Eq. (12) & (13) takes form as,

$$u(\rho) = a_0 + a_1Z(\rho) + a_2Z^2(\rho), \tag{16}$$

$$v(\rho) = b_0 + b_1Z(\rho) + b_2Z^2(\rho) + b_3Z^3(\rho) + b_4Z^4(\rho), \tag{17}$$

it is well known that Eqs. (10) & (11) has the solutions,

$$Z(\rho) = \frac{\Theta(\rho)}{\sqrt{p\Theta^2(\rho) + q}}, \tag{18}$$

where $(p\Theta^2(\rho) + q) \geqslant 0$ and $\Theta(\rho)$ is the solution of the Jacobian elliptic equation as,

$$\Theta'^2 = \delta_0 + \delta_2\Theta^2(\rho) + \delta_4\Theta^4(\rho), \tag{19}$$

where $\delta_i = 0, 2, 4$ are constants, while p and q are given by,

$$p = \frac{h_4(\delta_2 - h_2)}{(\delta_2 - h_2)^2 + 3\delta_0\delta_4 - 2\delta_2(\delta_2 - h_2)}, \tag{20}$$

$$q = \frac{3\delta_0 h_4}{(\delta_2 - h_2)^2 + 3\delta_0\delta_4 - 2\delta_2(\delta_2 - h_2)}, \tag{21}$$

under the constraint condition

$$h_4^2(\delta_2 - h_2)[9\delta_0\delta_4 - (\delta_2 - h_2)(2\delta_2 + h_2)] + 3h_6[3\delta_0\delta_4 - (\delta_2^2 - h_2^2)]^2 = 0. \tag{22}$$

Here JEFs of $\Theta(\rho)$ for the limits of m are taken as,

$\Theta(\rho, m)$	$m \to 1$	$m \to 0$	$\Theta(\rho, m)$	$m \to 1$	$m \to 0$
$sn(\rho, m)$	$\tanh(\rho)$	$\sin(\rho)$	$ns(\rho, m)$	$\coth(\rho)$	$csc(\rho)$
$cn(\rho, m)$	$sech(\rho)$	$\cos(\rho)$	$dn(\rho, m)$	$sech(\rho)$	1
$cd(\rho, m)$	1	$\cos(\rho)$	$cs(\rho, 1)$	$csch(\rho)$	$\cot(\rho)$
$sc(\rho, m)$	$sinh(\rho)$	$\tan(\rho)$	$sd(\rho, m)$	$\sinh(\rho)$	$\sin(\rho)$
$ds(\rho, m)$	$csch(\rho)$	$csc(\rho)$	$nc(\rho, 1)$	$\cosh(\rho)$	$\sec(\rho)$

Substitute Eq. (16) and its derivatives in equation Eqs. (10) & (11) and equating the co-efficient of the same power of $Z(\rho)$ equal to zero, get the system of equations easily. Further solve this system of equations by using the *Maple* to get the solutions set as follow,

Case 1: $a_0 = \sqrt{(-b + b_0a) - I + (b_0a - I) + b}, a_1 = 0, a_2 = -\frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}, h_0 = h_0, h_2 = h_2, h_4 = \frac{a(b_4b_2 - b_3^2)}{4b_4d_1}, h_6 = \frac{ab_4}{6d_1},$ $b_0 = \frac{k(eb_0a - m + r)}{r}, b_2 = -\frac{132ab_2ek}{43r}, h_0 = h_0, h_2 = -\frac{eb_0a - m + r}{2d_2}, h_4 = -\frac{aeb_2}{4d_2}, h_6 = 0.$

Substituting these values in Eqs. (16) & (17) along with Eq. (18). The different families of solution can be constructed by using JEFs from the table. The solutions of Eqs. (2) & (3) are summarized as under.

Family 1: If $\delta_0 = 1, \delta_2 = -(1 + m^2), \delta_4 = m^2$, then $\Theta(\rho) = sn(\rho)$ or $\Theta(\rho) = cd(\rho)$ and we get the JEFs as,

$$\phi_1(\rho) = \sqrt{(-b + b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{sn^2(\rho)}{psn^2(\rho) + q}\right),$$

$$\phi_2(\rho) = \sqrt{(-b + b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{cd^2(\rho)}{pcd^2(\rho) + q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2 - b_3^2)}{4b_4d_1}\right)^2(1 + m^2 + h_2)[9m^2 - (1 + m^2 + h_2)(2 + 2m^2 + h_2)] + \frac{ab_4}{2d_1}\left[3m^2 - ((1 + m^2)^2 - h_2^2)\right]^2 = 0.$$

or

$$\psi_1(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132ab_2ek}{43r}\left(\frac{sn^2(\rho)}{psn^2(\rho) + q}\right),$$

$$\psi_2(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132ab_2ek}{43r}\left(\frac{cd^2(\rho)}{pcd^2(\rho) + q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(-(1 + m^2) + \frac{eb_0a - m + r}{2d_2}) \times \left[9m^2 - (-(1 + m^2) + \frac{eb_0a - m + r}{2d_2})(-2(1 + m^2) - \frac{eb_0a - m + r}{2d_2})\right] = 0.$$

