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Original article

Fractional partial differential equations and novel double integral transform

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ABSTRACT

We propose in this study a combined expression mainly based on the double transformation of Laplace and Sumudu (DLST), by developing some results associated with this proposed transformation. We can apply this double transformation to certain functions to achieve interesting results which can be used to solve certain classes of fractional partial differential equations (FPDE). The numerical results show that this double transformation can lead to an exact solution of linear FPDEs. Laplace-Sumudu transform; Laplace transform; Sumudu transform; Fractional partial differential equations.

1. Introduction

The double integral transform is nevertheless a new and renewed study (see for example Alderremy et al., 2020; Alderremy and Elzaki, 2018; Elzaki, 2012; Waleed et al., 2021) where previous work considered definitions, simple theories of PDEs (Weatugala, 1993; Belgacem and Karaballi, 2002; Belgacem and Karaballi, 2006; Belgacem and Karaballi, 2006b). The combination of the integral transform with other computational techniques, such as Differential Transform Method, Homotopy Perturbation Method, Adomian Method and Variational Iteration Method has been the subject of much work (Ahmed et al., 2020; Elbadri et al., 2020; Mohamed and Elzaki, 2020; Alderremy et al., 2020, Chamekh et al., 2019; Chamekh et al., 2021) to solve differential equations. In this article, we will develop some results using the proposed double integral transformation. As this transform is still under study, we will compare the solutions with solutions obtained with standard methods. For non-linear FPDEs the work needs even more effort in the future to adapt this transformation to these types of equations.

Here we propose to study the FPDEs in the form:

Let c, d, e real constants and L is the linear differential operator. We consider resolving the following problem:

$$c\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}}+d\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}+eL\varphi(z,t)=s(z,t),z,t\geq 0, \quad (1)$$

with $m-1<\delta\leq m, n-1<\varsigma\leq n, m,n\in\mathbb{N}$,*with* $s(x,t)$ is a source term,*with the initial conditions ICs:*

$$\frac{\partial^{j}\varphi(x,0)}{\partial t^{j}}=f_{j}(x), j=0,1,...,m-1, \quad (2)$$

and the boundary conditions *BCs*:

$$\frac{\partial^{k}\varphi(0,t)}{\partial z^{k}}=h_{k}(t), k=0,1,...,n-1 \quad (3)$$

In what follows, we will consider DLST to solve the problem (1–3).

2. Preliminaries

Definition: 2.1

The DLST of $\varphi(z,t)$, denoted by:

$$L_zS_t[\varphi(z,t)]=\bar{\varphi}(\rho,\sigma)$$

and defined as:

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$$L_xS_t[\varphi(z,t)] = \bar{\varphi}(\rho,\sigma) = \frac{1}{\sigma}\int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(z,t)dzdt. \quad (4)$$

Clearly, the linearity of the (DLST) is shown in the relationship below:

$$\begin{aligned} L_zS_t[\eta\varphi(z,t)+\gamma\psi(z,t)] &= \frac{1}{\sigma}\int_0^\infty \\ &\times \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}[\eta\varphi(z,t)+\gamma\psi(z,t)]dzdt \\ &= \frac{1}{\sigma}\int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\eta\varphi(z,t)dzdt + \frac{1}{\sigma} \\ &\times \int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\gamma\psi(z,t)dzdt \\ &= \frac{\eta}{\sigma}\int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(z,t)dxdt + \frac{\gamma}{\sigma} \\ &\times \int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\psi(z,t)dzdt \\ &= \eta L_zS_t[\varphi(z,t)] + \gamma L_zS_t[\psi(z,t)], \end{aligned} \quad (5)$$

where η and γ are constants.

Definition: 2.2

The inverse (DLST)$L_z^{-1}S_t^{-1}[\varphi(\rho,\sigma)] = \varphi(z,t)$ is defined by:

$$\begin{aligned} L_x^{-1}S_t^{-1}\left[\bar{\varphi}(\rho,\sigma)\right] &= \varphi(z,t) \\ &= \left(\frac{1}{2\pi i}\right)\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\rho z}d\rho\left(\frac{1}{2\pi i}\right)\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{\sigma}e^{\frac{t}{\sigma}} \\ &\times \bar{\varphi}(\rho,\sigma)d\sigma. \end{aligned} \quad (6)$$

Definition: 2.3 The $\delta or \zeta$ th order Caputo derivative ($\delta > 0, \varsigma > 0$) of $\varphi(z,t)$ is given by:

$$\begin{aligned} \frac{\partial^\delta\varphi(z,t)}{\partial t^\delta} &= \frac{1}{\Gamma(n-\delta)}\int_0^t (t-\tau)^{n-\delta-1}\frac{\partial^n\varphi(z,\tau)}{\partial\tau^n}d\tau, p-1<\delta\leqslant p, p\in N, \frac{\partial^\varsigma\varphi(z,t)}{\partial z^\varsigma} \\ &= \frac{1}{\Gamma(q-\varsigma)}\int_0^z (z-\zeta)^{q-\varsigma-1}\frac{\partial^q\varphi(\zeta,t)}{\partial\zeta^q}d\zeta, q-1<\varsigma\leqslant q, q\in N \end{aligned} \quad (7)$$

