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Review

m-polar fuzzy q-ideals in BCI-algebras

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A B S T R A C T

In a BCI-algebra, the notion of m-polar $(\in,\in)$-fuzzy q-ideal is introduced, and its properties are investigated. Relations between m-polar $(\in,\in)$-fuzzy q-ideal and m-polar fuzzy ideal/subalgebra are discussed. Characterizations of m-polar $(\in,\in)$-fuzzy q-ideal are considered. The extension property about the m-polar $(\in,\in)$-fuzzy q-ideal is established. Homomorphic image and preimage of m-polar $(\in,\in)$-fuzzy q-ideal are discussed. Characterizations of a quasi-associative BCI-algebras are provided by using m-polar $(\in,\in)$-fuzzy q-ideal.

Contents

1. Introduction

Fuzzy sets, which were introduced by Zadeh (1965), deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. After the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields such as statistics, graph theory, medical and life science, engineering, business and social science, computer network, decision making, artificial intelligence, pattern recognition, robotics, and automata theory (see Kumar Singh, 2018; Feng et al., 2019; Irfan Ali et al., 2019; Irfan Ali, 2018). BCK/BCI-algebras, which are created from two distinct approaches: set theory and proposition calculus, first appeared in the mathematical literature in 1966 (see Imai and Iski, 1966; Iski, 1966). BCK and BCI algebras describe fragments of the propositional calculus involving implication known as BCK and BCI logics. The various attributes of BCK/BCI-algebras and their applications to different

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aspects are considered in Borzooei et al. (2020), Huang (2006), Meng and Jun (1994), Moussaei et al. (2018), Mohseni Takallo et al. (2019), Muhiuddin and Jun (2019, 2018), Muhiuddin and Al-roqi (2016, 2014), Muhiuddin and Aldhafeeri (2018, 2019) and Muhiuddin et al. (2014, 2017). Ideal theory in BCI-algebras, in particular q-ideal, is studied in Liu et al. (2000). As an extension of fuzzy set, Zhang (1994) introduced the notion of bipolar fuzzy sets. Bipolar fuzzy information is applied in many (algebraic) structures, for instance, Γ-semihypergroups (see Yaqoob et al., 2014), finite state machines (see Jun and Kavikumar, 2011; Subramaniyan and Rajasekar, 2012; Yang, 2014,), (ordered) semigroups (see Arulmozhi et al., 2019; Chinnadurai and Arulmozhi, 2018; Ibrar et al., 2019; Sardar et al., 2012), KU-algebras (see Muhiuddin, 2014), (hyper) BCK/BCI-algebras (see Al-Kadi and Muhiuddin, 2020; Al-Masarwah and Ahmad, 2018; Jun et al., 2012, 2011, 2009a,b; Lee, 2009; Muhiuddin et al., 2020). In many real problems, information sometimes comes from multi-factors and there are many multi-attribute data that cannot be processed using existing anomalies (e.g., fuzzy anomalies and bipolar fuzzy anomalies, etc.). In 2014, Chen et al. (Chen et al., 2014) introduced an m-polar fuzzy set which is an extension of bipolar fuzzy set. The m-polar fuzzy models provide more precision, flexibility, and compatibility to the system when more than one agreements are to be dealt with. The m-polar fuzzy set applied to decision making problem, graph theory and BCK/BCI-algebra (Akram et al., 2019; Al-Masarwah and Ahmad, 2019; Kumar Singh, 2018; Sarwar and Akram, 2017; Al-Masarwah and Ahmad, 2019).

In this paper, we introduce the notion of m-polar $(\in,\in)$-fuzzy q-ideal in BCI-algebra, and investigated its properties. We discuss relations between m-polar $(\in,\in)$-fuzzy q-ideal and m-polar fuzzy ideal/subalgebra, and consider characterizations of m-polar $(\in,\in)$-fuzzy q-ideal. We establish the extension property about the m-polar $(\in,\in)$-fuzzy q-ideal. We discuss homomorphic image and preimage of m-polar $(\in,\in)$-fuzzy q-ideal, and provide characterizations of a quasi-associative BCI-algebras are provided by using m-polar $(\in,\in)$-fuzzy q-ideal.

2. Preliminaries

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall u,w,v\in X)\ (((u*w)*(u*v))*(v*w)=0)$,
(II) $(\forall u,w\in X)\ ((u*(u*w))*w=0)$,
(III) $(\forall u\in X)\ (u*u=0)$,
(IV) $(\forall u,w\in X)\ (u*w=0,\ w*u=0\Rightarrow u=w)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(V) $(\forall u\in X)\ (0*u=0)$,

then X is called a *BCK-algebra*. A BCI-algebra X is said to be quasi-associative if

$$(\forall u,w,v\in X)((u*w)*v\leqslant u*(w*v)). \tag{2.1}$$

Lemma 2.1. (Huang, 2006) *A BCI-algebra X is quasi-associative if and only if $0*x=0*(0*x)$ for all $x\in X$.*

