Original article

The Marshall-Olkin extended inverted Kumaraswamy distribution: Theory and applications

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A B S T R A C T

A new three parametric distribution is proposed and analyzed, termed as the Marshall-Olkin extended inverted Kumaraswamy (MOEIK) distribution. This generalization has some renowned sub models such as the Beta type II, the Lomax and the Fisk distribution, stated in literature. Study includes the basic properties of the observed probabilistic model. Explicit expressions for major mathematical properties of this distribution such as quantile function, complete and incomplete moments, entropies and moments of order statistic are derived. Maximum likelihood estimation method is used to estimate the parameters. For different parameter values, a number of simulation studies are conducted for different sample sizes and compare the performance of the MOEIK distribution. Three real life applications are provided to explain the potentiality and reliability of the extended distribution with confidence that the generalized model have wider applications in hydrology and the associated fields.

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1. Introduction

From last few decades researchers have greater intention toward the inversion of univariate probability models and their applicability under inverse transformation i.e. inverted beta (Dubey, 1970), inverse Rayleigh (Voda, 1972), inverse Gaussian (Folks and Chhikara, 1978), inverse Weibull (Calabria and Pulcini, 1990), inverted Burr type XII also called Burr type III (Al-Dayian, 1999), inverse Rayleigh (Rosaiah and Kantam, 2005), inverted exponential (Dey, 2007), inverse Lindley (Sharma et al., 2015) and many other distributions.

Since two-parameter Kumaraswamy distribution has much importance as a substitute of the beta distribution introduced by Kumaraswamy (1980) during the explanation of hydrological applications. He had found that numerous important distributions such as log normal distribution, normal distribution, beta distribution and many more do not have better fit on hydrological data i.e. daily stream flow data and rain fall data. Moreover, the model
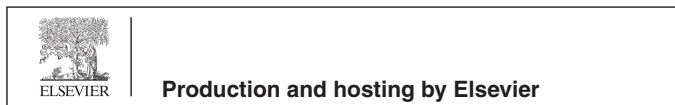
endorse wide range of applications including test scores, atmospheric temperature, height of individuals and many others (Jones, 2009; Sindhu et al., 2013; El-Deen et al., 2014). The distribution has two shape parameters $\alpha > 0$ and $\beta > 0$ while interval of random variable belong to distribution is [0, 1]. Many researchers such as Seifi et al. (2000), Ganji et al. (2006) and Nadarajah (2008) had presented the betterment of Kumaraswamy distribution over beta distribution. These models have much resemblance in their basic properties (Lemonte et al., 2013).

In literature, it had seen that IK distribution has long right tail as compared to other existing distributions. Therefore, it will have long term affect in passionate predictions of rare events and reliability predictions (Iqbal et al., 2017). To follow the concept of the inverted distribution, Al-Fattah et al. (2017) developed the inverted Kumaraswamy distribution by using the inverse transformation $T = \frac{1}{Y} - 1$, where Y follows the Kumaraswamy distribution with parameters α and β. The probability density function (pdf) and its corresponding cumulative distribution function (cdf) of the Inverted Kumaraswamy (IK) distribution with two shape parameters $\alpha > 0,\ \beta > 0$ are

$$g(t) = \alpha\beta(1+t)^{-\alpha-1}(1-(1+t)^{-\alpha})^{\beta-1};\ t \geqslant 0,\ \alpha,\ \beta > 0 \qquad (1)$$

$$G(t) = (1-(1+t)^{-\alpha})^{\beta} \qquad (2)$$

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The main purpose of the study is to develop the Marshall-Olkin extended inverted Kumaraswamy distribution to increase the flexibility of the parent model and study its behavior and basic properties. Recently Haq et al. (2017) used this transformation to develop Marshall-Olkin length biased moment exponential distribution. Marshall and Olkin (1997) introduced an innovative technique to add an additional shape parameter to the existing distribution and distribution resulting in the Marshall-Olkin distribution. If G(x) is the cdf and $\bar{G}(x) = 1 - G(x)$ is the survival rate function (srf) then the srf of Marshall-Olkin family takes the form

$$\bar{F}(x; \varphi) = \frac{\lambda \bar{G}(x; \varphi)}{1 - \bar{\lambda} \bar{G}(x; \varphi)} \tag{3}$$

where $\bar{\lambda} = 1 - \lambda$ and φ represents the parameters of the parent distribution. It is clear that we can obtain the base distribution for $\lambda = 1$. The parameter λ is generally called "tilt parameter". The probability density function of the Marshall Olkin generated family is

$$f(x; \varphi) = \frac{\lambda g(x; \varphi)}{[1 - \bar{\lambda} \bar{G}(x; \varphi)]^2} \tag{4}$$

The major motivation of the MO family is given as follows: Let $Z_N = \min\{T_1, \ldots, T_N\}$, where $T_1, \ldots, T_N$ are a sequence of IID random variables from $G(x)$ and N is a random variable with probability mass function $\gamma(1 - \gamma^{n-1})$, $n = 1, 2, \ldots$. Then, the CDF of Z_N is defined by;

$$P(Z_N \leqslant x) = 1 - \sum_{n=1}^{\infty} P(Z_N \leqslant x | N = n) P(N = n) = \frac{G(x)}{G(x) + \gamma \bar{G}(x)},$$

which is equivalent to (3).

Some explicit expressions of the moments, incomplete moments, mode, random number generator, order statistics and its moments are provided in Section 3. Section 4 explains the simulation study of true parameters estimated from maximum likelihood estimation. Three real life applications are also provided in Section 5 to explain the flexibility of the observed model as compared to some existing models. The Section 6 concludes the study.

2. The MOEIK distribution

By using the Marshall-Olkin generator, the pdf and cdf of the MOEIK distribution with a new parameter $\lambda > 0$ are given respectively.

$$f(x; \lambda, \alpha, \beta) = \frac{\lambda \alpha \beta (1 + x)^{-\alpha - 1} (1 - (1 + x)^{-\alpha})^{\beta - 1}}{[1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})]^2} \text{ for } \alpha > 0,$$
$$\beta > 0 \text{ and } x > 0 \tag{5}$$

$$F(x; \lambda, \alpha, \beta) = \frac{(1 - (1 + x)^{-\alpha})^{\beta}}{1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})} \tag{6}$$

Since the distribution function is complex therefore, to get the explicit expression of the distribution. We use some expansion functions follow the generalized binomial theorem (for 0 < a < 1),

$$(1 - a)^{-n} = \sum_{k=0}^{\infty} \frac{\Gamma(n + k)}{\Gamma(n)\Gamma(k+1)} a^k \text{ and } (1 + z)^b = \sum_{j=0}^{\infty} \binom{b}{j} z^j, \ |z| > 0$$

Under these expansions, the pdf of the MOEIK model becomes

$$f(x; \lambda, \alpha, \beta) = \lambda \alpha \beta (1 + x)^{-\alpha - 1} \sum_{i,j=0}^{\infty} w_{ij} (1 - (1 + x)^{-\alpha})^{\beta j + \beta - 1} \tag{7}$$

where

$$w_{i,j} = \frac{\Gamma(2 + i)(1 - \lambda)^i (-1)^j}{\Gamma(2)\Gamma(i + 1)} \binom{i}{j}$$

The survival rate function and the hazard rate function (hrf) of the observed models are

$$R(x; \lambda, \alpha, \beta) = \frac{\lambda(1 - (1 - (1 + x)^{-\alpha})^{\beta})}{1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})} \tag{8}$$

$$h(x; \lambda, \alpha, \beta) = \frac{\alpha \beta (1 + x)^{-\alpha - 1} (1 - (1 + x)^{-\alpha})^{\beta - 1}}{(1 - (1 - (1 + x)^{-\alpha})^{\beta})[1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})]} \tag{9}$$

respectively.