Using the results proven in Sakamoto et al., 2011, the DLST for the Caputo derivatives can give by:

$$\begin{aligned} L_zS_t\left[\frac{\partial^\delta\varphi(z,t)}{\partial t^\delta}\right] &= \sigma^{-\delta}\bar{\varphi}(\rho,\sigma) \\ &- \sum_{j=0}^{n-1}\sigma^{-\delta+j}L_z\left[\frac{\partial^j\varphi(z,0)}{\partial t^j}\right], L_zS_t\left[\frac{\partial^\varsigma\varphi(z,t)}{\partial z^\varsigma}\right] \\ &= \rho^\varsigma\bar{\varphi}(\rho,\sigma) - \sum_{k=0}^{q-1}\rho^{\varsigma-1-k}S_t\left[\frac{\partial^k\varphi(0,t)}{\partial x^k}\right] \end{aligned} \quad (8)$$

Definition: 2.4

The Mittag-Leffer functions $E_{\delta,\varsigma}(x)$, $\delta > 0$ and $\varsigma > 0$, is,

$$E_{\delta,\varsigma}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\delta k+\varsigma)}, z\in\mathbb{C}, \Re e(\delta)>0, \Re e(\varsigma)>0. \quad (9)$$

The single transform (ST) of $z^{\varsigma-1}E_{\delta,\varsigma}(\lambda z^\delta)$ and $t^{\varsigma-1}E_{\delta,\varsigma}(\lambda t^\delta)$ takes the form:

$$\begin{aligned} \mathscr{L}_x\left[z^{\varsigma-1}E_{\delta,\varsigma}(\lambda z^\delta),\right] &= \frac{\rho^{\delta-\varsigma}}{\rho^\delta-\lambda}, |\lambda|<\left|\rho^\delta\right|, S_t\left[t^{\varsigma-1}E_{\delta,\varsigma}(\lambda t^\delta),\right] \\ &= \frac{\sigma^{\varsigma-1}}{1-\lambda\sigma^\delta}, |\lambda|<\left|\sigma^\delta\right|. \end{aligned} \quad (10)$$

3. Fundamental properties of (DLST)

Sr. No	$\varphi(x,t)$	$L_xS_t[\varphi(x,t)] = \bar{\varphi}(\rho,\sigma)$
1	1	$\frac{1}{\rho}$
2	z^ct^d	$\frac{c!d!}{p^{c+1}}\sigma^d$
3	e^{cz+dt}	$\frac{1}{(\rho-c)(1-d\sigma)}$
4	$sin(cz+dt)$	$\frac{c+d\sigma\rho}{(\rho^2+c^2)(1+d^2\sigma^2)}$
5	$cos(cz+dt)$	$\frac{\rho-cd\sigma}{(\rho^2+c^2)(1+d^2\sigma^2)}$
6	$sinh(cz+dt)$	$\frac{c+d\sigma\rho}{(\rho^2-c^2)(1-d^2\sigma^2)}$
7	$cosh(cz+dt)$	$\frac{\rho+cd\sigma}{(\rho^2-c^2)(1-d^2\sigma^2)}$
8	$J_0(c\sqrt{zt})$, J_0 is the zero order Bessel function	$\frac{4}{4\rho+\sigma c^2}$
9	$\varphi(z-\delta,t-\varepsilon)H(z-\delta,t-\varepsilon)$	$e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\bar{\phi}(\rho,\sigma)$
10	$(\varphi**\psi)(z,t)$	$\sigma\bar{\phi}(\rho,\sigma)\bar{\psi}(\rho,\sigma)$
11	$f(z)g(t)$	$L_x[f(x)]S_t[g(t)]$

3.1. Existence condition for the DLST

If φ is an exponential order a and b; z,t tend to infinity and if it exist a nonnegative real,$K:\forall z>Z, t>T$;

Then:

$$|\varphi(z,t)| = Ke^{az+bt}, \quad (11)$$

and we write:

$$\varphi(z,t) = O(e^{az+bt})asz, ttendtoinfinity \quad (12)$$

Or,

$$\lim_{z\to\infty,t\to\infty} e^{-\rho z-\frac{t}{\sigma}}|\varphi(z,t)| = K\lim_{z\to\infty,t\to\infty} e^{-(\rho-a)z-\left(\frac{1}{\sigma}-b\right)t} = 0, \rho>a, \frac{1}{\sigma}>b. \quad (13)$$

Then φ is an exponential order as x,t tend to infinity.

Theorem 3.1.1

If a function φ on the interval $(0,Z)$ and $(0,T)$ of exponential order c and d, then the DLST of φ well-defined for all ρ and $\frac{1}{\sigma}$ supplied $Re[\rho]>a$ and $Re[\frac{1}{\sigma}]>b$.

Proof.

