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Original article

RBF collocation approach to calculate numerically the solution of the nonlinear system of qFDEs

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ABSTRACT

In this paper, we have implemented an efficient and high accurate radial basis function (RBF) collocation scheme for solving nonlinear systems of q-fractional differential equations. We firstly convert the problems under investigation into the equivalent systems of weakly singular q-integral equations by some essential results of fractional q-calculus. Secondly, we combine RBF collocation method and Newton–Raphson iterative algorithm to solve the latter systems of weakly singular q-integral equations. More precisely, applying RBF collocation scheme will transform the system of q-integral equations into the associated system of nonlinear algebraic equations that can be solved by iterative methods such as the Newton–Raphson algorithm. Finally, various numerical test problems including linear and nonlinear examples are listed to illustrate the robustness of the proposed global scheme with respect to the at least two recent methods in the literature.

1. Introduction

Researchers in the field of mathematical modelling have investigated models and problems in different branches of science and engineering through the notion of fractional operators deeply (for instance, see Kilbas et al. (2006) and the references therein). This is because of the specific conditions of fractional operators, such as fractional integrals and derivatives of Riemann–Liouville and Caputo types. However, validating with experimental data and also parameter estimation in fractional calculus need to patience and hard attempts since, in fractional calculus, the domain of parameters is very wide to integer-order calculus.In addition to the classical Caputo and Riemann–Liouville types, fractional operators, several other new definitions such as Caputo and Fabrizio (2015) and Atangana and Baleanu (2016) fractional derivatives have been proposed by researchers in recent years for modelling different kinds of phenomena in engineering and science. However, some criticisms are presented for these new fractional derivatives and conclude that because of nonsingularity (in other words, continuity) of the kernels of such these new tools, their applications for modelling in real-world are limited (Stynes, 2018). Therefore, nowadays two classical Caputo and Riemann–Liouville fractional derivatives are more popular and well-known between researchers and mathematicians.

In recent years, q-fractional calculus (as a particularl case of fractional calculus) was very popular among researchers. The definitions of q-fractional calculus originated from q-calculus (including q-integrals and q-derivatives) in classic calculus research works such as Jackson (1908) and Agarwal (1969). Caputo type q-fractional derivative is one of the most important definitions in

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this regard and has many applications in stochastic computations (Annaby and Mansour, 2012) and the mechanics of quantum (Kac and Cheung, 2002). As another typical example in the direction of applications of q-fractional derivatives, one can point out to the research work (Abdeljawad and Alzabut, 2018) that investigated the retarded logistic model. About recent studies about q-fractional differential equations, one can refer to the research articles that deal with the existence and uniqueness of the solutions and analytical methods. In Zhang and Guo (2020), the authors proved the existence and uniqueness of solutions of Catupo type q-fractional differential equations (with initial conditions) by using the q-analogy Gronwall inequality. Also, some investigations are made in the solution's existence of some classes of boundary value problems (BVPs) in Almeida and Martins (2014). Stability analysis of some classes of q-fractional initial value problems (IVPs) was examined in detail via a Lyapunov approach (Jarad et al., 2013). Moreover, Tang and Zhang (2019) have some discussions regarding the analytical method of q-fractional differential equations by implementing the q-beta functions. Applications of Mittag–Leffler functions (in its q-analogy) for presenting some sets of solutions were conveyed in Abdeljawad and Baleanu (2011), Abdeljawad et al. (2012). After that, Salahshour et al. (2015) have been studied the rigorous convergence of the method proposed in Abdeljawad and Baleanu (2011). Variational iteration method (VIM) is also generalized to solve difference equations of q-fractional types by Wu and Baleanu (2013).