A simple motivation of the MOEIK distribution follows by considering a parallel system with Z independent components and suppose that a random variable Z has the geometric distribution with the probability mass function $P(Z = z) = \gamma^{-1}(1 - \gamma^{-1})^{z-1}$, $z = 1, 2, \ldots$ and $\gamma > 1$. Let $T_1, T_2, \ldots$ represent the lifetimes of each component and suppose that they have the IK distribution given in (1). Then a random variable $X = \max(T_1, \ldots, T_z)$ represents the lifetime of the system. Therefore, the random variable X follows the MOEIK in (5).

2.1. Limiting behavior for the MOEIK density function

Lemma 1. For x approaches to origin, limits of the MOEIK density function is as follows

$$\lim_{x \to 0} f(x) = \begin{cases} \infty & \text{for } \beta < 1 \\ \frac{\alpha^{\beta} \beta}{\lambda} & \text{for } \beta = 1 \\ 0 & \text{for } \beta > 1 \end{cases} \tag{10}$$

Proof:. As the pdf of the MOEIK distribution is

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[\frac{\lambda \alpha \beta (1 + x)^{-\alpha - 1} (1 - (1 + x)^{-\alpha})^{\beta - 1}}{[1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})]^2} \right]$$

The quantity

$$\lim_{x \to 0} (1 + x)^{-\alpha - 1} \cong 1 \text{ and } \lim_{x \to 0} [1 - (1 - \lambda)(1 - (1 - (1 + x)^{-\alpha})^{\beta})]^2 \cong \lambda^2$$

The above expression becomes

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\left(\frac{\alpha \beta}{\lambda} \right) (1 - (1 + x)^{-\alpha})^{\beta - 1} \right)$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\left(\frac{\alpha \beta}{\lambda} \right) \left[1 - \left(1 - \alpha x + \frac{(\alpha x)^2}{2!} - \frac{(\alpha x)^3}{3!} + \ldots \right) \right]^{\beta - 1} \right)$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\left(\frac{\alpha^{\beta} \beta}{\lambda} \right) x^{\beta - 1} \left[1 - \frac{\alpha x}{2!} + \frac{(\alpha x)^2}{3!} + \ldots \right]^{\beta - 1} \right)$$

Now the results can be formed easily. □

2.2. Shapes of the density and hazard rate function

The Fig. 1(a) and (b) represent the behavior of the density function and explains the tractability and flexibility of the model graphically with its sub-families. The pdf plot shows that for $\beta < 1$, the newly developed model has exponentially decreasing
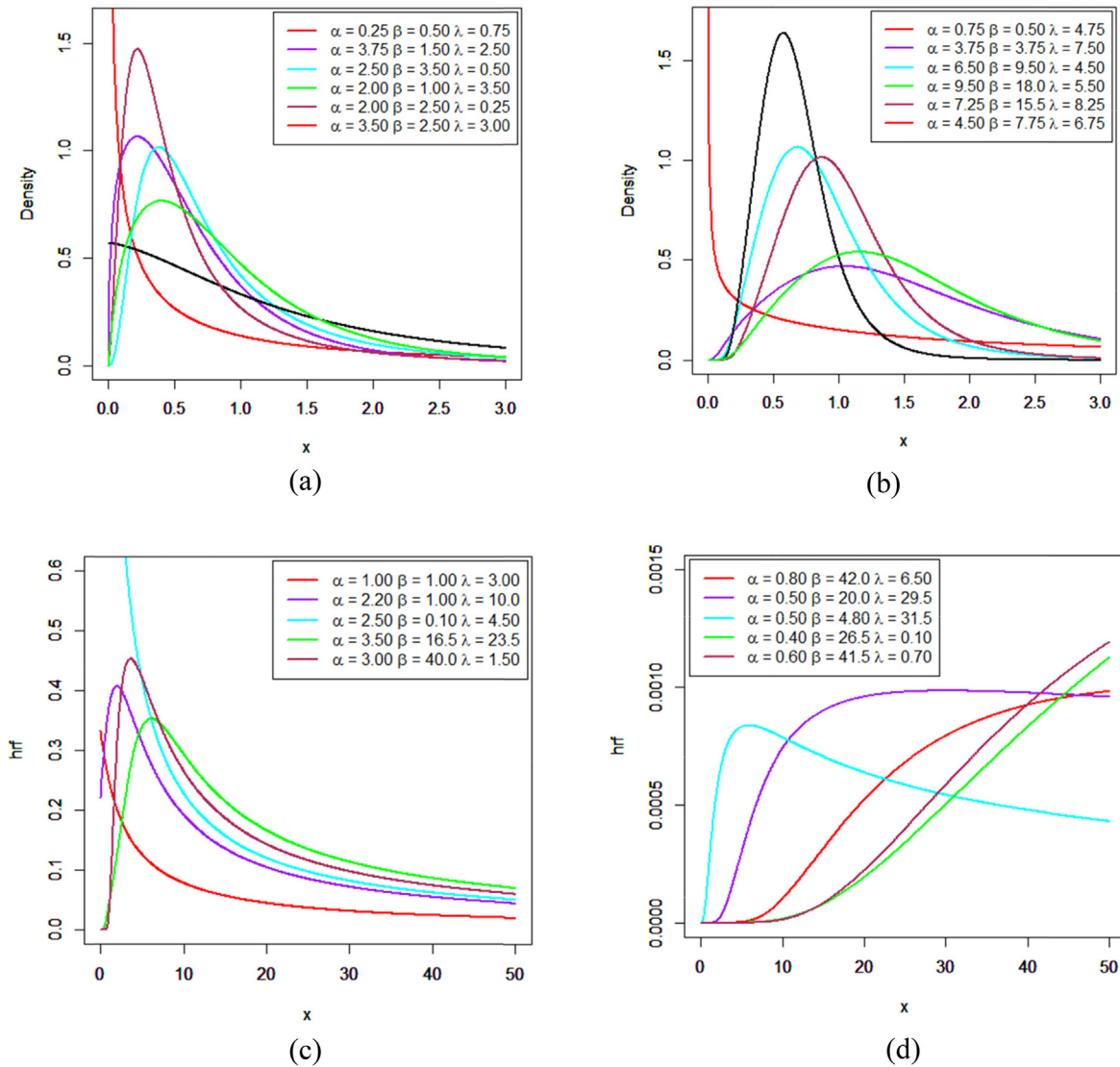
Fig. 1. Density function 1 (a) and 1 (b) and hazard rate function 1 (c) and 1 (d) at different parameter values.

behavior. For $\beta = 1$, the curves also have exponentially decreasing behavior but starts from y-axis while for $\beta > 1$, the distribution has unimodal positively skewed behavior. Fig. 1(c) and (d) represents behavior of hazard rate function (hrf) which is unimodal at various parameter combinations and also have bathtub and decreasing behavior for small α and β, while λ approaches to zero. Moreover, the observed model can have upside down bathtub behavior as the x-axis increases.