Type 1: If $m \to 1$, then we have solitary wave solutions as,

$$\phi_{(1,1)}(\rho) = \sqrt{(-b + b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{\tanh^2(\rho)}{p\tanh^2(\rho) + q}\right),$$

$$\psi_{(1,1)}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\tanh^2(\rho)}{p\tanh^2(\rho) + q}\right),$$

Type 2: If $m \to 0$, then we have periodic wave solutions as,

$$\phi_{(1,0)}(\rho) = \sqrt{(-b + b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{\sin^2(\rho)}{p\sin^2(\rho) + q}\right),$$

$$\psi_{(1,0)}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\sin^2(\rho)}{p\sin^2(\rho) + q}\right),$$

Family 2: If $\delta_0 = 1 - m^2, \delta_2 = 2m^2 - 1, \delta_4 = -m^2$, then $\Theta(\rho) = cn(\rho)$ and we get the JEFs as,

$$\phi_3(\rho) = \sqrt{(-b + b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{cn^2(\rho)}{pcn^2(\rho) + q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2 - b_3^2)}{4b_4d_1}\right)^2(2m^2 - 1 - h_2)[9m^2(1 - m^2) - (2m^2 - 1 - h_2)(4m^2 - 2 + h_2)] + \frac{ab_4}{2d_1}[3m^2(2m^2 - 1) - ((2m^2 - 1)^2 - h_2^2)]^2 = 0.$$

$$\psi_3(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{cn^2(\rho)}{pcn^2(\rho) + q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2m^2-1+\frac{eb_0a-m+r}{2d_2})[9m^2(1-m^2)-(2m^2-1+\frac{eb_0a-m+r}{2d_2})(-2(2m^2-1)-\frac{eb_0a-m+r}{2d_2})]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(3,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

$$\psi_{(3,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(3,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\cos^2(\rho)}{p\cos^2(\rho)+q}\right),$$

$$\psi_{(3,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\cos^2(\rho)}{p\cos^2(\rho)+q}\right),$$

Family 3: If $\delta_0=m^2-1,\delta_2=2-m^2,\delta_4=-1,$ then $\Theta(\rho)=dn(\rho)$ and we get the JEFs as,

$$\phi_4(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{dn^2(\rho)}{pdn^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2-m^2-h_2)[9(1-m^2)-(2-m^2-h_2)(4-2m^2+h_2)]+\frac{ab_4}{2d_1}[3(1-m^2)-((2-m^2)^2-h_2^2)]^2=0.$$

$$\psi_4(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{dn^2(\rho)}{pdn^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2-m^2+\frac{eb_0a-m+r}{2d_2})\times\left[9(1-m^2)-(2-m^2+\frac{eb_0a-m+r}{2d_2})(-2(2-m^2)-\frac{eb_0a-m+r}{2d_2})\right]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(4,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

$$\psi_{(4,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

Family 4: If $\delta_0=m^2,\delta_2=-(m^2+1),\delta_4=1,$ then $\Theta(\rho)=ns(\rho)$ and we get the JEFs as,

$$\phi_5(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{ns^2(\rho)}{pns^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(-m^2-1-h_2)[9m^2-(m^2+1+h_2)(2m^2+2+h_2)]+\frac{ab_4}{2d_1}[3m^2-((m^2+1)^2-h_2^2)]^2=0.$$

$$\psi_5(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{ns^2(\rho)}{pns^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(-m^2-1+\frac{eb_0a-m+r}{2d_2})[9m^2+(m^2+1+\frac{eb_0a-m+r}{2d_2})(-(m^2+1)-\frac{eb_0a-m+r}{2d_2})]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(5,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\coth^2(\rho)}{p\coth^2(\rho)+q}\right),$$

$$\psi_{(5,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\coth^2(\rho)}{p\coth^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(5,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\csc^2(\rho)}{p\csc^2(\rho)+q}\right),$$

$$\psi_{(5,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\csc^2(\rho)}{p\csc^2(\rho)+q}\right),$$