Despite extensive studies of q-fractional differential equations (qFDEs) by means of analytical methods, some very few research works were carried out to treat these types of equations numerically. Among the numerical approaches that are suggested and proposed to solve the Catupo type qFDEs, one can point out to Zhang and Tang (2019) as the first suggested local finite difference method (FDM). After that, as the second FDM in Lyu and Vong (2019), the authors proposed a second order convergent method with a full convergence analysis discussions in details. It should be noted, in Zhang and Tang (2019) just error of approximation is considered and convergence analysis was missed. As far as we know, fractional differential equations (FDEs) and specially qFDEs contain global fractional operators and solving them with local numerical techniques, such as FDM, will reduce the accuracy for computing the solutions. Therefore, researchers should provide some efficient global methods for calculating the solution with a high order of accuracy. This motivates us to investigate a very popular global numerical scheme in this regard. In this research work, we have considered a robust global numerical approach, radial basis function (RBF) collocation technique, for solving the following Caputo type qFDEs:

Problem (i) General form: The α-order $(0<\alpha<1)$ q-Caputo fractional order system

$$ {}^{c}D_q^{\alpha} y_i(t) = f_i(t, y_1, \ldots, y_n), \quad t_0 < t \leqslant 1, \quad i = 1, 2, \ldots, n, \tag{1}$$

together with the initial conditions of the form

$$D_q^j y_i(t_0) = b_{ij}, \quad (b_{ij} \in \mathbb{R} \quad i = 1, \ldots, n; j = 0, 1, \ldots, \lceil \alpha \rceil - 1). \tag{2}$$

Problem (ii) Special case of previous problem: The IVP of α-order $(0<\alpha<1)$ Caputo type q-fractional equation

$$ {}^{c}D_q^{\alpha} y(t) = f(t, y(t)), \quad t_0 < t \leqslant 1, \tag{3}$$

together with initial conditions

$$D_q^j y(t_0) = b_j, \quad (b_j \in \mathbb{R} \quad j = 0, 1, \ldots, \lceil \alpha \rceil - 1), \tag{4}$$

where $t_0 \in T_q, 0<q<1$ and f is continuous. In this research work, we will study the approximate solutions of test problems (i) and (ii) by RBF collocation approach. This type of global schemes are very popular and well-known among the researchers in computational mathematics and have been applied for solving an extensive set of problems, see for instance Wei et al. (2018) and Wang and Wang (2016) and the references therein. Under some mild conditions, RBF collocation techniques have an exponential rate of accuracy. Taking into account that their precision is high and by applying a short number of localization points, they achieve to spectral accuracy as fast as possible. Since the application of these methods has had no results for solving Caputo type system of nonlinear qFDEs, this motivates us to investigate such these numerical approaches for solving them. This remnant of this research work formed as follows. Some primary notions and definitions of q-calculus and fractional derivatives are provided in Section 2. Sections 3 is devoted to the transformation of the basic Problems (i) and (ii) to the equivalent nonlinear weakly singular integral equations in the q-analogy form. Section 4 contains the implementation of RBF collocation approach for solving the equivalent nonlinear integral equations in the q-analogy form. Section 5 is designed to examine the proposed numerical approach for extensive test problems. In this Section, the accuracy of the proposed numerical technique is investigated in details experimentally by making some comparisons with respect to two recent local FDM schemes and superior numerical results show the robustness of our suggested global technique. Conclusions about the proposed method and also the plans for our future research works are given in Section 6.

2. Preliminary remarks

2.1. Essential results about q-calculus, fractional q-derivatives and q-integrals

.

Definition 1. (Zhang and Tang, 2019) The q-analogy of $(t-x)^{(\gamma)}$ is defined by

$$(t-x)^{(\gamma)} = t^{\gamma} \Pi_{i=0}^{\infty} \frac{t - q^i x}{t - q^{\gamma+i} x}. \tag{5}$$

Definition 2. (Jackson, 1908) Let $g(t)$ be a real valued function on q-geometry set A and $|q|<1$. The q-integral of $g(t)$ on interval (a,b) is defined by

$$\int_a^b g(t) d_q t = \int_0^b g(t) d_q t - \int_0^a g(t) d_q t, \quad a, b \in A, \tag{6}$$

where

$$\int_0^x g(t) d_q t = (1-q) \sum_{n=0}^{\infty} x q^n g(x q^n), \quad x \in A. \tag{7}$$