Table: Some special models of MOEIK distribution.

Special Cases	Resultant model	Pdf for $x > 0$
$\beta = \lambda = 1$	Lomax distribution (α)	$\frac{\alpha}{(1+x)^{\alpha+1}}$ for $\alpha > 0$
$\beta = 1$	Marshall-Olkin Lomax (α, λ)	$\frac{\lambda\alpha(1+x)^{-\alpha-1}}{[1-(1-\lambda)(1+x)^{-\alpha}]^2}$ for $\lambda > 0, \alpha > 0$
$\alpha = \lambda = 1$	Beta type II (β)	$\frac{1}{B(\beta,1)} x^{\beta-1}(1+x)^{-(\beta+1)}$ for $\beta > 0$
$\alpha = 1$	Marshall-Olkin Beta type II (β, λ)	$\frac{\lambda x^{\beta-1}(1+x)^{-(\beta+1)}}{B(\beta,1)}\left[1-(1-\lambda)\left(1-\left(\frac{x}{1+x}\right)^\beta\right)\right]^{-2} \beta, \lambda > 0$
$\alpha = \beta = \lambda = 1$	The log-logistic (Fisk) distribution	$\frac{1}{(1+x)^2}$
$\alpha = \beta = 1$	Marshall-Olkin log-logistic (λ)	$\frac{\lambda}{(1+x)^2}\left[1-\frac{(1-\lambda)}{1+x}\right]^{-2}$ for $\lambda > 0$

2.3. Quantile function

Let random variable $X \sim MOEIK(\lambda, \alpha, \beta)$ having pdf in Eq. (5). The computation of the quantile function, say $x = Q(u)$ of the MOEIK distribution can be attained by inverting Eq. (6)

$$x = Q(u) = F^{-1}(u) = \left[1 - \left\{\frac{u\lambda}{1 - u(1-\lambda)}\right\}^{\frac{1}{\beta}}\right]^{-\frac{1}{\alpha}} - 1 \qquad (11)$$

where U follows the uniform distribution under limits [0, 1]. We can compute the first quartile, median and the third quartile of the observed distribution by inserting $u = 0.25, 0.5$ and 0.75 respectively.

3. Properties of MOEIK distribution

This section provides unambiguous expressions for some major properties of the observed distribution starting from the ordinary moments.

3.1. Moments

This subsection provides the unambiguous expressions of the rth non-central moment and cumulants. Numerous important features and characteristics such as tendencies, coefficient of skewness, coefficient of kurtosis and dispersions can be studied from moments. The given theorem provide moments.

The *rth* raw moments of the MOEIK distribution can be obtained by following the definition of the moments as

$$\mu'_r = \int_0^{\infty} x^r f(x; \alpha, \beta, \lambda) dx$$

$$= \lambda\alpha\beta \sum_{i,j=0}^{\infty} w_{i,j} \int_0^{\infty} x^r (1+x)^{-\alpha-1} (1 - (1+x)^{-\alpha})^{\beta j + \beta - 1} dx$$

$$\mu'_r = \lambda\beta \sum_{i,j,k=0}^{\infty} \pi_{i,j,k} B\left(1 - \frac{r}{\alpha}, \beta(j+1)\right) \text{ for } \alpha > r \qquad (12)$$

where $\pi_{i,j,k} = \frac{\Gamma(2+i)(1-\lambda)^i(-1)^{j+k+r}}{\Gamma(2)\Gamma(i+1)} \binom{i}{j}\binom{r}{k}$ under the conditions $i \geqslant j$ and $r \geqslant k$.

The cumulants (k_s) of the MOEIK distribution are attained from above expression as

$$k_s = \mu'_s - \sum_{k=1}^{s-1} \binom{s-1}{k-1} k_k \mu'_{s-k}, \quad \mu_s = \sum_{k=0}^{s} \binom{s}{k}(-1)^k \mu'^k_1 \mu'_{s-k}$$

where $\mu'_1 = k$. Thus,

$$k_2 = \mu'_2 - (\mu'_1)^2,$$

$$k_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3,$$

$$k_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4$$

The coefficient of skewness and the coefficient of kurtosis for the MOEIK distribution can also obtained from the third and the fourth standardized cumulants by using formulae $\gamma_1 = \frac{k_3}{k_2^{3/2}}$ and $\gamma_2 = \frac{k_4}{k_2^2}$ respectively. Table 1 represents the behavior of raw moments, skewness and kurtosis for MOEIK distribution.

3.2. Incomplete moments

Researchers have general interest to compute the rth incomplete moments of life-time models. The incomplete moments are mostly utilized to explain the Bonferroni and Lorenz curves while it also helps to evaluate the mean deviation. Let X follows the MOEIK distribution and its rth incomplete moment could be evaluated from

$$\varphi(x) = \int_0^x v^r f(v) dv$$

$$= \lambda\alpha\beta \sum_{i,j=0}^{\infty} w_{i,j} \int_0^x v^r (1+v)^{-\alpha-1} (1 - (1+v)^{-\alpha})^{\beta j + \beta - 1} dv$$

Let $(1+v)^{-\alpha} = z$, $v = z^{-\frac{1}{\alpha}} - 1$ and $dv = -\frac{1}{\alpha}z^{-\frac{1}{\alpha}-1}dz$

$$= ck\beta^r \sum_{i,j,k=0}^{\infty} w_{i,j}(-1)^{r+k}\binom{r}{k} \int_{(1+x)^{-\alpha}}^{1} z^{1-\frac{r}{\alpha}-1}(1-z)^{\beta j + \beta - 1} dz$$

$$\varphi(x) = ck\beta^r \sum_{i,j=0}^{\infty} w_{i,j,k} B\left(1 - \frac{r}{\alpha}, \beta(j+1), (1+x)^{-\alpha}\right) \text{ for } \alpha > r \qquad (13)$$

Table 1
Moments, skewness and Kurtosis for X at different parameter combination.