We find, from the **Def. 2.1,**

$$\begin{aligned} \left|\bar{\varphi}(\rho,\sigma)\right| &= \left|\frac{1}{\sigma}\int_0^\infty \int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(z,t)dzdt\right| \\ &\leq K\int_0^\infty e^{-(\rho-a)z}dz\int_0^\infty \frac{1}{\sigma}e^{-\left(\frac{1}{\sigma}-b\right)t}dt \\ &= \frac{K}{(\rho-a)(1-b\sigma)}, Re[\rho]>a, Re\left[\frac{1}{\sigma}\right]>b. \end{aligned} \quad (14)$$

So, from Eq. (14) we've got,

$$\lim_{z\to\infty,t\to\infty}\left|\bar{\varphi}(\rho,\sigma)\right| = 0, \text{or} \lim_{z\to\infty,t\to\infty}\bar{\varphi}(\rho,\sigma) = 0.$$

3.2. Basic derivatives properties of the DLST

If, $\bar{\varphi}(\rho,\sigma) = L_zS_t[\phi(z,t)]$, then:

$$L_zS_t\left[\frac{\partial\varphi(z,t)}{\partial z}\right] = \rho\,\bar{\varphi}(\rho,\sigma) - S[\varphi(0,t)]. \tag{15}$$

$$L_zS_t\left[\frac{\partial\varphi(z,t)}{\partial t}\right] = \frac{1}{\sigma}\bar{\varphi}(\rho,\sigma) - \frac{1}{\sigma}L(\varphi(z,0)) \tag{16}$$

$$L_zS_t\left[\frac{\partial^2\varphi(z,t)}{\partial z^2}\right] = \rho^2\bar{\varphi}(\rho,\sigma) - \rho S(\varphi(0,t)) - S(\varphi_z(0,t)). \tag{17}$$

$$L_zS_t\left[\frac{\partial^2\varphi(z,t)}{\partial t^2}\right] = \frac{1}{\sigma^2}\bar{\varphi}(\rho,\sigma) - \frac{1}{\sigma^2}L(\varphi(z,0)) - \frac{1}{\sigma}L(\varphi_t(z,0)). \tag{18}$$

$$L_zS_t\left[\frac{\partial^2\varphi(z,t)}{\partial z\partial t}\right] = \frac{\rho}{\sigma}\bar{\varphi}(\rho,\sigma) - \frac{\rho}{\sigma}L(\varphi(z,0)) - S(\varphi_t(0,t)). \tag{19}$$

Proof
For (16)

$$L_zS_t\left[\frac{\partial\varphi(z,t)}{\partial z}\right] = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\frac{\partial\varphi(z,t)}{\partial z}dzdt = \frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}}dt\left\{\int_0^\infty e^{-\rho z}\frac{\partial\varphi(z,t)}{\partial z}dz\right\}$$

Let, $u = e^{-\rho z}, dv = \frac{\partial\varphi(z,t)}{\partial z}dz$, thus:

$$L_zS_t\left[\frac{\partial\phi(z,t)}{\partial z}\right] = \frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}}dt\left\{-\varphi(0,t) + \rho\int_0^\infty e^{-\rho z}\varphi(z,t)dx\right\} = \rho\,\bar{\varphi}(\rho,\sigma) - S(\varphi(0,t)).$$

Concerning (16)

$$L_zS_t\left[\frac{\partial\varphi(z,t)}{\partial t}\right] = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\frac{\partial\varphi(z,t)}{\partial t}dzdt = \frac{1}{\sigma}\int_0^\infty e^{-\rho z}dz\left\{\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial\phi(z,t)}{\partial t}dt\right\}$$

Let, $u = e^{-\frac{t}{\sigma}}, dv = \frac{\partial\varphi}{\partial t}dt$, then:

$$L_zS_t\left[\frac{\partial\varphi(z,t)}{\partial t}\right] = \frac{1}{\sigma}\int_0^\infty e^{-\rho xz}dz\left\{-\varphi(z,0) + \frac{1}{\sigma}\int_0^\infty e^{-\frac{t}{\sigma}}\varphi(z,t)dt\right\} = \frac{1}{\sigma}\bar{\varphi}(\rho,\sigma) - \frac{1}{\sigma}L(\varphi(z,0)).$$

It is easy to prove (17), (18), and (19).

Theorem 3.2.1
If $\bar{\varphi}(\rho,\sigma) = L_zS_t[\varphi(z,t)]$, thus:

$$L_zS_t[\varphi(-\delta+z,-\varepsilon+t)H(-\delta+z,-\varepsilon+t)] = e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\bar{\varphi}(\rho,\sigma), \tag{20}$$

$$H(-\delta+z,-\varepsilon+t) = \left\{\begin{matrix}1, z>\delta, t>\varepsilon\\ 0, otherwise\end{matrix}\right\}. \tag{21}$$

Proof
Using **Def. 2.1**

$$L_zS_t[\varphi(-\delta+z,-\varepsilon+t)H(-\delta+z,-\varepsilon+t)] = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(-\delta+z,-\varepsilon+t)H(-\delta+z,-\varepsilon+t)dzdt = \frac{1}{\sigma}\int_\delta^\infty\int_\varepsilon^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(-\delta+z,-\varepsilon+t)dxdt,$$

Putting, $z-\delta = q, t-\varepsilon = w = e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho q-\frac{w}{\sigma}}\varphi(q,w)dqdw = e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\bar{\varphi}(\rho,\sigma)$.