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall u\in X)(u*0=u), \tag{2.2}$$

$$(\forall u,w,v\in X)(u\leqslant w\Rightarrow u*v\leqslant w*v,\ v*w\leqslant v*u), \tag{2.3}$$

$$(\forall u,w,v\in X)((u*w)*v=(u*v)*w) \tag{2.4}$$

where $u\leqslant w$ if and only if $u*w=0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $u*w\in S$ for all $u,w\in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0\in I, \tag{2.5}$$
$$(\forall u\in X)(\forall w\in I)(u*w\in I\Rightarrow u\in I). \tag{2.6}$$

A subset I of a BCI-algebra X is called

- a *p-ideal* of X if it satisfies (2.5) and
$$(\forall u,w,v\in X)((u*v)*(w*v)\in I,\ w\in I\Rightarrow u\in I). \tag{2.7}$$
- a *q-ideal* of X if it satisfies (2.5) and
$$(\forall u,w,v\in X)(u*(w*v)\in I,\ w\in I\ \Rightarrow u*v\in I). \tag{2.8}$$

See the books Huang (2006) and Meng and Jun (1994) for more information on BCK/BCI-algeebras.

By an *m-polar fuzzy set* on a set X (see Chen et al., 2014), we mean a function $\hat{\alpha}:X\to[0,1]^m$. The membership value of every element $x\in X$ is denoted by

$$\hat{\alpha}(x)=((\pi_1\circ\hat{\alpha})(x),(\pi_2\circ\hat{\alpha})(x),\ldots,(\pi_m\circ\hat{\alpha})(x)),$$

where $\pi_i:[0,1]^m\to[0,1]$ is the i-th projection for all $i=1,2,\ldots,m$.

Given an m-polar fuzzy set on a set X, we consider the set

$$U(\hat{\alpha};\hat{t}):=\{x\in X|\hat{\alpha}(x)\geqslant\hat{t}\}, \tag{2.9}$$

that is,

$$U(\hat{\alpha};\hat{t}):=\{x\in X|(\pi_1\circ\hat{\alpha})(x)\geqslant t_1,(\pi_2\circ\hat{\alpha})(x)\geqslant t_2,\ldots,(\pi_m\circ\hat{\alpha})(x)\geqslant t_m\}, \tag{2.10}$$

which is called an *m-polar level set* of $\hat{\alpha}$.

By an *m-polar fuzzy point* on a set X, we mean an m-polar fuzzy set $\hat{\alpha}$ on X of the form

$$\hat{\alpha}(y)=\begin{cases}\hat{r}=(r_1,r_2,\ldots,r_m)\in(0,1]^m & \text{if } y=x,\\ \hat{0}=(0,0,\ldots,0) & \text{if } y\neq x,\end{cases} \tag{2.11}$$

and it is denoted by $x_{\hat{r}}$. We say that x is the *support* of $x_{\hat{r}}$ and $\hat{r}$ is the *value* of $x_{\hat{r}}$.

We say that an m-polar fuzzy point $x_{\hat{r}}$ is contained in an m-polar fuzzy set $\hat{\alpha}$, denoted by $x_{\hat{r}}\in\hat{\alpha}$, if $\hat{\alpha}(x)\geqslant\hat{r}$, that is, $(\pi_i\circ\hat{\alpha})(x)\geqslant r_i$ for all $i=1,2,\ldots,m$.

Definition 2.2. (Al-Masarwah and Ahmad, 2019, Definition 3.1) An m-polar fuzzy set $\hat{\alpha}$ on a BCK/BCI-algebra X is called an *m-polar fuzzy subalgebra* of X if the following condition is valid.

$$(\forall x,y\in X)(\hat{\alpha}(x*y)\geqslant\inf\{\hat{\alpha}(x),\hat{\alpha}(y)\}), \tag{2.12}$$

that is,

$$(\forall x,y\in X)((\pi_i\circ\hat{\alpha})(x*y)\geqslant\inf\{(\pi_i\circ\hat{\alpha})(x),(\pi_i\circ\hat{\alpha})(y)\}) \tag{2.13}$$

for all $i=1,2,\ldots,m$.

Example 2.3. Let $X=\{0,a,b,c\}$ be a BCK-algebra with a Cayley table which is appeared in Table 1.

Define a 4-polar fuzzy set $\hat{\alpha}$ on X as follows;

$$\hat{\alpha}:X\to[0,1]^4,\ x\mapsto\begin{cases}(0.33,0.41,0.57,0.83) & \text{if } x=0,\\ (0.25,0.33,0.42,0.38) & \text{if } x=a,\\ (0.22,0.30,0.40,0.20) & \text{if } x=b,\\ (0.25,0.34,0.55,0.50) & \text{if } x=c\end{cases}$$

It is routine to check that $\hat{\alpha}$ is a 4-polar fuzzy subalgebra of X.

Table 1
Cayley table for the binary operation “$*$”.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table 2
Cayley table for the binary operation “$*$”.