Family 5: If $\delta_0=-m^2,\delta_2=2m^2-1,\delta_4=1-m^2,$ then $\Theta(\rho)=nc(\rho)$ and we get the JEFs as,

$$\phi_6(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{nc^2(\rho)}{pnc^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2m^2-1-h_2)[-9m^2(1-m^2)-(2m^2-1-h_2)(4m^2-2+h_2)]+\frac{ab_4}{2d_1}[-3m^2(1-m^2)-((2m^2-1)^2-h_2^2)]^2=0.$$

$$\psi_6(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{nc^2(\rho)}{pnc^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2m^2-1+\frac{eb_0a-m+r}{2d_2})[-9m^2(1-m^2)+(2m^2-1+\frac{eb_0a-m+r}{2d_2})(2m^2-1-\frac{eb_0a-m+r}{2d_2})]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(6,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\cosh^2(\rho)}{p\cosh^2(\rho)+q}\right),$$

$$\psi_{(6,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\cosh^2(\rho)}{p\cosh^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(6,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\sec^2(\rho)}{p\sec^2(\rho)+q}\right),$$

$$\psi_{(6,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\sec^2(\rho)}{p\sec^2(\rho)+q}\right),$$

Family 6: If $\delta_0=-1,\delta_2=2-m^2,\delta_4=-(1-m^2),$ then $\Theta(\rho)=nd(\rho)$ and we get the JEFs as,

$$\phi_7(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{nd^2(\rho)}{pnd^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2-m^2-h_2)[9(1-m^2)-(4-m^2-h_2)(2-2m^2+h_2)]+\frac{ab_4}{2d_1}[-3(2-m^2)-((2-m^2)^2-h_2^2)]^2=0.$$

$$\psi_7(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{nd^2(\rho)}{pnd^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2-m^2+\frac{eb_0a-m+r}{2d_2})\times\left[9(2-m^2)+(2-m^2+\frac{eb_0a-m+r}{2d_2})(2-m^2-\frac{eb_0a-m+r}{2d_2})\right]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(7,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

$$\psi_{(7,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{sech^2(\rho)}{psech^2(\rho)+q}\right),$$

Family 7: If $\delta_0=1,\delta_2=2-m^2,\delta_4=1-m^2$, then $\Theta(\rho)=sc(\rho)$ and we get the JEFs as,

$$\phi_8(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{sc^2(\rho)}{psc^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2-m^2-h_2)[9-9m^2-(2-m^2-h_2)(4-2m^2+h_2)]+\frac{ab_4}{2d_1}[3(2-m^2)-((2-m^2)^2-h_2^2)]^2=0.$$

$$\psi_8(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{sc^2(\rho)}{psc^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2-m^2+\frac{eb_0a-m+r}{2d_2})[9(2-m^2)+(2-m^2+\frac{eb_0a-m+r}{2d_2})(2-m^2-\frac{eb_0a-m+r}{2d_2})]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(8,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\sinh^2(\rho)}{p\sinh^2(\rho)+q}\right),$$

$$\psi_{(8,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\sinh^2(\rho)}{p\sinh^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(8,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\tan^2(\rho)}{p\tan^2(\rho)+q}\right),$$

$$\psi_{(8,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\tan^2(\rho)}{p\tan^2(\rho)+q}\right),$$

Family 8: If $\delta_0=1,\delta_2=2m^2-1,\delta_4=-m^2(1-m^2)$, then $\Theta(\rho)=sd(\rho)$ and we get the JEFs as,

$$\phi_9(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{sd^2(\rho)}{psd^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2m^2-1-h_2)[-9m^2(1-m^2)-(4m^2-2-h_2)(2m^2-1+h_2)]+\frac{ab_4}{2d_1}[-3m^2(1-m^2)-((2m^2-1)^2-h_2^2)]^2=0.$$

$$\psi_9(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{sd^2(\rho)}{psd^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2m^2-1+\frac{eb_0a-m+r}{2d_2})\times\left[9(2m^2-1)+(2m^2-1+\frac{eb_0a-m+r}{2d_2})(2m^2-1-\frac{eb_0a-m+r}{2d_2})\right]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(9,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\sinh^2(\rho)}{p\sinh^2(\rho)+q}\right),$$

$$\psi_{(9,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\sinh^2(\rho)}{p\sinh^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(9,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\sin^2(\rho)}{p\sin^2(\rho)+q}\right),$$

$$\psi_{(9,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\sin^2(\rho)}{p\sin^2(\rho)+q}\right),$$

Family 9: If $\delta_0=1-m^2,\delta_2=2-m^2,\delta_4=1$, then $\Theta(\rho)=cs(\rho)$ and we get the JEFs as,

$$\phi_{10}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{cs^2(\rho)}{pcs^2(\rho)+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2-m^2-h_2)[9(1-m^2)-(2-m^2-h_2)(4-2m^2+h_2)]+\frac{ab_4}{2d_1}[3(1-m^2)-((2-m^2)^2-h_2^2)]^2=0.$$

$$\psi_{10}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{cs^2(\rho)}{pcs^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2-m^2+\frac{eb_0a-m+r}{2d_2})[9(1-m^2)+(2-m^2+\frac{eb_0a-m+r}{2d_2})(2-m^2-\frac{eb_0a-m+r}{2d_2})]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(10,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{csch^2(\rho)}{pcsch^2(\rho)+q}\right),$$

$$\psi_{(10,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{csch^2(\rho)}{pcsch^2(\rho)+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(10,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\cot^2(\rho)}{p\cot^2(\rho)+q}\right),$$

$$\psi_{(10,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\cot^2(\rho)}{p\cot^2(\rho)+q}\right),$$