3.3. Convolution Theorem of DLST

3.4 Definition 3.3.1
The double convolution of $\phi(x,t)$ and $\psi(x,t)$ is given,

$$(\varphi**\psi)(z,t) = \int_0^z\int_0^t\varphi(-\delta+z,-\varepsilon+t)\psi(\delta,\varepsilon)d\delta d\varepsilon. \tag{22}$$

Theorem 3.3.1 (Convolution Theorem)
If $L_xS_t[\varphi(z,t)] = \bar{\varphi}(\rho,\sigma)$ and $L_zS_t[\psi(z,t)] = \bar{\psi}(\rho,\sigma)$ then:

$$L_zS_t[(\varphi**\psi)(z,t)] = \sigma\,\bar{\varphi}(\rho,\sigma)\,\bar{\psi}(\rho,\sigma). \tag{23}$$

Proof
Using **Def. 2.1**

$$L_zS_t[(\varphi**\psi)(z,t)] = \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}(\varphi**\psi)(z,t)dzdt$$
$$= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\left\{\int_0^z\int_0^t\varphi(-\delta+z,-\varepsilon+t)\psi(\delta,\varepsilon)d\delta d\varepsilon\right\}dzdt,$$
$$= \frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\left\{\int_0^\infty\int_0^\infty\varphi(-\delta+z,-\varepsilon+t)H(-\delta+z,-\varepsilon+t)\psi(\delta,\varepsilon)d\delta d\varepsilon\right\}dzdt$$
$$= \int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)d\delta d\varepsilon\left\{\frac{1}{\sigma}\int_0^\infty\int_0^\infty e^{-\rho z-\frac{t}{\sigma}}\varphi(-\delta+z,-\varepsilon+t)H(-\delta+z,-\varepsilon+t)dzdt\right\},$$
$$= \int_0^\infty\int_0^\infty\psi(\delta,\varepsilon)d\delta d\varepsilon\left\{e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\bar{\varphi}(\rho,\sigma)\right\} = \bar{\varphi}(\rho,\sigma)\int_0^\infty\int_0^\infty e^{-\rho\delta-\frac{\varepsilon}{\sigma}}\psi(\delta,\varepsilon)d\delta d\varepsilon$$
$$= \sigma\,\bar{\phi}(\rho,\sigma)\,\bar{\psi}(\rho,\sigma)$$

Using **Theorem 3.2.1** to find

4. Principle of DLST method

In this part, we exercised DLST to find the solution to the Eq. (1). First, apply DLST to Eq. (1), we get:

$$c\left[\sigma^{-\delta}\bar{\phi}(\rho,\sigma) - \sum_{j=0}^{n-1}\sigma^{-\delta+j}L_x\left[\frac{\partial^j\phi(x,0)}{\partial t^j}\right]\right] + d\left[\rho^{\varsigma}\bar{\phi}(\rho,\sigma) - \sum_{k=0}^{m-1}\rho^{\varsigma-1-k}S_t\left[\frac{\partial^k\phi(0,t)}{\partial x^k}\right]\right] + eL_xS_t[L\phi(x,t)] = \bar{g}(\rho,\sigma) \tag{24}$$

Furthermore, applying single (LT) to the $ICs(2)$ and single (ST) to the $BCs(3)$, we've got,

$$L_x\left[\frac{\partial^j\phi(x,0)}{\partial t^j}\right] = \bar{f}_j(\rho), j = 0,1,...,n-1, S_t\left[\frac{\partial^k\phi(0,t)}{\partial x^k}\right] = \bar{h}_k(\sigma), k = 0,1,\cdots,m-1. \tag{25}$$

By replacing (25) in (24), we get

$$\bar{\phi}(\rho,\sigma) = \frac{1}{(c\sigma^{-\delta}+d\rho^{\varsigma})}\left[\begin{matrix}c\sum_{j=0}^{n-1}\sigma^{-\delta+j}\bar{f}_j(\rho) + d\sum_{k=0}^{m-1}\rho^{\varsigma-1-k}\bar{h}_k(\sigma) - eL_xS_t[L\phi(x,t)]\\ +\bar{g}(\rho,\sigma)\end{matrix}\right] \tag{26}$$

Taking $L_x^{-1}S_t^{-1}\left[\bar{\phi}(\rho,\sigma)\right]$ of (26) to find the solution of (1);

$$\phi(x,t) = L_x^{-1}S_t^{-1}\left[\frac{1}{(c\sigma^{-\delta}+d\rho^{\varsigma})}\left[\begin{array}{c} c\sum_{j=0}^{n-1}\sigma^{-\delta+j}\bar{f}_j(\rho)+d\sum_{k=0}^{m-1}\rho^{\varsigma-1-k}\bar{h}_k(\sigma) \\ -eL_xS_t[L\phi(x,t)]+\bar{g}(\rho,\sigma)\end{array}\right]\right] \quad (27)$$

5. Elucidative examples

Without forgetting the work of Podlubny 1999, who has proposed some analytical solutions for certain linear equations which can be calculated. We treat in this numerical part some physical applications using the DLST method. The objective is to illustrate the method and prove its applicability.