$*$	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

Definition 2.4. (Al-Masarwah and Ahmad, 2019, Definition 3.7) An m-polar fuzzy set $\hat{\alpha}$ on a BCK/BCI-algebra X is called an *m-polar fuzzy ideal* of X if the following conditions are valid.

$$(\forall x \in X)(\hat{\alpha}(0) \geqslant \hat{\alpha}(x)), \tag{2.14}$$

$$(\forall x, y \in X)(\hat{\alpha}(x) \geqslant \inf\{\hat{\alpha}(x * y), \hat{\alpha}(y)\}), \tag{2.15}$$

that is,

$$(\forall x \in X)((\pi_i \circ \hat{\alpha})(0) \geqslant (\pi_i \circ \hat{\alpha})(x)), \tag{2.16}$$

$$(\forall x, y \in X)((\pi_i \circ \hat{\alpha})(x) \geqslant \inf\{(\pi_i \circ \hat{\alpha})(x * y), (\pi_i \circ \hat{\alpha})(y)\}) \tag{2.17}$$

for all $i = 1, 2, \ldots, m$.

Example 2.5. Let $X = \{0, a, b, c, d\}$ be a BCI-algebra with a Cayley table which is appeared in Table 2.

Define a 4-polar fuzzy set $\hat{\alpha}$ on X as follows;

$$\hat{\alpha} : X \to [0,1]^4,\ x \mapsto \begin{cases} (0.50, 0.60, 0.60, 0.70) & \text{if } x = 0, \\ (0.40, 0.50, 0.50, 0.70) & \text{if } x = a, \\ (0.20, 0.30, 0.30, 0.20) & \text{if } x = b, d, \\ (0.30, 0.40, 0.40, 0.50) & \text{if } x = c \end{cases}$$

It is routine to check that $\hat{\alpha}$ is a 4-polar fuzzy ideal of X.

Lemma 2.6 Mohseni Takallo et al., 2019, Lemma 1. *An m-polar fuzzy set $\hat{\alpha}$ on a BCK/BCI-algebra X is an m-polar fuzzy ideal of X if and only if the following conditions are valid.*

$$(\forall x \in X)(\forall \hat{r} \in [0,1]^m)(x_{\hat{r}} \in \hat{\alpha} \Rightarrow 0_{\hat{r}} \in \hat{\alpha}), \tag{2.18}$$

$$(\forall x, y \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m)\Big((x * y)_{\hat{r}} \in \hat{\alpha},\ y_{\hat{t}} \in \hat{\alpha} \Rightarrow x_{\inf\{\hat{r}, \hat{t}\}} \in \hat{\alpha}\Big). \tag{2.19}$$

Definition 2.7. An m-polar fuzzy set $\hat{\alpha}$ on a BCI-algebra X is called an *m-polar* $(\in, \in)$*-fuzzy p-ideal* of X if it satisfies (2.18) and

$$(\forall x, y, z \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m)\Big(((x * z) * (y * z))_{\hat{r}} \in \hat{\alpha},\ y_{\hat{t}} \in \hat{\alpha} \Rightarrow x_{\inf\{\hat{r}, \hat{t}\}} \in \hat{\alpha}\Big). \tag{2.20}$$

Note that the condition (2.20) is equivalent to the following condition.

Table 3
Cayley table for the binary operation “$*$”.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$$(\forall x, y, z \in X)(\hat{\alpha}(x) \geqslant \inf\{\hat{\alpha}((x * z) * (y * z)), \hat{\alpha}(y)\}), \tag{2.21}$$

that is,

$$(\pi_i \circ \hat{\alpha})(x) \geqslant \inf\{(\pi_i \circ \hat{\alpha})((x * z) * (y * z)), (\pi_i \circ \hat{\alpha})(y)\} \tag{2.22}$$

for all $x, y, z \in X$ and $i = 1, 2, \ldots, m$.

Example 2.8. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation $*$ which is given in Table 3.

Then X is a BCI-algebra (see Huang, 2006). Define a 5-polar fuzzy set $\hat{\alpha}$ on X as follows:

$$\hat{\alpha} : X \to [0,1]^5, x \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.5, 0.9) & \text{if } x = 0, \\ (0.5, 0.6, 0.7, 0.4, 0.7) & \text{if } x = 1, \\ (0.3, 0.4, 0.6, 0.2, 0.5) & \text{if } x = 2, \\ (0.3, 0.4, 0.6, 0.2, 0.5) & \text{if } x = 3 \end{cases}$$

It is routine to check that $\hat{\alpha}$ is a 5-polar $(\in, \in)$-fuzzy p-ideal of X.

3. *m*-polar fuzzy *q*-ideals

Definition 3.1. An m-polar fuzzy set $\hat{\alpha}$ on a BCI-algebra X is called an *m-polar* $(\in, \in)$*-fuzzy q-ideal* of X if it satisfies (2.18) and

$$(\forall x, y, z \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m)\Big((x * (y * z))_{\hat{r}} \in \hat{\alpha},\ y_{\hat{t}} \in \hat{\alpha} \Rightarrow (x * z)_{\inf\{\hat{r}, \hat{t}\}} \in \hat{\alpha}\Big). \tag{3.1}$$

It is routine to verify that the condition (3.1) is equivalent to the following condition.

$$(\forall x, y, z \in X)(\hat{\alpha}(x * z) \geqslant \inf\{\hat{\alpha}(x * (y * z)), \hat{\alpha}(y)\}), \tag{3.2}$$

that is,

$$(\pi_i \circ \hat{\alpha})(x * z) \geqslant \inf\{(\pi_i \circ \hat{\alpha})(x * (y * z)), (\pi_i \circ \hat{\alpha})(y)\} \tag{3.3}$$

for all $x, y, z \in X$ and $i = 1, 2, \ldots, m$.

