Journal of King Saud University (Science) (2011) **23**, 111–114

King Saud University
Journal of King Saud University (Science)
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ORIGINAL ARTICLE

Solving two-dimensional integral equations

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Received 14 June 2010; accepted 25 June 2010
Available online 30 June 2010

KEYWORDS
Homotopy analysis method; two-dimensional integral equations

Abstract This paper applies the homotopy analysis method proposed by Liao to obtain approximate analytic solutions for integral equations of two-dimensional. Some examples are presented to show the ability of the method for integral equations of two-dimensional. The results reveal that the method is very effective and simple.

1. Introduction

Homotopy analysis method (HAM), first proposed by Liao (1992). HAM properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problem. Liao, in his book (Liao, 2003a), proves that this method is a generalization of some previously used techniques such as d-expansion method, artificial small parameter method and ADM. This method has proven to be very effective and result in considerable saving in computation time (Liao, 1995, 2003b,c, 2004; Liao and Chwang, 1998; Sami et al., 2008; Abbasbandy, 2007).

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doi:10.1016/j.jksus.2010.06.017

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2. Analysis of the method

Consider the following linear integral equation of two-dimensional

$$u(x,y) = f(x,y) + \int_c^y \int_a^b k(x,y,s,t)u(s,t)dsdt.$$

To illustrate the homotopy analysis method, we consider

$$N[u(x,y)] = u(x,y) - f(x,y) - \int_c^y \int_a^b k(x,y,s,t)u(s,t)dsdt = 0, \quad (1)$$

where $u(x,y)$ is an unknown function, respectively. By means of generalizing the traditional homotopy method, Liao (2003a) constructs the so-called zeroth-order deformation equation

$$(1-p)L(\varphi(r,t;p) - u_0(x,y)) = p\hbar H(r,t)N(r,t;p), \quad (2)$$

where $p \in [0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter, $H(x,y) \neq 0$ is nonzero auxiliary function, L is an auxiliary linear operator, $u_0(x,y) = f(x,y)$ and $u(x,y;p)$ is a unknown function, respectively It is important that one has great freedom to choose auxiliary parameter $\hbar$ in HAM. If $p = 0$ and 1, it holds

$$\varphi(x,y;0)=u_0(x,y)=f(x,y), \qquad \varphi(x,y;1)=u(x,y). \tag{3}$$

Thus, as p increases from 0 to 1, the solution $\varphi(r,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\varphi(r,t;p)$, in Taylor series with respect to p, we have

$$\varphi(x,y;p)=f(x,y)+\sum_{m=1}^{\infty}u_m(x,y)p^m, \tag{4}$$

where

$$u_m(x,y)=\frac{1}{m!}\frac{\partial^m\varphi(x,y;p)}{\partial p^m}\bigg|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p=1$, then we have

$$u(x,y)=f(x,y)+\sum_{m=1}^{\infty}u_m(r,t). \tag{6}$$

Define the vector $\vec{u}_n=\{u_0=f,u_1,\ldots u_n\}$. Differentiating Eq. (2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation

$$L[u_m-\chi_m u_{m-1}]=\hbar H(x,y)R_m(\vec{u}_{m-1}), \tag{7}$$

where

$$R_m(\vec{u}_m-1)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N(x,y;p)}{\partial p^{m-1}}, \tag{8}$$

and

$$\chi_m=\begin{cases}0 & m\leqslant 1,\\ 1 & m>1.\end{cases} \tag{9}$$

Applying L^{-1} on both side of Eq. (7), we get

$$u_m(x,y)=\chi_m u_{m-1}(x,y)+\hbar L^{-1}[H(x,y)R_m(\vec{u}_{m-1})]. \tag{10}$$

In this way, it is easily to obtain u_m form $m\geqslant 1$, at Mth-order, we have

$$u(x,y)=\sum_{m=0}^{M}u_m(x,y). \tag{11}$$

When $M\to\infty$ we get an accurate approximation of the original Eq. (1). For the convergence of the above method we refer the reader to Liao (2003a).

3. Numerical example

Example 1. Consider the following linear Volterra–Fredholm equation

$$u(x,y)=f(x,y)+\int_{-1}^{y}\int_{-1}^{1}xys^2t^2u(s,t)dsdt, \tag{12}$$

where

$$f(x,y)=x^2-\frac{13}{15}xy-\frac{2}{15}xy^4,$$

with the exact solution $u(x,y)=x^2+xy$.

To solve the Eq. (12) by means of homotopy analysis method we choose

$$\varphi_0(x,y)=f(x,y). \tag{13}$$

We now define a nonlinear operator as

$$N[u(x,y)]=\varphi(x,y;p)-f(x,y)-\int_{-1}^{y}\int_{-1}^{1}\varphi(x,y;p)dsdt.$$

Using above definition, with assumption $H(x,t)=1$. We construct the zeroth-order deformation equation

$$(1-p)L(\varphi(x,t;p)-u_0(x,t))=p\hbar N(\varphi(x,t;p)),$$

obviously, when $p=0$ and 1,

$$\varphi(x,t;0)=u_0(x,t), \quad \varphi(x,t;0)=u(x,t).$$

Thus, we obtain the mth-order deformation equations

$$L[u_m-\chi_m u_{m-1}]=\hbar R_m(\vec{u}_{m-1}), \tag{14}$$

where

$$R_m(\vec{u}_{m-1})=u_{m-1}(x,y)-\int_{-1}^{y}\int_{-1}^{1}xys^2t^2u_{m-1}(s,t)dsdt.$$

Now, the solution of the mth-order order deformation Eq. (14)

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{15}$$

Finally, we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t).$$

From Eqs. (13) and (15) and subject to initial condition

$$u_m(x,0)=0, \quad m\geqslant 1.$$

We obtain

$$u_0(x,y)=x^2-\frac{13}{15}xy-\frac{2}{15}xy^4,$$
$$u_1(x,y)=-\hbar\frac{2}{15}xy(y^3+1),$$
$$u_2(x,y)=0.$$

And by repeating this approach, we obtain

$$u_3(x,t)=u_4(x,t)=\cdots=0.$$

When $\hbar=-1$ we have

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$$
$$=x^2-\frac{13}{15}xy-\frac{2}{15}xy^4+\frac{2}{15}xy(y^3+1)+0+0+\cdots,$$

$$u(x,y)=x^2+xy,$$

which is an exact solution.