5.1. Fractional advection-diffusion equation

Putting, $c=1, n=1, s=0, m=2, d=-q_1, e=-q_2, L=\frac{\partial}{\partial x}$, to find,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = q_1\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}} + q_2\frac{\partial\varphi(z,t)}{\partial z}, 0<\delta\le 1, 1<\varsigma\le 2, \quad (28)$$

WithICs and BCs:

$$\varphi(z,0)=f_0(z), \varphi(0,t)=h_0(t), \varphi_z(0,t)=h_1(t), \quad (29)$$

So,(27) yields the solution of (28)

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta}-q_1\rho^{\varsigma}-q_2\rho)}\left[\begin{array}{c}\sigma^{-\delta}\bar{f}_0(\rho)-q_1\rho^{\varsigma-1}\bar{h}_0(\sigma)-q_1\rho^{\varsigma-2}\bar{h}_1(\sigma)\\ -q_2\bar{h}_0(\sigma)\end{array}\right]\right] \quad (30)$$

5.2. Fractional reaction-diffusion equation

Putting, $s=0, m=2, n=1, a=1, b=-q$, and $L\varphi(z,t)=\varphi(z,t)$ in Eq. (1), to get,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} + e\varphi(z,t) = q\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}, 0<\delta\le 1, 1<\varsigma\le 2, \quad (31)$$

With, ICs and BCs:

$$\varphi(z,0)=f_0(z), \varphi(0,t)=h_0(t), \varphi_z(0,t)=h_1(t), \quad (32)$$

So,(27) gives a solution of (31) as:

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta}+e-q\rho^{\varsigma})}\left[\sigma^{-\delta}\bar{f}_0(\rho)-q\rho^{\varsigma-1}\bar{h}_0(\sigma)-q\rho^{\varsigma-2}\bar{h}_1(\sigma)\right]\right]. \quad (33)$$

5.2.1. Fractional heat (diffusion) equation

If we choose, $e=0$ in (31), we got,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = q\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}, 0<\delta\le 1, 1<\varsigma\le 2, \quad (34)$$

With, ICs and BCs:

$$\varphi(z,0)=f_0(z), \varphi(0,t)=h_0(t), \varphi_z(0,t)=h_1(t), \quad (35)$$

So, (33) yields the solution of (34)

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta}-q\rho^{\varsigma})}\left[\sigma^{-\delta}\bar{f}_0(\rho)-q\rho^{\varsigma-1}\bar{h}_0(\sigma)-q\rho^{\varsigma-2}\bar{h}_1(\sigma)\right]\right]. \quad (36)$$

Example 1: Putting $q=1$ and $\varsigma=2$ in (34), to yield,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = \frac{\partial^{2}\varphi(z,t)}{\partial z^{2}}, 0<\delta\le 1, \quad (37)$$

with;

$$\varphi(z,0)=sinz=f_0(z), \varphi(0,t)=0=h_0(t), \varphi_z(0,t)=E_{\delta}(-t^{\delta})=h_1(t) \quad (38)$$

Replacing,

$$\bar{f}_0(\rho)=\frac{1}{\rho^2+1}, \ \bar{h}_0(\sigma)=0, \ \bar{h}_1(\sigma)=\frac{1}{1+\sigma^{\delta}},$$

in (36) and simplifying, we obtain,

$$\varphi(z,t)=L_z^{-1}S_t^{-1}\left[\frac{1}{(\rho^2+1)(1+\sigma^{\delta})}\right]=sinzE_{\delta}(-t^{\delta}). \quad (39)$$

When, $\delta=1$, we get,

$$\varphi(z,t)=e^{-t}sinz. \quad (40)$$

5.3. Fractional telegraph equation

Putting $m=2, n=2, d=-q, s=0$, and $L=c_0+c_1\frac{\partial}{\partial t}$, we've got,

$$c\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} + c_1\frac{\partial\varphi(z,t)}{\partial t} + c_0\varphi(z,t) = q\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}, 1<\delta,\varsigma\le 2, \quad (41)$$

With, ICs and BCs:

$$\varphi(z,0)=f_0(z), \varphi_t(z,0)=f_1(z), \varphi(0,t)=h_0(t), \varphi_z(0,t)=h_1(t), \quad (42)$$

So, (27) gives the solution of (41) in the form,

$$\varphi(z,t)=L_z^{-1}S_t^{-1}\left[\frac{1}{(c_0+c_1\sigma^{-1}+c\sigma^{-\delta}-q\rho^{\varsigma})}\left[\begin{array}{c}c_1\sigma^{-1}\bar{f}_0(\rho)+c\sigma^{-\delta}\bar{f}_0(\rho)+c\sigma^{-\delta+1}\bar{f}_1(\rho)\\ -q\rho^{\varsigma-1}\bar{h}_0(\sigma)-q\rho^{\varsigma-2}\bar{h}_1(\sigma)\end{array}\right]\right]. \quad (43)$$