Moments→	μ'_1	μ'_2	μ'_3	μ'_4	Skewness	Kurtosis
$\lambda \downarrow$		$\alpha = 5$	$\beta = 1.5$			
0.25	0.1567	0.0755	0.0985	0.3760	0.4097	65.961
0.75	0.2826	0.1896	0.2825	1.1204	0.7920	31.164
1.50	0.3974	0.3274	0.5398	2.2225	1.2115	20.731
2.50	0.5026	0.4808	0.8609	3.6718	1.6616	15.885
4.50	0.6478	0.7334	1.4549	6.5181	2.3928	12.118
7.50	0.7962	1.0410	2.2696	10.692	3.2844	9.8651
$\beta \downarrow$		$\alpha = 5$	$\lambda = 2.5$			
0.25	0.1643	0.1050	0.1564	0.6219	0.5890	56.385
0.75	0.3418	0.2746	0.4504	1.8523	1.1005	24.559
1.50	0.5026	0.4807	0.8609	3.6718	1.6616	15.886
2.50	0.6459	0.7087	1.3711	6.0602	2.2654	12.066
4.50	0.8370	1.0801	2.3096	10.741	3.2496	9.2058
7.50	1.0266	1.5265	3.5879	17.585	4.4521	7.5467
$\alpha \downarrow$		$\beta = 0.5$	$\lambda = 2.5$			
4.25	0.3281	0.3213	0.8082	9.3416	2.2608	90.457
4.75	0.2836	0.2267	0.4076	2.1659	1.1475	42.108
5.50	0.2356	0.1477	0.1859	0.5050	0.5427	23.146
6.50	0.1922	0.0935	0.0841	0.1374	0.2608	15.702
7.50	0.1623	0.0645	0.0451	0.0527	0.1486	12.661
9.50	0.1237	0.0360	0.0174	0.0129	0.0639	10.005

where $w_{i,j,k} = \sum_{i,j,k=0}^{\infty} w_{i,j}(-1)^{r+k}\binom{r}{k}$ and $r \geqslant k$

3.3. Mean deviation

A most appropriate tool used to evaluate the average absolute deviation of the observation from their mean is mean deviation. We can find the mean deviation about the mean and median from the following expressions given respectively.

$$\delta_\mu(y) = E(|Y - \mu_1'|) = \int_0^\beta |y - \mu_1'|f(y)dy$$
$$= 2\mu_1'F(\mu_1') - 2\varphi_1(\mu_1') \tag{14}$$

$$\delta_M(y) = E(|y - M|) = \int_0^\beta |y - M|f(y)dy = \mu_1' - 2\varphi_1(M) \tag{15}$$

where $\mu_1' = E(Y)$ can be obtained from Eq. (11), $F(\mu_1')$ can be calculated from Eq. (6) and $\varphi_1(\mu_1')$ is the first incomplete moment computed from Eq. (13).

Mean deviation is used in numerous important fields such as economics, demography, insurance, medicines and reliability.

3.4. Mode

For symmetrical distributions such as the standard normal, t-distribution and Cauchy distribution, the mean, median and mode are same say 0. But for the asymmetrical distribution we have to follow the general method with necessary and sufficient conditions to find the mode. The mode of the MOEIK distribution can be obtained by taking the first derivative and equate it to zero.
$f'(x) = 0$.
The density function of the MOEIK in Eq. (6) is given as

$$f(x) = \frac{\lambda\alpha\beta(1+x)^{-\alpha-1}(1-(1+x)^{-\alpha})^{\beta-1}}{[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)]^2}$$

Following the definition of mode

$$f'(x) = \lambda\alpha^2\beta(1+x)^{-2\alpha-2}(1-(1+x)^{-\alpha})^{\beta-2}$$
$$\times \left[\frac{(\beta-1)(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))+2\beta(1-(1+x)^{-\alpha})^\beta}{\{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)\}^3}\right] = 0$$

Since $(1+x)^{-2\alpha-2} > 0$, $(1-(1+x)^{-\alpha})^{\beta-2} > 0$ and $[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)]^3 > 0$, $\alpha, \beta, \theta > 0$ for all $x > 0$. Therefore,

$$(\beta-1)(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)) + 2\beta(1-(1+x)^{-\alpha})^\beta = 0$$

$$x = \left[1 - \left\{\frac{\lambda(\beta+1)}{(\beta-1)(1-\lambda)}\right\}^{\frac{1}{\beta}}\right]^{-\frac{1}{\alpha}} - 1 \tag{16}$$

is the mode of the MOEIK distribution.

3.5. Rényi entropy

Entropy is generally belongs to statistical mechanics, more specifically microscopic variables. Therefore, more precisely we can say that Entropy is the measure of disorder in a microscopic system. A common measure of entropy is the Rényi entropy and has much importance in many fields such as statistical inference, classification, problems identification in statistics, econometrics and pattern recognition in computer sciences. The given theorem provides expression for Rényi entropy.

Theorem 3. *If the random variable X is defined as Eq. (6), then the Rényi entropy is given under the necessary condition that* $\delta > 0$ *and* $\delta \neq 1$

$$I_R(\delta) = \frac{\log \lambda}{1-\delta} - \log\alpha + \frac{\log\beta}{1-\delta} + \frac{1}{1-\delta}$$
$$\times \log\left[\sum_{i,j=0}^{\infty}\varphi_{i,j}B\left(\delta + \frac{\delta}{\alpha} + \frac{1}{\alpha}, \beta(\delta+j) - \delta + 1\right)\right] \tag{17}$$

Proof: If r.v X follows MOEIK distribution then Rényi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta}\log[I(\delta)]$$

where $I(\delta) = \int_0^\infty f^\delta(x)dx$

$$I(\delta) = \int_0^\infty \lambda^\delta\alpha^\delta\beta^\delta(1+x)^{-\alpha\delta-\delta}(1-(1+x)^{-\alpha})^{\beta\delta-\delta}[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)]^{-2\delta}dx$$

After simplification, the expression becomes

$$I(\delta) = \lambda^\delta\alpha^\delta\beta^\delta\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^\infty (1+x)^{-\alpha\delta-\delta}(1-(1+x)^{-\alpha})^{\beta j+\beta\delta-\delta}dx$$

where $\varphi_{i,j} = \frac{\Gamma(i+2\delta)}{\Gamma(2\delta)\Gamma(i+1)}(1-\lambda)^i(-1)^j\binom{i}{j}$

$$I(\delta) = \lambda^\delta\alpha^{\delta-1}\beta^\delta\sum_{i,j=0}^{\infty}\varphi_{i,j}\int_0^1 z^{\delta+\frac{\delta}{\alpha}+\frac{1}{\alpha}-1}(1+z)^{\beta(\delta+j)-\delta+1-1}dy$$

$$I(\delta) = \lambda^\delta\alpha^{\delta-1}\beta^\delta\sum_{i,j=0}^{\infty}\varphi_{i,j}B\left(\delta + \frac{\delta}{\alpha} + \frac{1}{\alpha}, \beta(\delta+j) - \delta + 1\right)$$

Substitution completes the proof. □

3.6. Order statistics

Let the random variables and its ordered values are denoted as $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$. The probability density function (pdf) of order statistics is obtained using the function