Family 10: If $\delta_0 = -m^2(1-m^2), \delta_2 = 2m^2-1, \delta_4 = 1,$ then $\Theta(\rho) = ds(\rho)$ and we get the JEFs as,

$$\phi_{11}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{ds^2(\rho)}{pds^2(\rho)+q}\right)},$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(2m^2-1-h_2) \times [-9m^2(1-m^2) - (2m^2-1-h_2)(4m^2-2+h_2)] + \frac{ab_4}{2d_1}[-3m^2(1-m^2) - ((2m^2-1)^2 - h_2^2)]^2 = 0.$$

$$\psi_{11}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{ds^2(\rho)}{pds^2(\rho)+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(2m^2-1+\frac{eb_0a-m+r}{2d_2}) \times \left[-9m^2(1-m^2) + (2m^2-1+\frac{eb_0a-m+r}{2d_2})(2m^2-1-\frac{eb_0a-m+r}{2d_2})\right] = 0.$$

Type 1: If $m \to 1$, then we have solitary wave solutions as,

$$\phi_{(11,1)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{csch^2(\rho)}{pcsch^2(\rho)+q}\right)},$$
$$\psi_{(11,1)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{csch^2(\rho)}{pcsch^2(\rho)+q}\right),$$

Type 2: If $m \to 0$, then we have periodic wave solutions as,

$$\phi_{(11,0)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{csc^2(\rho)}{pcsc^2(\rho)+q}\right)},$$
$$\psi_{(11,0)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{csc^2(\rho)}{pcsc^2(\rho)+q}\right),$$

Family 11: If $\delta_0 = \frac{1-m^2}{4}, \delta_2 = \frac{1+m^2}{2}, \delta_4 = \frac{1-m^2}{4},$ then $\Theta(\rho) = ns(\rho) \pm sc(\rho)$ or $\frac{cn(\rho)}{1\pm sn(\rho)}$and we get the JEFs as,

$$\phi_{12}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{(ns(\rho)\pm sc(\rho))^2}{p(ns(\rho)\pm sc(\rho))^2+q}\right)},$$

or

$$\phi_{13}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{(\frac{cn(\rho)}{1\pm sn(\rho)})^2}{p(\frac{cn(\rho)}{1\pm sn(\rho)})^2+q}\right)},$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(\frac{1+m^2}{2}-h_2) \times [9(\frac{1-m^2}{4})^2 - (\frac{1+m^2}{2}-h_2)(1+m^2+h_2)] + \frac{ab_4}{2d_1}[-3m^2(1-m^2) - ((\frac{1+m^2}{2})^2 - h_2^2)]^2 = 0.$$

$$\psi_{12}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(ns(\rho)\pm sc(\rho))^2}{p(ns(\rho)\pm sc(\rho))^2+q}\right),$$

or

$$\psi_{13}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(\frac{cn(\rho)}{1\pm sn(\rho)})^2}{p(\frac{cn(\rho)}{1\pm sn(\rho)})^2+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(\frac{1+m^2}{2}+\frac{eb_0a-m+r}{2d_2}) \times \left[9(\frac{1-m^2}{4})^2 + (\frac{1+m^2}{2}+\frac{eb_0a-m+r}{2d_2})(\frac{1+m^2}{2}-\frac{eb_0a-m+r}{2d_2})\right] = 0.$$

Type 1: If $m \to 1$, then we have solitary wave solutions as,

$$\phi_{(12,1)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{(\coth(\rho)\pm\sinh(\rho))^2}{p(\coth(\rho)\pm\sinh(\rho))^2+q}\right)},$$
$$\psi_{(12,1)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(\coth(\rho)\pm\sinh(\rho))^2}{p(\coth(\rho)\pm\sinh(\rho))^2+q}\right),$$

or

$$\phi_{(13,1)} = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{(\frac{sech(\rho)}{1\pm\tanh(\rho)})^2}{p(\frac{sech(\rho)}{1\pm\tanh(\rho)})^2+q}\right)},$$
$$\psi_{(13,1)} = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(\frac{sech(\rho)}{1\pm\tanh(\rho)})^2}{p(\frac{sech(\rho)}{1\pm\tanh(\rho)})^2+q}\right),$$