Example 2: If we choose, $c_0=c_1=c=q=1$ and $\delta=2$ in (41), we get,

$$\varphi(z,t)+\frac{\partial\varphi(z,t)}{\partial t}+\frac{\partial^2\varphi(z,t)}{\partial t^2}=\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}, 1<\varsigma\le 2, \quad (44)$$

$$\varphi(0,t)=h_0(t)=e^{-t}, \varphi_z(0,t)=h_1(t)=e^{-t}, \quad (45)$$

Replacing,

$$\bar{f}_0(\rho)=\left(\frac{1}{\rho}+\frac{1}{\rho^2}\right)\frac{\rho^{\varsigma}}{\rho^{\varsigma}-1}, \ \bar{f}_1(\rho)=-\left(\frac{1}{\rho}+\frac{1}{\rho^2}\right)\frac{\rho^{\varsigma}}{\rho^{\varsigma}-1},$$

$$\bar{h}_0(\sigma)=\bar{h}_1(\sigma)=\frac{1}{1+\sigma},$$

in (43) and simplify it, to find:

$$\varphi(z,t)=L_z^{-1}S_t^{-1}\left[\frac{1}{(1+\sigma)}\left(\frac{1}{\rho}+\frac{1}{\rho^2}\right)\frac{\rho^{\varsigma}}{\rho^{\varsigma}-1}\right] = e^{-t}\left[E_{\varsigma}(z^{\varsigma})+zE_{\varsigma,2}(z^{\varsigma})\right], \quad (46)$$

When $\varsigma=2$, we get

$$\varphi(z,t)=e^{z-t}. \quad (47)$$

5.3.1. Fractional wave equation

If we choose, $c_0 = c_1 = 0, c = 1$ in (41), then, we get,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = q\frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}}, 1 < \delta, \varsigma \le 2, \quad (48)$$

With ICs and BCs:

$$\varphi(z,0) = f_0(z), \varphi_t(z,0) = f_1(z), \varphi(0,t) = h_0(t), \varphi_z(0,t) = h_1(t), \quad (49)$$

Then, (43) gives a solution of (48) as;

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta} - q\rho^{\varsigma})}\left[\sigma^{-\delta}\bar{f}_0(\rho) + \sigma^{-\delta+1}\bar{f}_1(\rho) - q\rho^{\varsigma-1}\bar{h}_0(\sigma) - q\rho^{\varsigma-2}\bar{h}_1(\sigma)\right]\right]. \quad (50)$$

Fractional Klein–Gordon equation

Putting, $n = m = 2, c = 1, d = -1$, and $L\varphi = \varphi$, to find,

$$\frac{\partial^{\delta}\varphi}{\partial t^{\delta}} - \frac{\partial^{\varsigma}\varphi}{\partial z^{\varsigma}} + e\varphi(z,t) = s(z,t), 1 < \delta, \varsigma \le 2. \quad (51)$$

With ICs and BCs:

$$\varphi(z,0) = f_0(z), \varphi_t(z,0) = f_1(z), \varphi(0,t) = h_0(t), \varphi_z(0,t) = h_1(t). \quad (52)$$

So, (27) yields the solution of (51),

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(e + \sigma^{-\delta} - \rho^{\varsigma})}\left[\begin{array}{c}\sigma^{-\delta}\bar{f}_0(\rho) + \sigma^{-\delta+1}\bar{f}_1(\rho) - \rho^{\varsigma-1}\bar{h}_0(\sigma) \\ -\rho^{\varsigma-2}\bar{h}_1(\sigma) + \bar{s}(\rho,\sigma)\end{array}\right]\right]. \quad (53)$$

Example 3: If we choose, $e = -1$, $\varsigma = 2$, and $g(x,t) = 0$ in (51), we have got

$$\frac{\partial^{\delta}\varphi(x,t)}{\partial t^{\delta}} - \frac{\partial^{2}\varphi(x,t)}{\partial z^{2}} - \varphi(z,t) = 0, 1 < \delta \le 2, \quad (54)$$

with;

$$\varphi(z,0) = sinz + 1 = f_0(x), \varphi_t(z,0) = 0 = f_1(z),$$
$$\varphi(0,t) = E_{\delta}(t^{\delta}) = h_0(t), \varphi_z(0,t) = 1 = h_1(t). \quad (55)$$

Substituting, $\bar{f}_0(\rho) = \frac{1}{\rho^2+1} + \frac{1}{\rho}$, $\bar{f}_1(\rho) = 0$, $\bar{h}_0(\sigma) = \frac{1}{1-\sigma^{\delta}}$, $\bar{h}_1(\sigma) = 1$, in (53) and simplifying, to obtain,

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{\rho^2+1} + \frac{1}{\rho(1-\sigma^{\delta})}\right] = sinz + E_{\delta}(t^{\delta}). \quad (56)$$

5.4. Fractional Burger's equation

Choosing, $= 2, n = 1, d = -1, c = 1, e = 1, L = \frac{\partial}{\partial z}$, we obtain,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} - \frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}} + \frac{\partial\varphi(z,t)}{\partial z} = s(z,t), 0 < \delta \le 1, 1 < \varsigma \le 2, \quad (57)$$

and

$$\varphi(z,0) = f_0(z), \varphi(0,t) = h_0(t), z(0,t) = h_1(t), \quad (58)$$

then we have the following solution,

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta} - \rho^{\varsigma} + \rho)}\left[\begin{array}{c}\sigma^{-\delta}\bar{f}_0(\rho) + \bar{h}_0(\sigma) - \rho^{\varsigma-1}\bar{h}_0(\sigma) \\ -\rho^{\varsigma-2}\bar{h}_1(\sigma) + \bar{s}(\rho,\sigma)\end{array}\right]\right]. \quad (59)$$