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!}[F(x)]^{s-1}[1-F(x)]^{n-s}f(x)$$

The density of the nth ordered statistics follows the MOEIK distribution is

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!}\left[\frac{(1-(1+x)^{-\alpha})^\beta}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)}\right]^{s-1}$$
$$\times \left[1 - \frac{(1-(1+x)^{-\alpha})^\beta}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)}\right]^{n-s}$$
$$\times \frac{\lambda\alpha\beta(1+x)^{-\alpha-1}(1-(1+x)^{-\alpha})^{\beta-1}}{[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)]^2}$$

$$f_{s:n}(x) = \frac{n!\lambda\alpha\beta(1+x)^{-\alpha-1}}{(s-1)!(n-s)!}\sum_{i,j,k=0}^{\infty}\phi_{i,j,k}(1-(1+x)^{-\alpha})^{\beta(s+i+k)-1} \tag{18}$$

where

$$\phi_{i,j,k} = \binom{n-s}{i}\frac{\Gamma(s+i+j+1)}{\Gamma(s+i+1)\Gamma(j+1)}(1-\lambda)^j(-1)^{i+k}\binom{j}{k}$$

The rth moment of order statistic is obtained as

$$E(X^r_{s:n}) = \frac{n!\lambda\alpha\beta}{(s-1)!(n-s)!} \sum_{i,j,k=0}^{\infty} \phi_{i,j,k}$$
$$\times \int_0^\infty x^r (1+x)^{-\alpha-1}(1-(1+x)^{-\alpha})^{\beta(s+i+k)-1} dx$$

On simplifications, we get

$$= \frac{n!\lambda\beta}{(s-1)!(n-s)!} \sum_{i,j,k,m=0}^{\infty} \psi_{i,j,k,m} \int_0^1 z^{1-\frac{m}{\alpha}-1}(1-z)^{\beta(s+i+k)-1} dz$$

where $\psi_{i,j,k,m} = \phi_{i,j,k} \sum_{m=0}^{\infty} (-1)^{m+r} \binom{r}{m}$

The expressions for rth moment of order statistic becomes

$$E(X^r_{s:n}) = \frac{n!\lambda\beta}{(s-1)!(n-s)!} \sum_{i,j,k,m=0}^{\infty} \psi_{i,j,k,m} B\left(1-\frac{m}{\alpha}, \beta(s+i+k)\right) for\ \alpha>m$$
$$(19)$$

3.7. Maximum likelihood estimation

This study adopts the maximum likelihood estimation method so that it is mostly used and provides the maximum information about the properties of the estimated parameters. Moreover, the normal approximation of these estimators can frankly be managed systematically and mathematically for large sample theory. Consequently, the ML estimation has adopted to estimate the unknown parameters (α, β and λ) of the MOEIK distribution.

Let random variable X belong to the observed distribution and have vector of the parameters (α, β, λ)^T with size n. The sample likelihood function is achieved as

$$\prod_{i=1}^{n} f(x;\alpha,\beta,\lambda) = \alpha^n \beta^n \lambda^n \prod_{i=1}^{n} \frac{(1+x)^{-\alpha-1}(1-(1+x)^{-\alpha})^{\beta-1}}{[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^{\beta})]^2}$$

The log-likelihood function is

$$L = n\log\alpha + n\log\beta + n\log\lambda - (\alpha+1)\sum\log(1+x)$$
$$+ (\beta-1)\sum\log[1-(1+x)^{-\alpha}]$$
$$- 2\sum\log[1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^{\beta})]$$
$$(20)$$

Now we have to maximize the log-likelihood function given in Eq. (20) to get the ML estimates of unknown parameters of the Marshall-Olkin extended Kumaraswamy distribution. For this purpose, we take the first derivative of the log-likelihood equation with respect to parameters and equate to zero respectively.

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} - \sum\log(1+x) + (\beta-1)\sum\frac{\log(1+x)(1+x)^{-\alpha}}{1-(1+x)^{-\alpha}}$$
$$- 2\sum\frac{\beta(1-\lambda)\log(1+x)(1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{\beta-1}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^{\beta})}$$
$$(21)$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} + \sum\log(1-(1+x)^{-\alpha})$$
$$- 2\sum\frac{(1-\lambda)\log(1-(1+x)^{-\alpha})(1-(1+x)^{-\alpha})^{\beta}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^{\beta})}$$
$$(22)$$

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} - 2\sum\frac{1-(1-(1+x)^{-\alpha})^{\beta}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^{\beta})}$$
$$(23)$$

The exact solution of the derived ML estimator for unknown parameters in Eqs. (21)–(23) is genuinely not possible. Therefore, it is more appropriate to use the nonlinear optimization algorithms such as a Newton-Raphson algorithm for maximizing the likelihood function numerically. We can use R (optimal function or maxBFGS function), or MATHEMATICA (Maximize function). After application of large sample property of the ML Estimates, MLE $\hat{\theta}$ can be treated as being approximately normal with the mean θ and the variance-covariance matrix equal to the inverse of the expected information matrix, i.e. $\sqrt{n(\hat{\theta}-\theta)} \to N(0, nI^{-1}(\theta))$, $I(\theta)$

Table 2
Summaries of the estimates for the MOEIK distribution.

True parameters			Sample size	Parameters	Summaries of Parameters		
α	β	λ			Mean	Bias	MSE
1.25	2.50	0.75	10	α	1.625	0.375	111.6
				β	2.828	0.328	1.330
				λ	0.777	0.027	0.026
			50	α	1.305	0.055	0.085
				β	2.556	0.056	0.162
				λ	0.755	0.005	0.004
			100	α	1.276	0.026	0.036
				β	2.532	0.032	0.078
				λ	0.753	0.003	0.002
			200	α	1.262	0.012	0.017
				β	2.515	0.015	0.037
				λ	0.751	0.001	0.001
2.5	3	1.5	10	α	6.065	3.565	709.1
				β	3.264	0.264	1.131
				λ	1.556	0.056	0.108
			50	α	2.665	0.165	0.601
				β	3.047	0.047	0.157
				λ	1.510	0.010	0.016
			100	α	2.582	0.082	0.236
				β	3.025	0.025	0.079
				λ	1.505	0.005	0.008
			200	α	2.536	0.036	0.098
				β	3.013	0.013	0.038
				λ	1.503	0.003	0.004

is the information matrix then its inverse of matrix is $I^{-1}(\theta)$ provide the variances and covariance's. The $I(\hat{\theta})$ variance-covariance matrix is actually equal to the inverse of the expected information matrix $I^{-1}(\hat{\theta})$ is given as

$$I(\hat{\theta}) = \begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\lambda} \\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\lambda} \\ J_{\lambda\alpha} & J_{\lambda\beta} & J_{\lambda\lambda} \end{bmatrix} \tag{24}$$

The elements of the variance-covariance information matrix are given in Appendix.