Type 2: If $m \to 0$, then we have periodic wave solutions as,

$$\phi_{(12,0)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{(\csc(\rho)\pm\tan(\rho))^2}{p(\csc(\rho)\pm\tan(\rho))^2+q}\right)},$$
$$\psi_{(12,0)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(\csc(\rho)\pm\tan(\rho))^2}{p(\csc(\rho)\pm\tan(\rho))^2+q}\right),$$

or

$$\phi_{(13,0)} = \sqrt{(-b+b_0a) - I + (b_0a-I) + b - \frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{(\frac{\cos(\rho)}{1\pm\sin(\rho)})^2}{p(\frac{\cos(\rho)}{1\pm\sin(\rho)})^2+q}\right)},$$
$$\psi_{(13,0)} = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(\frac{\cos(\rho)}{1\pm\sin(\rho)})^2}{p(\frac{\cos(\rho)}{1\pm\sin(\rho)})^2+q}\right),$$

Family 12: If $\delta_0 = \frac{-(1-m^2)^2}{4}, \delta_2 = \frac{1+m^2}{2}, \delta_4 = \frac{-1}{4},$ then $\Theta(\rho) = mcn(\rho) \pm dn(\rho)$ and we get the JEFs as,

$$\phi_{14}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{(mcn(\rho)\pm dn(\rho))^2}{p(mcn(\rho)\pm dn(\rho))^2+q}\right)},$$

under the constraint condition

$$-\left(\frac{a(b_4b_2-b_3^2)}{4b_4d_1}\right)^2(\frac{1+m^2}{2}-h_2) \times [-9\frac{1-m^2}{16} - (\frac{1+m^2}{2}-h_2)(1+m^2+h_2)] + \frac{ab_4}{2d_1}[-3\frac{1-m^2}{8} - ((\frac{1+m^2}{2})^2 - h_2^2)]^2 = 0.$$

$$\psi_{14}(\rho) = \frac{k(eb_0a - m + r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(mcn(\rho)\pm dn(\rho))^2}{p(mcn(\rho)\pm dn(\rho))^2+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2(\frac{1+m^2}{2}+\frac{eb_0a-m+r}{2d_2}) \times \left[-9\frac{1-m^2}{8} + (\frac{1+m^2}{2}+\frac{eb_0a-m+r}{2d_2})(\frac{1+m^2}{2}-\frac{eb_0a-m+r}{2d_2})\right] = 0.$$

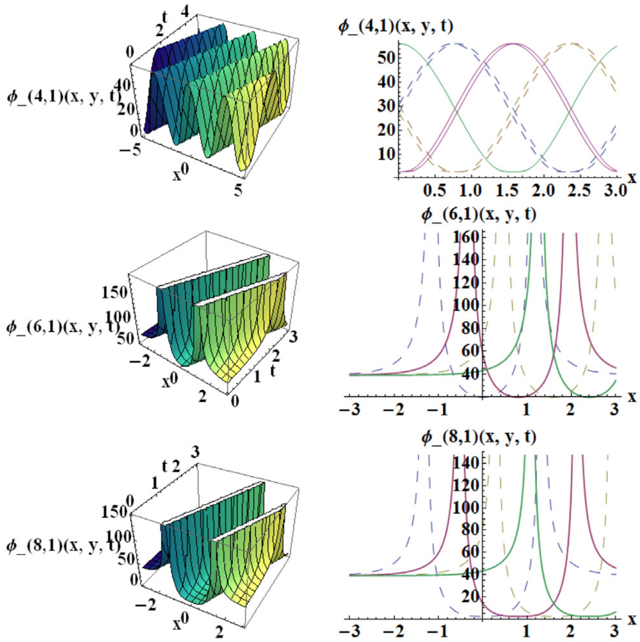

Fig. 1. The 3D and 2D graphs for the values of $\phi_4(\rho,1), \phi_6(\rho,1)$ and $\phi_8(\rho,1)$ for the values of parameters $c = 0.8, a = 4.7, b = 0.8, b_0 = 4.8, b_1 = 2.5, b_2 = 2, b_3 = 3, b_4 = 4, I = 3.4, h_2 = 2, h_6 = 2.3$ $e = 0.2, d_1 = 2$, and $r = 100$.

Type 1: If $m \to 1$, then we have solitary wave solutions as,

$$\phi_{(14,1)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{(msech(\rho)\pm sech(\rho))^2}{p(msech(\rho)\pm sech(\rho))^2+q}\right),$$

$$\psi_{(14,1)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(msech(\rho)\pm sech(\rho))^2}{p(msech(\rho)\pm sech(\rho))^2+q}\right),$$

Type 2: If $m \to 0$, then we have periodic wave solutions as,

$$\phi_{(14,0)}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{(m\cos(\rho)\pm 1)^2}{p(m\cos(\rho)\pm 1)^2+q}\right),$$

$$\psi_{(14,0)}(\rho) = \frac{k(eb_0a-m+r)}{r} - \frac{132}{43}\frac{ab_2ek}{r}\left(\frac{(m\cos(\rho)\pm 1)^2}{p(m\cos(\rho)\pm 1)^2+q}\right),$$