Example 3.2. Let $X = \{0, 1, a\}$ be a set with a binary operation $*$ which is given in Table 4.

Then X is a BCI-algebra (see Huang, 2006). Define a 3-polar fuzzy set $\hat{\alpha}$ on X as follows:

$$\hat{\alpha} : X \to [0,1]^3,\ x \mapsto \begin{cases} (0.75, 0.63, 0.82) & \text{if } x = 0, \\ (0.55, 0.63, 0.72) & \text{if } x = 1, \\ (0.35, 0.43, 0.62) & \text{if } x = a \end{cases}$$

Table 4
Cayley table for the binary operation “$*$”.

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

It is routine to check that $\hat{\alpha}$ is a 3-polar $(\in,\in)$-fuzzy q-ideal of X.

Theorem 3.3. *Every m-polar $(\in,\in)$-fuzzy q-ideal of a BCI-algebra X is an m-polar fuzzy ideal and an m-polar fuzzy subalgebra of X.*

Proof. Let $\hat{\alpha}$ be an m-polar $(\in,\in)$-fuzzy q-ideal of a BCI-algebra X. Putting $z=0$ in (3.3) and using (2.2) implies that

$$\begin{aligned}(\pi_i\circ\hat{\alpha})(x) = (\pi_i\circ\hat{\alpha})(x*0) &\geqslant \inf\{(\pi_i\circ\hat{\alpha})(x*(y*0)),(\pi_i\circ\hat{\alpha})(y)\}\\ &= \inf\{(\pi_i\circ\hat{\alpha})(x*y)),(\pi_i\circ\hat{\alpha})(y)\}\end{aligned}$$

for all $x,y\in X$ and $i=1,2,\ldots,m$. Hence $\hat{\alpha}$ is an m-polar fuzzy ideal of X. Putting $z=y$ in (3.3) and using (III) and (2.2) implies that

$$\begin{aligned}(\pi_i\circ\hat{\alpha})(x*y) &\geqslant \inf\{(\pi_i\circ\hat{\alpha})(x*(y*y)),(\pi_i\circ\hat{\alpha})(y)\}\\ &= \inf\{(\pi_i\circ\hat{\alpha})(x*0),(\pi_i\circ\hat{\alpha})(y)\}\\ &= \inf\{(\pi_i\circ\hat{\alpha})(x),(\pi_i\circ\hat{\alpha})(y)\}\end{aligned}$$

for all $x,y\in X$ and $i=1,2,\ldots,m$. Thus $\hat{\alpha}$ is an m-polar fuzzy subalgebra of X. □

In the following example, we show that the converse of Theorem 3.3 is not true in general.

Example 3.4. Let $X=\{0,1,b,c\}$ be a set with a binary operation $*$ which is given in Table 5.

Then X is a BCI-algebra (see Huang, 2006). Define a 4-polar fuzzy set $\hat{\alpha}$ on X as follows:

$$\hat{\alpha}: X\to[0,1]^4,\ x\mapsto\begin{cases}(0.7,0.6,0.8,0.5) & \text{if } x=0,\\ (0.3,0.4,0.5,0.2) & \text{if } x=1,\\ (0.3,0.5,0.6,0.2) & \text{if } x=b,\\ (0.3,0.4,0.5,0.2) & \text{if } x=c\end{cases}$$

It is routine to check that $\hat{\alpha}$ is an 4-polar fuzzy ideal and a 4-polar fuzzy subalgebra of X. But it is not a 4-polar $(\in,\in)$-fuzzy q-ideal of X since

$$\begin{aligned}(\pi_2\circ\hat{\alpha})(c*1) = (\pi_2\circ\hat{\alpha})(b) = 0.5 &< 0.6\\ &= \inf\{(\pi_2\circ\hat{\alpha})(c*(0*1)),(\pi_2\circ\hat{\alpha})(0)\}.\end{aligned}$$

We provide conditions for an m-polar fuzzy ideal to be an m-polar $(\in,\in)$-fuzzy q-ideal, and consider characterization of m-polar $(\in,\in)$-fuzzy q-ideal.

Lemma 3.5. (Al-Masarwah and Ahmad, 2019, Proposition 3.9) *If $\hat{\alpha}$ is an m-polar fuzzy ideal of a BCI-algebra X, then*

$$(\forall x,y\in X)(x\leqslant y\Rightarrow\hat{\alpha}(x)\geqslant\hat{\alpha}(y)),$$

that is, $(\pi_i\circ\hat{\alpha})(x)\geqslant(\pi_i\circ\hat{\alpha})(y)$ for all $x,y\in X$ with $x\leqslant y$ and $i=1,2,\ldots,m$.

Lemma 3.6. (Al-Masarwah and Ahmad, 2019, Proposition 3.14) *If $\hat{\alpha}$ is an m-polar fuzzy ideal of a BCI-algebra X, then*

$$(\forall x,y,z\in X)(x*y\leqslant z\Rightarrow\hat{\alpha}(x)\geqslant\inf\{\hat{\alpha}(y),\hat{\alpha}(z)\}),$$

that is, $(\pi_i\circ\hat{\alpha})(x)\geqslant\inf\{(\pi_i\circ\hat{\alpha})(y),(\pi_i\circ\hat{\alpha})(z)\}$ for all $x,y,z\in X$ with $x*y\leqslant z$ and $i=1,2,\ldots,m$.