4. Simulation

To inspect the performance of the Marshall-Olkin extended inverted Kumaraswamy distribution, we conduct a simulation study by using the Monte Carlos simulation method with 10,000 repetitions on the basis of the bias and the mean square error of the estimated parameters from the maximum likelihood estimation method. The simulation is done as follows:

Data is generated from $F(x) = u$, where u is uniformly distributed (0, 1).
(1.25, 2.50, 0.75 *and* (2.5, 3, 1.5) are taken as the true parameter values (α, β and *lambda*) .
Simulation is conducted for the sample sizes $n = 10,\ 50,\ 100\ and\ 200$.
The repetition of the experiment is 10,000 times for each sample size.

Table 2 represents the outcomes of the Monte Carlos simulation study. We evaluate the mean of estimated parameters, mean square errors (MSE) and biases. These findings based on the expected first order asymptotic theory as the bias and the MSE's decreases toward zero with the increase in sample size.

The observations in Table 2 direct the MSE of the ML estimators of α, β and λ decreases and their biases decay towards 0 as sample size increases. While the increase in shape parameters, MSE of estimated parameters increases.

5. Applications

This section represents the potentiality and flexibility of the new model as compared to some other existing life-time models by using some real-life examples.

(i) The first real-life data was originally reported by Hinkley (1977). Data consists of 30 observations of the March precipitation (in inches) in Minneapolis/St Paul. The observed values are

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20	3.00	3.09
1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81	2.81	1.87	1.18	1.35
4.75	2.48	0.96	1.89	0.90	2.05						

(ii) The second data also consists of 30 values for the failure time of repairable objects used by Murthy et al. (2004). The observed values are

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74	1.23	0.94
4.36	0.40	1.74	4.73	2.23	0.45	0.70	1.06	1.46	0.30	1.82	2.37
0.63	1.23	1.24	1.97	1.86	1.17						

(iii) The third application is initially used by Bhaumik et al. (2009). The observed values

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6
0.9	0.4	2.0	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1.0	0.2
0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2		

represents vinyl chloride used for monitoring wells in mg/L. Table 3 provides the discriptive statistics for all given data sets.

By using these data sets, we have made comparison for the new generated Marshall-Olkin extended inverted Kumaraswamy (MOEIK) distribution with the Generalized Inverted Kumaraswamy (GIK), Transmuted Exponentiated Inverse Rayleigh (TEIR), Logistic Weibull (LW), Transmuted Power Lindley (TPL), Marshall Olkin Frechet (MOFr) and Inverted Kumaraswamy (IK) distribution. We use numerous goodness of fit measures to compare the new developed model with other existing models such as the log likelihood function ($-2l$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hanna-Quinn Information Criterion (HQIC), Kolmogrov-Smirnov (K-S), Anderson Darling (A*) and Cramer-von Mises (W*). The pdf of other existing models are stated below.

Generalized Inverted Kumaraswamy Distribution (Iqbal et al., 2017)

$$f(x) = \alpha\beta\gamma x^{\gamma-1}(1 + x^{\gamma})^{-(\alpha+1)}[1 - (1 + x^{\gamma})^{-\alpha}]^{\beta-1}$$

where $\alpha > 0,\ \beta > 0,\ \gamma > 0$ are shape parameters.
Transmuted Exponentiated Inverse Rayleigh Distribution (Haq, 2016)

$$f(x) = \frac{2\alpha\theta}{x^3} e^{-\frac{\alpha\theta}{x^2}}\left[1 + \lambda - \lambda e^{\frac{\alpha\theta}{x^2}}\right]$$

where $\alpha > 0$ is shape parameter, $\beta > 0$ is scale parameter and $\lambda \leqslant 1$ is transmuted parameter.
Logistic Weibull Distribution (Tahir et al., 2016)

$$f(x) = \frac{\lambda\alpha}{\beta^{\lambda}} x^{-(\lambda\alpha+1)}(1 + (\beta x^{\alpha})^{-\lambda})^{-2}$$

where $\alpha > 0,\ \beta > 0,\ \lambda > 0$ are shape parameters.
Transmuted Power Lindley Distribution (Granzotto et al., 2016)

$$f(x) = \frac{\alpha\theta^2}{(1+\theta)}(1 + x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}\left[1 + \lambda - 2\lambda\left(1 - \left(1 + \frac{\theta x^{\alpha}}{1+\theta}\right)e^{-\theta x^{\alpha}}\right)\right]$$

where $\alpha > 0,\ \beta > 0\ and\ \lambda \leqslant 1$.
Marshall Olkin Fréchet Distribution (Krishna et al., 2013)

$$f(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-(\frac{\sigma}{x})^{\beta}}}{\left(\alpha + (1-\alpha)e^{-(\frac{\sigma}{x})^{\beta}}\right)^2}$$

where $\alpha > 0, \beta > 0$ are positive shape parameters while $\sigma > 0$ is scale parameter.

Tables 4–6 provide the estimated parameters and goodness of fit measures of three real life data sets. It is obvious from the tables that the goodness of fit measures such as *AIC, BIC, CAIC, HQIC, K-S, A** and *W** of the new developed MOEIK distribution are less than that of the *GIK, TEIR, LW* and *IK* distribution, therefore, considered as a best fitted model. Figs. 2, 3 and 4 show the histogram of data with estimated pdf curves (left side) and also explain the estimated and the empirical cdf curves (right side). It is clear from the above tables and figures that the MOEIK distribution provides better fit as compared to other existing models.

Table 3
Descriptive statistics for Data 1, Data 2 and Data 3.

	Min.	Q_1	Median	Q_3	Mean	Max.
Data 1	0.320	0.915	1.470	2.088	1.675	4.750
Data 2	0.110	0.717	1.235	1.943	1.543	4.730
Data 3	0.100	0.500	1.150	2.475	1.879	8.000

Table 4
ML estimates for parameters with the goodness of fit measures (March precipitation).

Models	Estimates (Std. Error)			$-2l$	AIC	BIC	CAIC	HQIC	K-S	A^*	W^*
MOEIK (α, β, λ)	**4.3228 (0.9387)**	**6.5798 (5.0017)**	**6.9226 (9.6043)**	**76.69**	**82.69**	**86.89**	**84.79**	**84.03**	**0.068**	**0.137**	**0.019**
GIK (α, β, γ)	1.9552 (1.8117)	3.9501 (5.6951)	1.4202 (1.0211)	78.63	84.63	88.84	86.77	85.98	0.110	0.318	0.052
TEIR $(\alpha, \theta, \lambda)$	6.5630 (80.152)	0.0958 (1.1693)	−0.6700 (0.2661)	84.20	90.20	94.41	92.30	91.55	0.182	1.139	0.212
LW (α, β, λ)	2.7709 (243.47)	0.3635 (32.322)	1.0061 (88.405)	77.86	83.86	88.06	85.96	85.21	0.074	0.191	0.025
TPL $(\alpha, \theta, \lambda)$	1.5965 (0.2054)	0.4801 (0.1643)	0.5812 (0.6548)	77.30	83.31	89.05	86.18	86.19	0.091	0.146	0.029
MOFr (α, β, σ)	42.598 (2.8490)	2.6975 (0.4482)	0.3548 (0.2109)	77.60	83.60	87.80	85.70	84.94	0.095	0.224	0.024
IK (α, β)	2.9872 (0.4730)	8.5899 (3.1222)	–	78.85	83.85	87.65	85.25	84.74	0.114	0.338	0.055

These bold values highlight the model which have better fit as compared to competitor models.