Family 13: If $\delta_0 = \frac{1}{4}, \delta_2 = \frac{1-2m^2}{2}, \delta_4 = \frac{1}{4}$, then $\Theta(\rho) = \frac{sn(\rho)}{1\pm cn(\rho)}$ and we get the JEFs as,

$$\phi_{15}(\rho) = \sqrt{(-b+b_0a) - I + (b_0a - I) + b} - \frac{4c(3ab_4 - 38h_6d_1)}{a^2b_3^2}\left(\frac{\left(\frac{sn(\rho)}{1\pm cn(\rho)}\right)^2}{p\left(\frac{sn(\rho)}{1\pm cn(\rho)}\right)^2+q}\right),$$

under the constraint condition

$$-\left(\frac{a(b_4b_2 - b_3^2)}{4b_4d_1}\right)^2\left(\frac{1-2m^2}{2} - h_2\right) \times \left[\frac{9}{16} - \left(\frac{1-2m^2}{2} - h_2\right)(1-2m^2+h_2)\right] + \frac{ab_4}{2d_1}\left[3\frac{1}{16} - \left(\left(\frac{1-2m^2}{2}\right)^2 - h_2^2\right)\right]^2 = 0.$$

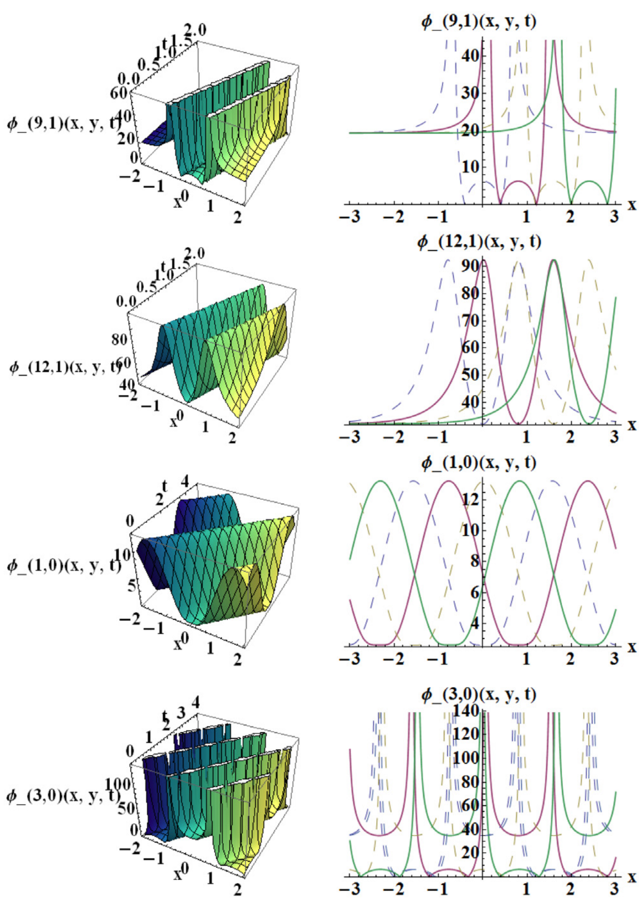

Fig. 2. The 3D and 2D graphs for the values of $\phi_9(\rho,1), \phi_{12}(\rho,1), \phi_1(\rho,0)$ and $\phi_3(\rho,0)$ for the values of parameters $c = 0.8, a = 4.7, b = 0.8, b_0 = 4.8, b_1 = 2.5, b_2 = 2, b_3 = 3, b_4 = 4, I = 3.4, h_2 = 2, h_6 = 2.3$ $e = 0.2, d_1 = 2$, and $r = 100$.