Example 4: If we choose, $\varsigma = 2$, and $s(z,t) = 0$ in (57) to get,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} - \frac{\partial^{2}\varphi(z,t)}{\partial z^{2}} + \frac{\partial\varphi(z,t)}{\partial z} = 0, 0 < \delta \le 1, \quad (60)$$

with;

$$\varphi(z,0) = e^{-z} = f_0(z), \varphi(0,t) = E_{\delta}(2t^{\delta}) = h_0(t),$$
$$\varphi_z(0,t) = -E_{\delta}(2t^{\delta}) = h_1(t). \quad (61)$$

Replacing, $\bar{f}_0(\rho) = \frac{1}{\rho+1}$, $\bar{h}_0(\sigma) = \frac{1}{1-2\sigma^{\delta}}$, $\bar{h}_1(\sigma) = -\frac{1}{1-2\sigma^{\delta}}$, in (59) and simplifying, to obtain,

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\rho+1)(1-2\sigma^{\delta})}\right] = e^{-z}E_{\delta}(2t^{\delta}). \quad (62)$$

5.5. Fractional Fokker-Planck equation

Putting, $m = 2, n = 1, d = -1, c = 1, e = -1, L = \frac{\partial}{\partial z}$, and $s(z,t) = 0$, to get,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = \frac{\partial^{\varsigma}\varphi(z,t)}{\partial z^{\varsigma}} + \frac{\partial\varphi(z,t)}{\partial z}, 0 < \delta \le 1, 1 < \varsigma \le 2, \quad (63)$$

with ICs and BCs:

$$\varphi(z,0) = f_0(z), \varphi(0,t) = h_0(t), \varphi_z(0,t) = h_1(t), \quad (64)$$

Then:

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta} - \rho^{\varsigma} - \rho)}\left[\sigma^{-\delta}\bar{f}_0(\rho) - \rho^{\varsigma-1}\bar{h}_0(\sigma) - \rho^{\varsigma-2}\bar{h}_1(\sigma) - \bar{h}_0(\sigma)\right]\right]. \quad (65)$$

Example 5: If we choose, $\varsigma = 2$ in (63) then, we find:

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} = \frac{\partial^{2}\varphi(z,t)}{\partial z^{2}} + \frac{\partial\varphi(z,t)}{\partial z}, 0 < \delta \le 1, \quad (66)$$

with,

$$\varphi(z,0) = z = f_0(z), \varphi(0,t) = \frac{t^{\delta}}{\Gamma(1+\delta)} = h_0(t), \varphi_z(0,t) = 1 = h_1(t). \quad (67)$$

Replacing, $\bar{f}_0(\rho) = \frac{1}{\rho^2}$, $\bar{h}_0(\sigma) = \sigma^{\delta}$, $\bar{h}_1(\sigma) = 1$, in (65) and simplify it, to find:

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{\rho^2} + \frac{\sigma^{\delta}}{\rho}\right] = z + \frac{t^{\delta}}{\Gamma(1+\delta)}. \quad (68)$$

5.6. Fractional Korteweg-de Vries (KdV) equation

Putting, $m = 1, n = 1, c = 1$, and $L = \frac{\partial^3}{\partial z^3}$ in (1), to get,

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} + d\frac{\partial^{\varsigma}\varphi(x,t)}{\partial z^{\varsigma}} + e\frac{\partial^{3}\varphi(\varphi,t)}{\partial z^{3}} = s(z,t), 0 < \delta, \varsigma \le 1, \quad (69)$$

with ICs and BCs:

$$\varphi(z,0) = f_0(z), \varphi(0,t) = h_0(t), \varphi_z(0,t) = h_1(t), \varphi_{zz}(0,t) = h_2(t), \quad (70)$$

Then:

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\sigma^{-\delta} + d\rho^{\varsigma} + e\rho^3)}\left[\begin{array}{c}\sigma^{-\delta}\bar{f}_0(\rho) + e\rho^2\bar{h}_0(\sigma) + e\rho\bar{h}_1(\sigma) \\ + e\bar{h}_2(\sigma) + d\rho^{\varsigma-1}\bar{h}_0(\sigma) + \bar{s}(\rho,\sigma)\end{array}\right]\right]. \quad (71)$$

Example 6: If we choose, $d = 2, e = 1, \varsigma = 1$, and $s(x,t) = 0$ in (69) we have:

$$\frac{\partial^{\delta}\varphi(z,t)}{\partial t^{\delta}} + 2\frac{\partial\varphi(z,t)}{\partial z} + \frac{\partial^{3}\varphi(z,t)}{\partial z^{3}} = 0, 0 < \delta \le 1, \quad (72)$$

with,

$$\varphi(z,0) = f_0(z) = sinz, \varphi(0,t) = -t^{\delta}E_{2\delta,\delta+1}(-t^{2\delta}) = h_0(t), \varphi_z(0,t) = E_{2\delta,1}(-t^{2\delta}) = h_1(t), \varphi_{zz}(0,t) = t^{\delta}E_{2\delta,\delta+1}(-t^{2\delta}) = h_2(t). \quad (73)$$