Table 5
ML estimates of the parameters with the goodness of fit measures (failure time data).

Models	Estimates (Std. Error)			$-2l$	AIC	BIC	CAIC	HQIC	K-S	A^*	W^*
MOEIK (α, β, λ)	**3.8031 (0.8166)**	**2.1861 (2.0393)**	**9.6185 (14.059)**	**79.60**	**85.60**	**89.80**	**87.70**	**86.94**	**0.078**	**0.119**	**0.017**
GIK (α, β, γ)	1.1623 (1.0228)	1.4462 (1.3918)	1.8481 (1.1813)	81.64	87.64	91.84	89.74	88.98	0.109	0.284	0.048
TEIR $(\alpha, \theta, \lambda)$	7.2940 (25.055)	0.0254 (0.0871)	−0.8803 (0.1142)	117.0	123.0	127.2	125.1	124.3	0.376	1.366	1.471
LW (α, β, λ)	2.8927 (293.52)	0.5531 (33.237)	0.7775 (78.895)	80.83	86.83	91.03	88.93	88.17	0.083	0.177	0.023
TPL $(\alpha, \theta, \lambda)$	1.3380 (0.1743)	0.6301 (0.1927)	0.5356 (0.5587)	80.73	86.73	90.93	88.83	88.07	0.089	0.180	0.025
MOFr (α, β, σ)	63.165 (6.7032)	2.1070 (0.3087)	0.1669 (0.0608)	81.53	87.53	91.73	89.63	88.87	0.104	0.283	0.043
IK (α, β)	2.4609 (0.4214)	4.1716 (1.2783)	–	82.48	86.48	90.28	88.88	87.37	0.111	0.340	0.052

These bold values highlight the model which have better fit as compared to competitor models.

Table 6
ML estimates of the parameters with the goodness of fit measures (vinyl chloride data).

Models	Estimates (Std. Error)			$-2l$	AIC	BIC	CAIC	HQIC	K-S	A^*	W^*
MOEIK (α, β, λ)	**2.1193 (0.5717)**	**1.7026 (0.8020)**	**2.2471 (2.3570)**	**110.9**	**116.9**	**121.5**	**119.2**	**118.5**	**0.082**	**0.233**	**0.035**
GIK (α, β, γ)	2.0236 (1.8839)	2.6369 (3.8684)	0.8858 (0.6642)	111.5	117.5	122.1	119.8	119.1	0.095	0.281	0.039
TEIR $(\alpha, \theta, \lambda)$	6.9784 (12.604)	0.0138 (0.0247)	−0.7798 (0.1350)	170.8	176.8	181.4	179.1	178.3	0.429	18.01	2.294
LW (α, β, λ)	2.7709 (193.53)	0.7056 (17.189)	0.5526 (38.601)	111.9	117.9	122.5	120.2	119.5	0.085	6.327	0.633
TPL $(\alpha, \theta, \lambda)$	0.9265 (0.1156)	0.7453 (0.2414)	0.4094 (0.5958)	111.2	117.2	122.5	120.5	119.7	0.091	0.280	0.048
MOFr (α, β, σ)	29.053 (2.3853)	1.4730 (0.2397)	0.1124 (0.1098)	111.4	117.4	121.9	119.7	119.3	0.089	0.289	0.043
IK (α, β)	1.7409 (0.3238)	2.1058 (0.5373)	–	112.5	117.5	122.6	120.1	119.6	0.096	0.279	0.039

These bold values highlight the model which have better fit as compared to competitor models.
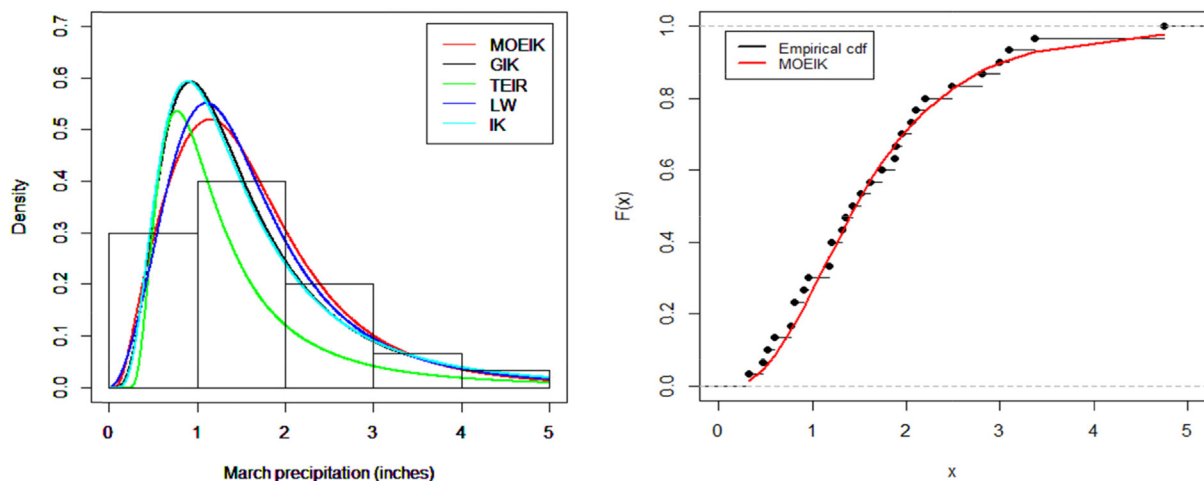