Example 2. Consider the following Volterra–Fredholm equation

$$u(x,y)=f(x,y)+\int_{0}^{y}\int_{-1}^{1}x^2e^{-s}u(s,t)dsdt, \tag{16}$$

where

$$f(x,y)=y^2e^x-\frac{2}{3}x^2y^3,$$

with the exact solution is $u(x,y)=y^2e^x$.

To solve Eq. (16) by means of homotopy analysis method, we have

$$\varphi_0(x,y) = f(x,y), \tag{17}$$

We now define a nonlinear operator as

$$N[u(x,y)] = \varphi(x,y;p) - f(x,y) - \int_0^y \int_{-1}^1 x^2 e^{-s} u(s,t) ds dt.$$

Using above definition, with assumption $H(x,t) = 1$. We construct the zeroth-order deformation equations

$$(1-p)L(\varphi(x,t;p) - u_0(x,t)) = p\hbar N(\varphi(x,t;p)),$$

obviously, when $p = 0$ and 1,

$$\varphi(x,t;0) = u_0(x,t), \quad \varphi(x,t;0) = u(x,t).$$

Thus, we obtain the mth-order deformation equations

$$L[u_m - \chi_m u_{m-1}] = \hbar R_m(\vec{u}_{m-1}), \tag{18}$$

where

$$R_m(\vec{u}_{m-1}) = u_{m-1}(x,y) - \int_{-1}^y \int_{-1}^1 x^2 e^{-s} u(s,t) ds dt.$$

Now, the solution of the mth-order order deformation Eq. (16)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{19}$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From Eqs. (17) and (18) and subject to initial condition

$$u_m(x,0) = 0, \quad m \geqslant 1.$$

We obtain

$$u_0(x,y) = y^2 e^x - \frac{2}{3} x^2 y^3,$$

$$u_1(x,y) = \hbar \frac{1}{6} y^4 x^2 e - \hbar \frac{5}{6} y^4 x^2 e^{-1} - \hbar \frac{2}{3} x^2 y^3,$$

$$\vdots$$

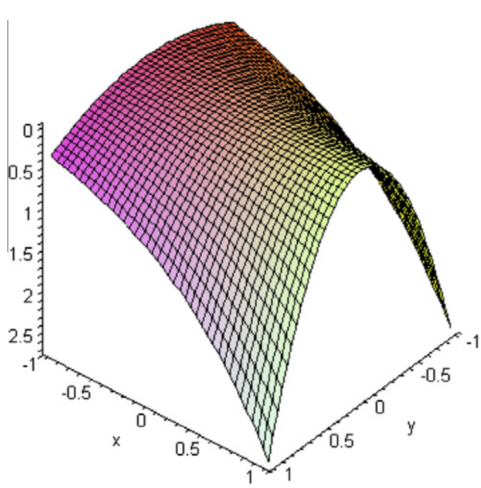

Figure 1 HAM solution for Example 2.

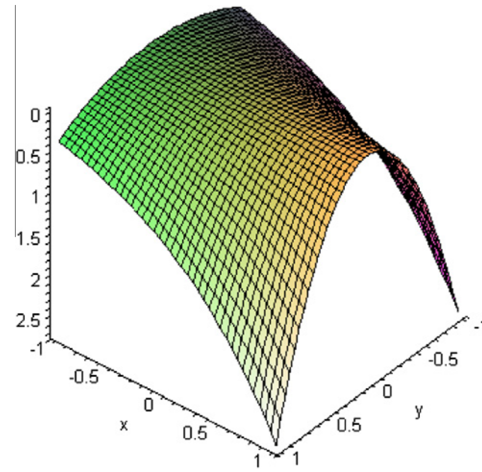

Figure 2 Exact solution for Example 2.

When $\hbar = -1$, we have

$$u = \sum_{i=0}^{\infty} u_i$$
$$= y^2 e^x - \frac{2}{3} x^2 y^3 - \frac{1}{6} y^4 x^2 e + \frac{5}{6} y^4 x^2 e^{-1} + \frac{2}{3} x^2 y^3 + \frac{1}{3} y^5 x^2 - \frac{5}{6} y^5 x^2 e^{-2} - \frac{1}{30} y^5 x^2 e^2 - \frac{5}{6} y^4 x^2 e^{-1} + \frac{1}{6} y^4 x^2 e + \cdots.$$

The exact solution and HAM results are shown in Figs. 1 and 2, respectively.

4. Conclusion

In this work we applied homotopy analysis method for solving integral equations of two-dimensional. The approximate solutions obtained by the homotopy analysis method are compared with exact solutions. It can be concluded that the homotopy analysis method is very powerful and efficient technique in finding exact solutions for wide classes of problems. In our work, we use the MAPLE 11 package to carry the computations.

Acknowledgement

With special thank full from Azad university of Gonbad Kavos Branch that this article is resulting from research project Homotopy analysis method for integral equations of mixed type published in this university.

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