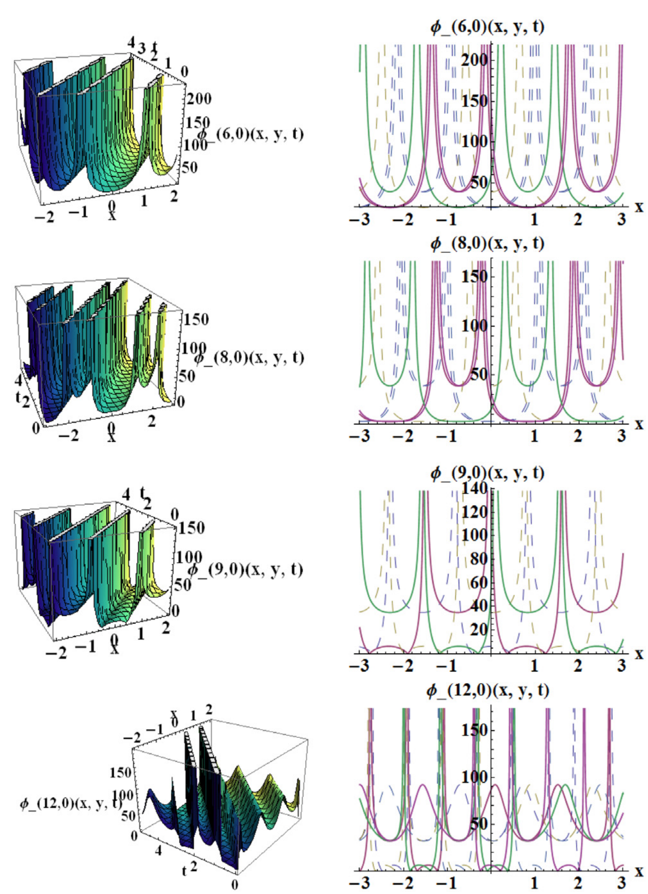

Fig. 3. The 3D and 2D graphs for the values of $\phi_6(\rho,0), \phi_8(\rho,0), \phi_9(\rho,0)$ and $\phi_{12}(\rho,0)$ for the values of parameters $c = 0.8, a = 4.7, b = 0.8, b_0 = 4.8, b_1 = 2.5, b_2 = 2, b_3 = 3, b_4 = 4, I = 3.4, h_2 = 2, h_6 = 2.3$ $e = 0.2, d_1 = 2$, and $r = 100$.

$$\psi_{15}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\left(\frac{sn(\rho)}{1\pm cn(\rho)}\right)^2}{p\left(\frac{sn(\rho)}{1\pm cn(\rho)}\right)^2+q}\right),$$

under the constraint condition

$$\left(\frac{aeb_2}{4d_2}\right)^2\left(\frac{1-2m^2}{2}+\frac{eb_0a-m+r}{2d_2}\right)\times\left[\frac{9}{16}+\left(\frac{1-2m^2}{2}+\frac{eb_0a-m+r}{2d_2}\right)\left(\frac{1-2m^2}{2}-\frac{eb_0a-m+r}{2d_2}\right)\right]=0.$$

Type 1: If $m\to 1$, then we have solitary wave solutions as,

$$\phi_{(15,1)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\left(\frac{\tanh(\rho)}{1\pm sech(\rho)}\right)^2}{p\left(\frac{\tanh(\rho)}{1\pm sech(\rho)}\right)^2+q}\right),$$

$$\psi_{(15,1)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\left(\frac{\tanh(\rho)}{1\pm sech(\rho)}\right)^2}{p\left(\frac{\tanh(\rho)}{1\pm sech(\rho)}\right)^2+q}\right),$$

Type 2: If $m\to 0$, then we have periodic wave solutions as,

$$\phi_{(15,0)}(\rho)=\sqrt{(-b+b_0a)-I+(b_0a-I)+b}-\frac{4c(3ab_4-38h_6d_1)}{a^2b_3^2}\left(\frac{\left(\frac{\sin(\rho)}{1\pm\cos(\rho)}\right)^2}{p\left(\frac{\sin(\rho)}{1\pm\cos(\rho)}\right)^2+q}\right),$$

$$\psi_{(15,0)}(\rho)=\frac{k(eb_0a-m+r)}{r}-\frac{132}{43}\frac{ab_2ek}{r}\left(\frac{\left(\frac{\sin(\rho)}{1\pm\cos(\rho)}\right)^2}{p\left(\frac{\sin(\rho)}{1\pm\cos(\rho)}\right)^2+q}\right),$$

4. Graphical representation:

This section shows the graphical behaviors of the nutrients concentration and algae density. The different types of solutions are derived, which shows the nutrient concentrations and algae density in the hyperbolic, trigonometric, and rational forms are observed. These results are new and may be helpful to understand the interaction of nutrients and algae. These results show the input rate I of nutrients flowing into the water has an important influence on the density and algae populations. To understand the physical description of these solutions, the graphs are shown in 3D and line representations, for the different choices of parameters. Hence, these results are fruitful in mechanisms of eutrophication and nonlinear wave phenomenon in applied sciences.Fig. 1–5.