Table 5
Cayley table for the binary operation "$*$".

$*$	0	1	b	c
0	0	c	b	1
1	1	0	c	b
b	b	1	0	c
c	c	b	1	0

Theorem 3.7. *Given an m-polar fuzzy ideal $\hat{\alpha}$ of a BCI-algebra X, the following are equivalent.*

(1) *$\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X.*
(2) *$\hat{\alpha}$ satisfies $\hat{\alpha}(x*y)\geqslant\hat{\alpha}(x*(0*y))$, that is,*

$$(\pi_i\circ\hat{\alpha})(x*y)\geqslant(\pi_i\circ\hat{\alpha})(x*(0*y))$$

for all $x,y\in X$ and $i=1,2,\ldots,m$.
(3) *$\hat{\alpha}$ satisfies $\hat{\alpha}((x*y)*z)\geqslant\hat{\alpha}(x*(y*z))$, that is,*

$$(\pi_i\circ\hat{\alpha})((x*y)*z)\geqslant(\pi_i\circ\hat{\alpha})(x*(y*z))$$

for all $x,y,z\in X$ and $i=1,2,\ldots,m$.

Proof. (1) ⇒ (2). If we replace y and z by 0 and y, respectively, in (3.3) and use (2.16), then

$$\begin{aligned}(\pi_i\circ\hat{\alpha})(x*y) &\geqslant \inf\{(\pi_i\circ\hat{\alpha})(x*(0*y)),(\pi_i\circ\hat{\alpha})(0)\}\\ &= (\pi_i\circ\hat{\alpha})(x*(0*y)),\end{aligned}$$

and so for $\hat{\alpha}(x*y)\geqslant\hat{\alpha}(x*(0*y))$ for all $x,y\in X$.(2) ⇒ (3). Note that

$$\begin{aligned}((x*y)*(0*z))*(x*(y*z)) &= ((x*y)*(x*(y*z)))*(0*z)\\ &\leqslant((y*z)*y)*(0*z)\\ &=(0*z)*(0*z)=0,\end{aligned}$$

that is, $(x*y)*(0*z)\leqslant x*(y*z)$ for all $x,y,z\in X$. It follows from (2) and Lemma 3.5 that

$$\begin{aligned}(\pi_i\circ\hat{\alpha})((x*y)*z) &\geqslant (\pi_i\circ\hat{\alpha})((x*y)*(0*z))\\ &\geqslant (\pi_i\circ\hat{\alpha})(x*(y*z)),\end{aligned}$$

that is, $\hat{\alpha}((x*y)*z)\geqslant\hat{\alpha}(x*(y*z))$ for all $x,y,z\in X$.

(3) ⇒ (1). Note that $(x*z)*((x*y)*z)\leqslant x*(x*y)\leqslant y$ for all $x,y,z\in X$. Using (3) and Lemma 3.6, we have

$$\begin{aligned}(\pi_i\circ\hat{\alpha})(x*z) &\geqslant \inf\{(\pi_i\circ\hat{\alpha})((x*y)*z),(\pi_i\circ\hat{\alpha})(y)\}\\ &\geqslant \inf\{(\pi_i\circ\hat{\alpha})(x*(y*z)),(\pi_i\circ\hat{\alpha})(y)\}\end{aligned}$$

and so $\hat{\alpha}(x*z)\geqslant\inf\{\hat{\alpha}(x*(y*z)),\hat{\alpha}(y)\}$ for all $x,y,z\in X$. Therefore $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X. □

Theorem 3.8. *An m-polar fuzzy set $\hat{\alpha}$ on a BCI-algebra X is an m-polar $(\in,\in)$-fuzzy q-ideal of X if and only if the m-polar level set $U(\hat{\alpha};\hat{r})$ of $\hat{\alpha}$ is a q-ideal of X for all $\hat{r}\in[0,1]^m$.*

Proof. Suppose that $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X and let $\hat{r}=(r_1,r_2,\ldots,r_m)\in(0,1]^m$. It is clear that $0\in U(\hat{\alpha};\hat{r})$. Let $x,y,z\in X$ be such that $x*(y*z)\in U(\hat{\alpha};\hat{r})$ and $y\in U(\hat{\alpha};\hat{r})$. Then $(\pi_i\circ\hat{\alpha})(x*(y*z))\geqslant r_i$ and $(\pi_i\circ\hat{\alpha})(y)\geqslant r_i$ for all $i=1,2,\ldots,m$. It follows from (3.3) that

$$(\pi_i\circ\hat{\alpha})(x*z)\geqslant\inf\{(\pi_i\circ\hat{\alpha})(x*(y*z)),(\pi_i\circ\hat{\alpha})(y)\}\geqslant r_i$$

for $i=1,2,\ldots,m$. Hence $x*z\in U(\hat{\alpha};\hat{r})$, and therefore $U(\hat{\alpha};\hat{r})$ is a q-ideal of X.