Replacing, $\bar{f}_0(\rho) = \frac{1}{\rho^2+1}$, $\bar{h}_0(\sigma) = -\frac{\sigma^{\delta}}{1+\sigma^{2\delta}}$, $\bar{h}_1(\sigma) = \frac{1}{1+\sigma^{2\delta}}$, $\bar{h}_2(\sigma) = \frac{\sigma^{\delta}}{1+\sigma^{2\delta}}$, in (71) and simplifying, to obtain,

$$\varphi(z,t) = L_z^{-1}S_t^{-1}\left[\frac{1}{(\rho^2+1)(1+\sigma^{2\delta})} - \frac{\rho\sigma^{\delta}}{(\rho^2+1)(1+\sigma^{2\delta})}\right] = (\sin z)E_{2\delta,1}(-t^{2\delta}) - (cosz)t^{\delta}E_{2\delta,\delta+1}(-t^{2\delta}). \quad (74)$$

6. Discussions

We are going look in this section the numeric evaluation of obtained results of fractional equations that have proposed to be solve. Moreover, we will discuss the numerical behavior of the solution resulting from a fractional differential equation and compare it with that of the equation with integer derivative.

When $\delta = 1$, the closed form solution of each Example 3, 4, 5 and 6 is simply calculated. We have opted to examine the numerical results for the values of $\delta = 0.95$, 0.85 and 0.75. We have noticed that the solutions obtained for the different fractional values of δ are in coordination with the solution of closed form for $\alpha = 1$, as shown in Figs. 1–4. It suffices to clearly notice that when α approaches 1, the solution resulting from the fractional equation approaches this exact solution.

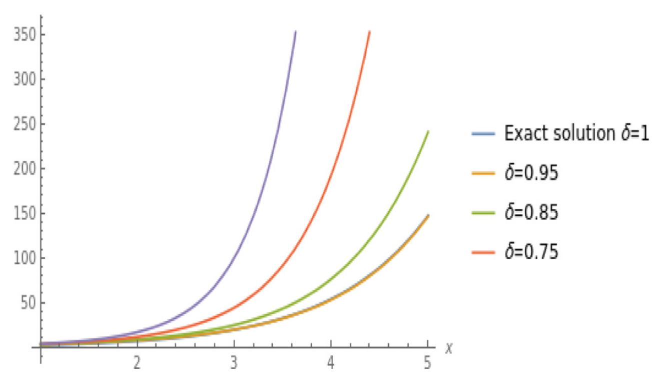

Fig. 1. Solutions of Fractional Klein–Gordon.

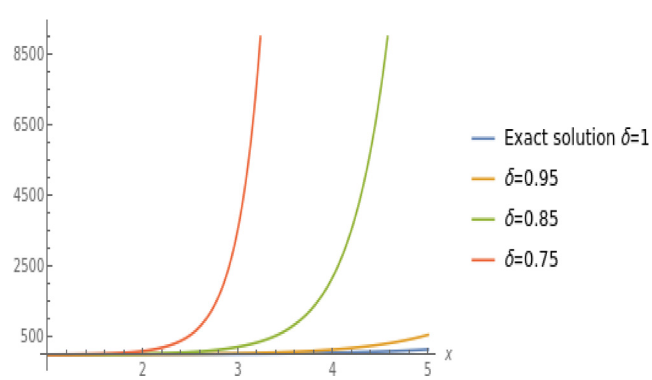

Fig. 2. Solutions of Fractional Burger's.

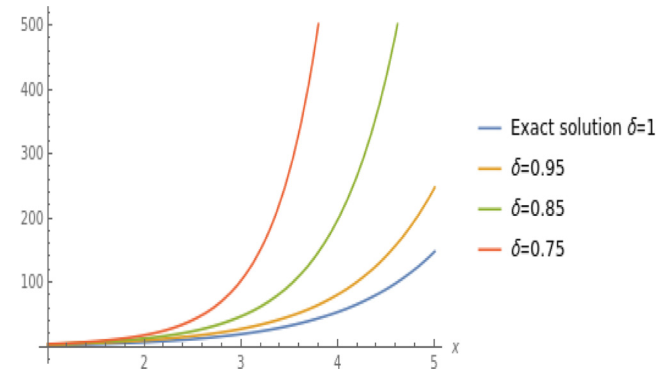

Fig. 3. Solutions of Fractional Fokker-Planck.

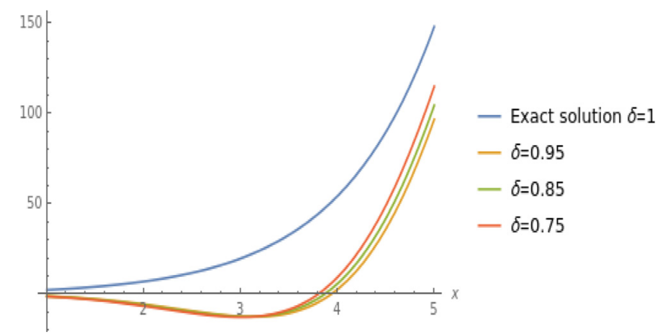

Fig. 4. Solutions of Fractional Korteweg-de Vries (KdV).

7. Conclusions

This article deals with the approach of a new double transformation. The objective is based on the simplicity of the technique used to solve a large class of fractional equations of mathematical physics. The examples show that DLST can be a good alternative to treat many LFPDEs that appear in science. However, it should be noted that the solutions obtained are valid while the inverse of this transform exists.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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