Special cases include

$$\int_0^1 t d_q t = \frac{1}{1+q},$$

$$\int_0^1 t^2 d_q t = \frac{1}{1+q+q^2},$$

$$\int_0^1 t^n d_q t = \frac{q-1}{q^{n+1}-1}.$$

Definition 3. (Jackson, 1908) Let $g(t)$ be a real valued function on q-geometry set A and $|q|<1$. The q-derivative of $g(t)$ is defined by

$$D_q g(t) = \frac{g(qt) - g(t)}{(q-1)t}, \quad t \neq 0, \tag{8}$$

and

$$D_q g(t) = \lim_{n\to\infty} \frac{g(tq^n) - g(0)}{tq^n}, \quad | q |< 1.$$

Definition 4. (Atici and Eloe, 2007) For $0 <| q |< 1$, the q-gamma function is defined By

$$\Gamma_q(\alpha) = \frac{\Pi_{i=0}^{K_1}(1-q^{i+1})}{\Pi_{i=0}^{K_2}(1-q^{i+\alpha})}(1-q)^{1-\alpha}, \tag{9}$$

where $\alpha \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}^*\}, 0 < |q| < 1$, and $K_1, K_2 \longrightarrow \infty$.

Definition 5. (Atici and Eloe, 2007) Let $\alpha \geqslant 0$, the α-order fractional q-integral of the Riemann–Liouville type is $(I_q^0 g)(x) = g(x)$ and

$$(I_{q,a}^{\alpha} g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} g(t) d_q t, \tag{10}$$

Definition 6. (Abdeljawad and Baleanu, 2011) The α-order version of Caputo q-fractional derivative of a function $g(x) : (a,\infty) \longrightarrow \mathbb{R}$ is defined by

$$({}^c D_{q,a}^{\alpha} g)(x) = \begin{cases} (I_{q,a}^{-\alpha} g)(x), & \alpha \leqslant 0 \\ (I_{q,a}^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} g)(x), & \alpha > 0. \end{cases} \tag{11}$$

2.2. Radial basis functions collocation method

Given data at nodes $t_0, t_1, \ldots, t_n$ in d dimension, the basic form for RBF interpolating is

$$S(t) = \sum_{j=0}^{n} \lambda_j \phi(||t - t_j||),$$

where $||\cdot||$ denotes the Euclidean norm, λ_j are the set of unknown coefficients to be determined. For scalar function values $g_i = g(t_i)$, the coefficients λ_j are obtained by solving the following system of equations

$$[A]\begin{bmatrix}\lambda_0\\ \vdots \\ \lambda_n\end{bmatrix} = \begin{bmatrix} g_0 \\ \vdots \\ g_n\end{bmatrix}$$

where the interpolation matrix A satisfies $a_{ij} = \phi(||t_i - t_j||)$.

Some common types of RBF: Commonly used types of RBFs include the following forms in which $r = ||t - t_i||$ and ϵ is the shape parameter.

- $\phi_\epsilon(r) = \sqrt{1+(\epsilon r)^2}$ Multiquadrics(MQ).
- $\phi_\epsilon(r) = e^{-(\epsilon r)^2}$ Gaussian.
- $\phi(r) = r^3$ Cubic.
- $\phi(r) = r^2 Lnr$ Thin plate splines (TPS).

In this paper we focus on the popular choice of multiquadrics.