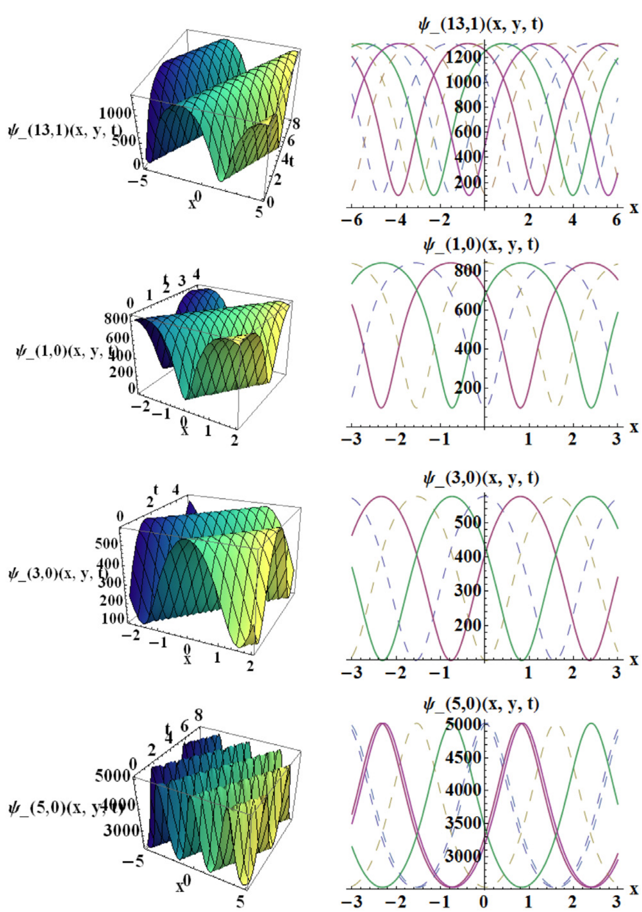

Fig. 4. The 3D and 2D graphs for the values of $\phi_{13}(\rho,0),\psi_1(\rho,0),\psi_3(\rho,0)$ and $\psi_5(\rho,0)$ for the values of parameters $c=0.8,a=0.7,b=8,b_0=40.8,b_1=2.5,b_2=20,b_3=30,b_4=4,k=3.4,h_6=2.3$ $e=20,d_2=30$, and $r=10$.

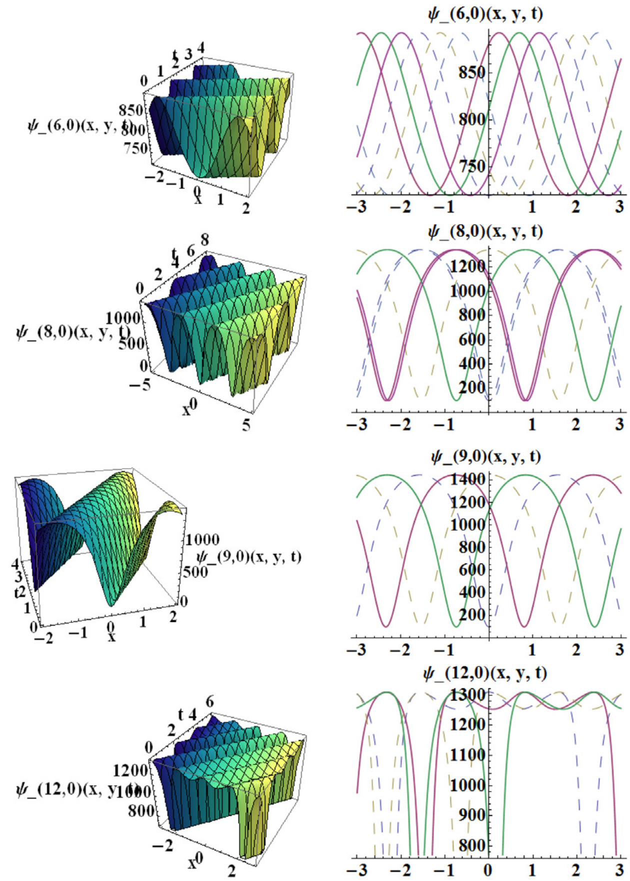

Fig. 5. The 3D and 2D graphs for the values of $\psi_6(\rho,0),\psi_8(\rho,0),\psi_9(\rho,0)$ and $\psi_{12}(\rho,0)$ for the values of parameters $c=0.8,a=0.7,b=8,b_0=40.8,b_1=2.5,b_2=20,b_3=30,b_4=4,k=3.4,h_6=2.3$ $e=20,d_2=30$, and $r=10$.

5. Conclusion

In this research, Reaction Diffusive Nutrient-Algae model is studied which describes the concentration of nutrients and algae density. Nutrients are important for the growth of algae and water eutrophication factors. The oxygen is produced by algae through photosynthesis in day time and is consumed in night time. Thus high concentration algae may lead to low dissolved oxygen concentration. Thus it is important to discus the interaction of nutrients concentration and algae density variables, and hence, the computational approach is applied to discus its exact solutions. This model is describes on the Sanyang wetland. The adopted computational technique gives new families of different kind of exact solutions of nutrient concentration and algae density.It is also observed that existence of these solutions are also discussed under different constraint conditions and solutions of nutrient concentration and algae density are represented in hyperbolic, trigonometric and rational forms. The graphical behavior of these nutrient concentrations and algae density are also depicted for different values of parameters in 2D and 3D shapes.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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