Conversely, suppose that the m-polar level set $U(\hat{\alpha};\hat{r})$ of $\hat{\alpha}$ is a q-ideal of X for all $\hat{r}\in[0,1]^m$. If $\hat{\alpha}(0)<\hat{\alpha}(a)$ for some $a\in X$ and take $\hat{r}:=\hat{\alpha}(a)$, then $a\in U(\hat{\alpha};\hat{r})$ and $0\notin U(\hat{\alpha};\hat{r})$. This is a contradiction, and so $\hat{\alpha}(0)\geqslant\hat{\alpha}(x)$ for all $x\in X$. Now, suppose that there exist $a,b,c\in X$ such that $\hat{\alpha}(a*c)<\inf\{\hat{\alpha}(a*(b*c)),\hat{\alpha}(b)\}$. If we take

$$\hat{r}:=\inf\{\hat{\alpha}(a*(b*c)),\hat{\alpha}(b)\},$$

then $a*(b* c) \in U(\hat{\alpha};\hat{r})$ and $b \in U(\hat{\alpha};\hat{r})$. Since $U(\hat{\alpha};\hat{r})$ is a q-ideal of X, it follows that $a* c \in U(\hat{\alpha};\hat{r})$. Hence $\hat{\alpha}(a* c) \geqslant \hat{r}$, which is a contradiction. Thus $\hat{\alpha}(x* z) \geqslant \inf\{\hat{\alpha}(x*(y* z)), \hat{\alpha}(y)\}$ for all $x,y,z \in X$. Therefore $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X. □

Corollary 3.9. *If $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of a BCI-algebra X, then the set*

$$J := \{x \in X | \hat{\alpha}(x) = \hat{\alpha}(0)\}$$

is a q-ideal of X.

We give an extension property about the m-polar $(\in,\in)$-fuzzy q-ideal of a BCI-algebra X.

Theorem 3.10. *Let $\hat{\alpha}$ and $\hat{\beta}$ be m-polar fuzzy ideals of a BCI-algebra X such that $\hat{\alpha} \leqslant \hat{\beta}$ and $\hat{\alpha}(0) = \hat{\beta}(0)$. If $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X, then so is $\hat{\beta}$.*

Proof. Assume that $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X. Since

$$(x*(x*(0 * y)))*(0 * y) = (x*(0 * y))*(x*(0 * y)) = 0$$

for all $x,y \in X$, we have

$$(\pi_i \circ \hat{\alpha})((x*(x*(0 * y)))*(0 * y)) = (\pi_i \circ \hat{\alpha})(0) = (\pi_i \circ \hat{\beta})(0)$$

for $i = 1,2,\ldots,m$. It follows from Theorem 3.7 that

$$\begin{aligned}(\pi_i \circ \hat{\alpha})((x*(x*(0 * y)))* y) &\geqslant (\pi_i \circ \hat{\alpha})((x*(x*(0 * y))) \\ &\quad *(0 * y)) = (\pi_i \circ \hat{\beta})(0).\end{aligned}$$

Hence

$$\begin{aligned}(\pi_i \circ \hat{\beta})((x* y)*(x*(0 * y))) &= (\pi_i \circ \hat{\beta})((x*(x*(0 * y)))* y) \\ &\geqslant (\pi_i \circ \hat{\alpha})((x*(x*(0 * y)))* y) \\ &\geqslant (\pi_i \circ \hat{\beta})(0) \\ &\geqslant (\pi_i \circ \hat{\beta})(x*(0 * y)),\end{aligned}$$

which implies from (2.17) that

$$\begin{aligned}(\pi_i \circ \hat{\beta})(x* y) &\geqslant \inf\{(\pi_i \circ \hat{\beta})((x* y)*(x*(0 * y))), \\ &\qquad (\pi_i \circ \hat{\beta})(x*(0 * y))\} \\ &= (\pi_i \circ \hat{\beta})(x*(0 * y))\end{aligned}$$

for all $x,y \in X$ and $i = 1,2,\ldots,m$. Therefore $\hat{\beta}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X by Theorem 3.7. □

Theorem 3.11. *Let $f : X \to Y$ be an epimorphism of BCI-algebras. If $\hat{\beta}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of Y, then the m-polar fuzzy set $\hat{\alpha}$ on X defined by*

$$\hat{\alpha} : X \to [0,1]^m,\ x \mapsto \hat{\beta}(f(x)),$$

that is, $(\pi_i \circ \hat{\alpha})(x) = (\pi_i \circ \hat{\beta})(f(x))$ for $x \in X$ and $i = 1,2,\ldots,m$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X.