Fig. 2. The fitted densities (left) and estimated cdf (right) on Data set 1.
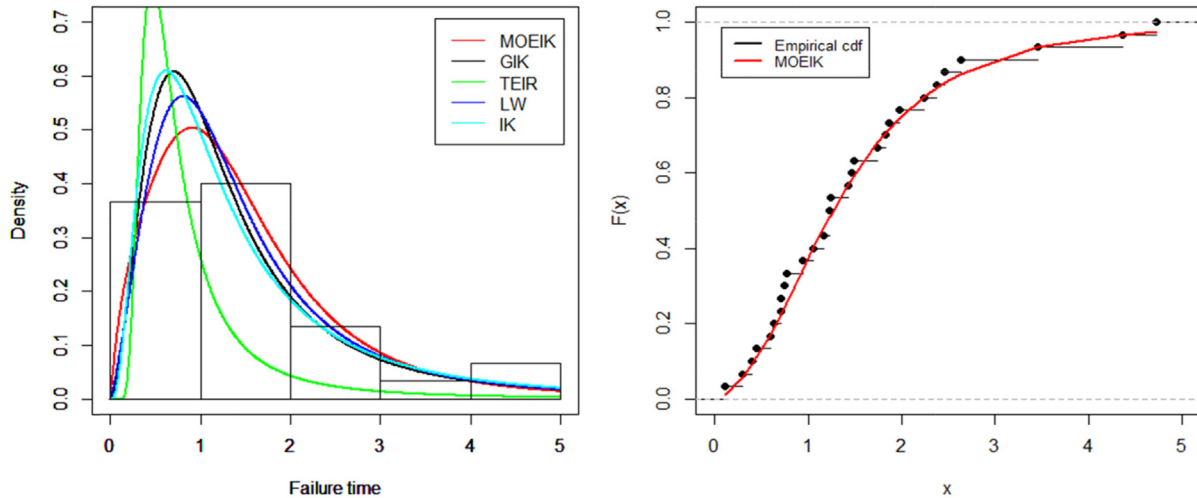
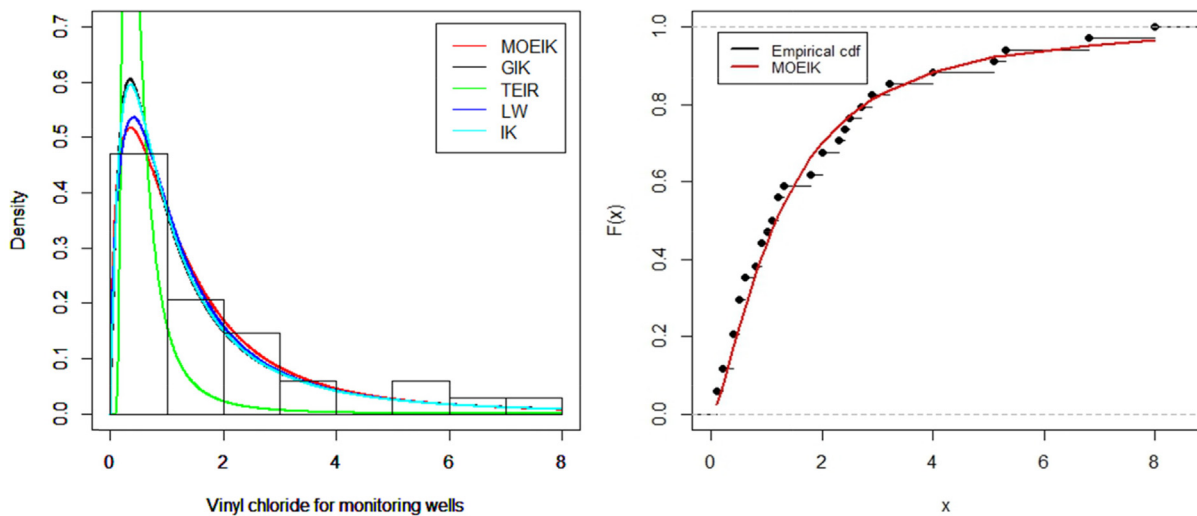**Fig. 3.** The fitted densities (left) and estimated cdf (right) on Data set 2.

Fig. 4. The fitted densities (left) and estimated cdf (right) on Data set 3.

6. Conclusion

The study reveals a generalization of the Inverted Kumaraswamy distribution and elaborates explicit expression for some of its major properties. The study also explains the behavior of the estimated parameters by using the Monte Carlos simulation approach. Three real-life applications have also presented for the demonstration of enhanced flexibility and better fit of the observed model as compared to the other considered existing models.

Appendix

$$
J_{\alpha\alpha} = -\frac{n}{\alpha^2} - (\beta - 1) \sum \left(\frac{\text{Log}[1+x]^2 (1+x)^{-2\alpha}}{(1 - (1+x)^{-\alpha})^2} + \frac{\text{Log}[1+x]^2 (1+x)^{-\alpha}}{1 - (1+x)^{-\alpha}} \right) + 2 \sum \left(\frac{\beta^2 (1-\lambda)^2 \text{Log}[1+x]^2 (1+x)^{-2\alpha} (1 - (1+x)^{-\alpha})^{2\beta-2}}{(1 - (1-\lambda)(1 - (1 - (1+x)^{-\alpha})^{\beta}))^2} \right.
$$

$$
\left. - \frac{\beta(\beta-1)(1-\lambda)\text{Log}[1+x]^2 (1+x)^{-2\alpha} (1 - (1+x)^{-\alpha})^{\beta-2}}{1 - (1-\lambda)(1 - (1 - (1+x)^{-\alpha})^{\beta})} + \frac{\beta(1-\lambda)\text{Log}[1+x]^2 (1+x)^{-\alpha} (1 - (1+x)^{-\alpha})^{\beta-1}}{1 - (1-\lambda)(1 - (1 - (1+x)^{-\alpha})^{\beta})} \right)
$$

$$J_{\alpha\beta} = \sum \frac{\text{Log}[1+x](1+x)^{-\alpha}}{1-(1+x)^{-\alpha}} - 2\sum \left(-\frac{\beta(1-\lambda)^2\text{Log}[1+x]\text{Log}[1-(1+x)^{-\alpha}](1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{2\beta-1}}{(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))^2} \right.$$

$$\left. + \frac{(1-\lambda)\text{Log}[1+x](1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{\beta-1}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)} + \frac{\beta(1-\lambda)\text{Log}[1+x]\text{Log}[1-(1+x)^{-\alpha}](1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{-1+\beta}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)} \right)$$

$$J_{\alpha\lambda} = 2\sum \left(\frac{\beta(1-\lambda)\text{Log}[1+x](1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{\beta-1}(1-(1-(1+x)^{-\alpha})^\beta)}{(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))^2} + \frac{\beta\text{Log}[1+x](1+x)^{-\alpha}(1-(1+x)^{-\alpha})^{\beta-1}}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)} \right)$$

$$J_{\beta\beta} = -\frac{n}{\beta^2} + 2\sum \left(\frac{(1-\lambda)^2\text{Log}[1-(1+x)^{-\alpha}]^2(1-(1+x)^{-\alpha})^{2\beta}}{(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))^2} - \frac{(1-\lambda)\text{Log}[1-(1+x)^{-\alpha}]^2(1-(1+x)^{-\alpha})^\beta}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)} \right)$$

$$J_{\beta\lambda} = 2\sum \left(\frac{(1-\lambda)\text{Log}[1-(1+x)^{-\alpha}](1-(1+x)^{-\alpha})^\beta(1-(1-(1+x)^{-\alpha})^\beta)}{(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))^2} + \frac{\text{Log}[1-(1+x)^{-\alpha}](1-(1+x)^{-\alpha})^\beta}{1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta)} \right)$$

$$J_{\lambda\lambda} = -\frac{n}{\lambda^2} + 2\sum \frac{(1-(1-(1+x)^{-\alpha})^\beta)^2}{(1-(1-\lambda)(1-(1-(1+x)^{-\alpha})^\beta))^2}$$

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