Proof. Let $\hat{\beta}$ be an m-polar $(\in,\in)$-fuzzy q-ideal of Y. For any $x \in X$, we have

$$\begin{aligned}(\pi_i \circ \hat{\alpha})(x) &= (\pi_i \circ \hat{\beta})(f(x)) \leqslant (\pi_i \circ \hat{\beta})(0) = (\pi_i \circ \hat{\beta})(f(0)) \\ &= (\pi_i \circ \hat{\alpha})(0)\end{aligned}$$

for all $i = 1,2,\ldots,m$. Let $x,y,z \in X$. Then

$$\begin{aligned}(\pi_i \circ \hat{\alpha})(x* z) &= (\pi_i \circ \hat{\beta})(f(x* z)) = (\pi_i \circ \hat{\beta})(f(x)* f(y)) \\ &\geqslant \inf\{(\pi_i \circ \hat{\beta})(f(x)*(f(y)* f(z))), (\pi_i \circ \hat{\beta})(f(y))\} \\ &= \inf\{(\pi_i \circ \hat{\beta})(f(x*(y* z))), (\pi_i \circ \hat{\beta})(f(y))\} \\ &= \inf\{(\pi_i \circ \hat{\alpha})(x*(y* z)), (\pi_i \circ \hat{\alpha})(y)\}\end{aligned}$$

for all $i = 1,2,\ldots,m$. Therefore $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X. □

Theorem 3.12. *Let $f : X \to Y$ be an epimorphism of BCI-algebras. If $\hat{\alpha}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of X such that*

$$(\forall T \subseteq X)(\exists x_0 \in T)\left(\hat{\alpha}(x_0) = \sup_{a \in T} \hat{\alpha}(a)\right),$$

then the image $\hat{\beta}$ of $\hat{\alpha}$ under f which is defined by

$$\hat{\beta} : Y \to [0,1]^m, y \mapsto \sup_{x \in f^{-1}(y)} \hat{\alpha}(x)$$

is an m-polar $(\in,\in)$-fuzzy q-ideal of Y.

Proof. Since $0 \in f^{-1}(0)$, we have

$$\hat{\beta}(0) = \sup_{x \in f^{-1}(0)} \hat{\alpha}(x) = \hat{\alpha}(0) \geqslant \hat{\alpha}(x)$$

for all $x \in X$, and so

$$\hat{\beta}(0) = \sup_{x \in f^{-1}(y)} \hat{\alpha}(x) = \hat{\beta}(y)$$

for all $y \in Y$. For any $a,b,c \in Y$, let $x_0 \in f^{-1}(a), y_0 \in f^{-1}(b)$ and $z_0 \in f^{-1}(c)$ satisfying $\hat{\alpha}(x_0 * z_0) = \sup_{u \in f^{-1}(a* c)} \hat{\alpha}(u)$, $\hat{\alpha}(y_0) = \sup_{u \in f^{-1}(b)} \hat{\alpha}(u)$ and $\hat{\alpha}(x_0 *(y_0 * z_0)) = \sup_{u \in f^{-1}(a*(b* c))} \hat{\alpha}(u)$. Then

$$\begin{aligned}\hat{\beta}(a* c) &= \sup_{u \in f^{-1}(a* c)} \hat{\alpha}(u) = \hat{\alpha}(x_0 * z_0) \\ &\geqslant \inf\{\hat{\alpha}(x_0 *(y_0 * z_0)), \hat{\alpha}(y_0)\} \\ &= \inf\left\{\sup_{u \in f^{-1}(a*(b* c))} \hat{\alpha}(u), \sup_{u \in f^{-1}(b)} \hat{\alpha}(u)\}\right\} \\ &= \inf\left\{\hat{\beta}(a*(b* c)), \hat{\beta}(b)\right\}.\end{aligned}$$

Therefore $\hat{\beta}$ is an m-polar $(\in,\in)$-fuzzy q-ideal of Y. □

Lemma 3.13. *Let I be a subset of a BCI-algebra X and let $\hat{\alpha}_I$ be an m-polar fuzzy set on X defined by*

$$\hat{\alpha}_I : X \to [0,1]^m,\ x \mapsto \begin{cases}\hat{1} & \text{if } x \in I, \\ \hat{0} & \text{otherwise}\end{cases}$$

Then $\hat{\alpha}_I$ is an m-polar $(\in,\in)$-fuzzy ideal (resp., m-polar $(\in,\in)$-fuzzy q-ideal) of X if and only if I is an ideal (resp., q-ideal) of X.

Proof. Straightforward. □

We provide characterizations of a quasi-associative BCI-algebras.

Theorem 3.14. *Given a BCI-algebra X, the following assertions are equivalent.*

(1) X is quasi-associative.
(2) *Every m-polar fuzzy ideal of X is an m-polar $(\in,\in)$-fuzzy q-ideal of X.*

(3) *Every m-polar fuzzy ideal* $\hat{\alpha}$ *of X with* $\hat{\alpha}(0) = \hat{1}$ *is an m-polar* $(\in, \in)$*-fuzzy q-ideal of X.*

(4) *Every zero m-polar fuzzy ideal* $\hat{\alpha}_{\{0\}}$ *of X is an m-polar* $(\in, \in)$*-fuzzy q-ideal of X.*

(5) *Every m-polar fuzzy ideal* $\hat{\alpha}_{X_+}$ *of X is an m-polar* $(\in, \in)$*-fuzzy q-ideal of X, where* X_+ *is the BCK-part of X.*

(6) *The m-polar fuzzy ideal* $\hat{\beta}$ *of X with* $\hat{\beta} \leqslant \hat{\alpha}_{X_+}$ *and* $\hat{\beta}(0) = \hat{1}$ *is an m-polar* $(\in, \in)$*-fuzzy q-ideal of X.*

Proof. (1) ⇒ (2). Let $\hat{\alpha}$ be m-polar fuzzy ideal of X. Using (2.1) and Lemma 3.5, we have $\hat{\alpha}((x * y) * z) \geqslant \hat{\alpha}(x * (y * z))$ for all $x, y, z \in X$. It follows from (2.15) and (2.4) that

$$\hat{\alpha}(x * z) \geqslant \inf\{\hat{\alpha}((x * z) * y), \hat{\alpha}(y)\} = \inf\{\hat{\alpha}((x * y) * z), \hat{\alpha}(y)\}$$
$$\geqslant \inf\{\hat{\alpha}(x * (y * z)), \hat{\alpha}(y)\}$$

for all $x, y, z \in X$. Hence $\hat{\alpha}$ is an m-polar $(\in, \in)$-fuzzy q-ideal of X.

(2) ⇒ (3), (3) ⇒ (4) and (2) ⇒ (6) are straighrforward.

(4) ⇒ (5). Note that $\hat{\alpha}_{\{0\}} \leqslant \hat{\alpha}_{X_+}$ and $\hat{\alpha}_{\{0\}}(0) = \hat{1} = \hat{\alpha}_{X_+}(0)$. Thus $\hat{\alpha}_{X_+}$ is an m-polar $(\in, \in)$-fuzzy q-ideal of X by Theorem 3.10.

(5) ⇒ (1). If $\hat{\alpha}_{X_+}$ is an m-polar $(\in, \in)$-fuzzy q-ideal of X, then X_+ is a q-ideal of X by Lemma 3.13. Since $(0 * x) * (0 * x) = 0 \in X_+$ for all $x \in X$, it follows from (2.8) that $(0 * x) * x \in X_+$. Hence $0 * x = 0 * (0 * x)$ for all $x \in X$, and so X is quasi-associative by Lemma 2.1.

(6) ⇒ (5). Since $\hat{\beta} \leqslant \hat{\alpha}_{X_+}$ and $\hat{\beta}(0) = \hat{1} = \hat{\alpha}_{X_+}(0)$, we know that $\hat{\alpha}_{X_+}$ is an m-polar $(\in, \in)$-fuzzy q-ideal of X by Theorem 3.10. □

Corollary 3.15. *If a BCI-algebra X meets any of the following conditions*

(1) $(\forall x, y \in X)\ (0 * (x * y) = 0 * (y * x))$,
(2) $(\forall x, y \in X)\ ((0 * x) * y = 0 * (x * y))$,
(3) $(\forall x, y \in X)\ (x * (0 * y) = 0 \Rightarrow x * y = 0)$,
(4) $(\forall x \in X)\ (S_x := \{0, 0 * x\} \text{isasubalgebraofX})$,
(5) *The p-semisimple part of X is an associative subalgebra of X,*

then X is a quasi-associative and so has all of five other properties of Theorem 3.14.

Proof.

By Lemma 2.1, a BCI-algebra X is quasi-associative if and only if $0 * x = 0 * (0 * x)$, for all $x \in X$. Now, it is easy to prove that each of the properties (1) to (5) are equivalent to the condition $0 * x = 0 * (0 * x)$, for all $x \in X$. Hence by Lemma 2.1, X is a quasi-associative and so by Theorem 3.14, X has all of five other properties of Theorem 3.14. □

4. Conclusions

The traditional fuzzy set expression cannot distinguish between elements unrelated to the opposite. It is difficult to express differences in components unrelated to the opposing elements in the fuzzy set only if the membership extends over the interval [0, 1]. If a set expression can express this kind of difference, it will be more beneficial than a traditional fuzzy set expression. Based on these observations, Lee introduced an extension of the fuzzy set called the bipolar value fuzzy set in his paper [Lee, K.M. Bipolar-valued fuzzy sets and their operations. *Proc. Int. Conf. on Intelligent Technologies*, Bangkok, Thailand 2000, 307–312]. This concept is being applied from various angles to algebraic structure and applied science etc. An m-polar fuzzy model is a generalized form of a bipolar fuzzy model. The m-polar fuzzy models provide more precision, flexibility and compatibility to the system when more than one agreements are to be dealt with. The purpose of this paper is to study m-polar fuzzy q-ideals of BCI-algebras. We have first introduced the notion of m-polar $(\in, \in)$-fuzzy q-ideals of BCI-algebras and have investigated several properties. we have discussed relations between an m-polar fuzzy ideal/subalgebra and an m-polar $(\in, \in)$-fuzzy q-ideal, and considered the characterization of m-polar $(\in, \in)$-fuzzy q-ideal. We discussed homomorphic image and preimage of m-polar $(\in, \in)$-fuzzy q-ideal, and provided characterizations of a quasi-associative BCI-algebras are provided by using m-polar $(\in, \in)$-fuzzy q-ideal. There are several kinds of ideals in BCI-algebras, for example, p-ideal, q-ideal, a-ideal, BCI-implicative ideal, BCI-positive implicative ideal, BCI-commutative ideal, sub-implicative ideal, etc. These different kinds of ideals are basically very relevant to the ideal. Thus, the polarity of q-ideals as studied in this paper will be the basic step in the polarity study of other ideals.

The purpose of our research in the future is study on set of all m-polar fuzzy q-ideals of BCI-algebras. Can we construct a lattice structure on this set? Can we define an algebraic structure on this set such that it be a BCI-algebra again?.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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