Instantly, we briefly introduce the RBFs collocation method. Consider the following boundary value problem when $\Omega \subset \mathbb{R}^d$:

$$Lu = f \ \ in\,\Omega \tag{12}$$

$$u = g \ \ on\,\partial\Omega \tag{13}$$

where L is a linear differential operator. We distinguish in our notation center $X = \{x_1, \ldots, x_N\}$ and the collocation points $E = \{\alpha_1, \ldots, \alpha_N\}$. Then we have the approximate solution of (12)–(13) in the form:

$$\tilde{u}(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|), \tag{14}$$

where $\lambda_i, i = 1, 2, \ldots, N$, are unknown coefficients that determined by collocation, ϕ is a RBF, $\|.\|$ is the Euclidean norm and x_i is the center of the RBF.

Now, let E divided into two subsets. One subset contains N_I centers, E_1, where Eq.(12) is enforced and the other subset contains N_B centers, E_2, where boundary conditions are enforced. So the collocation matrix has the following form:

$$M = \begin{bmatrix} M_I \\ M_B \end{bmatrix},$$

in which, $M_I = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}, \alpha_i \in E_1, x_j \in X$, and $M_B = L\phi(\|\alpha - x_j\|)_{\alpha=\alpha_i}, \alpha_i \in E_2, x_j \in X$. The unknown coefficients λ_i are determined by solving the linear system $M\lambda = F$, where F is a vector included $f(\alpha_i), \alpha_i \in E_1$, and $g(\alpha_i), \alpha_i \in E_2$.

3. Discussion on equivalent problem

In this section, we deal with the equivalent form of the q-fractional models (1)–(2) and (3)–(4). For convenience, we need a Lemma for providing the proof of the main Theorem.

Lemma 1. *Let* $\alpha > 0$. *If there exists* $\gamma \leqslant \alpha - \lceil\alpha\rceil + 1$ *such that* $f \in \mathbf{C}_\gamma[0,b]$, *then* $I_q^\alpha f \in \mathbf{C}_q^{\lceil\alpha\rceil}[0,b]$.

Proof. See Annaby and Mansour (2012).

Theorem 1. *Let* $\alpha > 0$ *and* G_i *be an open set in* $\mathbb{C}$ *for* $i = 1, \ldots, n$. *Let* $f_i(t, y_1, \ldots, y_n)$ *be a function defined for* $t \in (0,1]$, *and* y_i *in the domain* G_i, *such that* $f_i(t, y_1, \ldots, y_n) \in \mathbf{C}_\gamma[0,1], \gamma \leqslant \alpha - \lceil\alpha\rceil + 1$. *If* $y_i \in \mathbf{C}_q^{\lceil\alpha\rceil}[0,1]$ *then* y_i $(i = 1, \ldots, n)$ *satisfy* (1)–(2) *for all* $t \in [0,1]$ *if only if* $y_i(t)$ *satisfies the q-integral system*

$$y_i(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{b_{ij}}{\Gamma_q(j+1)} t^j + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f_i(s, y_1(s), \ldots, y_n(s)) d_q s \tag{15}$$

for $i = 1, \ldots, n$ *and all* $t \in [0,1]$.

Proof. Let $y_i(t)$ satisfy (1)–(2), then ${}^c D_q^\alpha y_i(t) \in \mathbf{C}_\gamma[0,1]$. Hence,

$$I_q^\alpha({}^c D_q^\alpha y_i(t)) = y_i(t) - \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{b_{ij}}{\Gamma_q(j+1)} t^j, \quad (0 \leqslant t \leqslant 1). \tag{16}$$

On the other hand,

$$I_q^\alpha({}^c D_q^\alpha y_i(t)) = I_q^\alpha f_i(t, y_1, \ldots, y_n) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s, y_1(s), \ldots, y_n(s)) d_q s \tag{17}$$

for all $t \in [0,1]$. With combining (16) and (17) the necessity condition is proved. Now we prove the sufficiency condition. Let $y_i(t)$ satisfies (15) for all $t \in [0,1]$, then from Lemma 1, $y_i(t) \in \mathbf{C}_q^{\lceil\alpha\rceil}[0,1]$. Consequently, acting on the two sides of (15) by the operator ${}_c D_q^\alpha$ we have

$${}^c D_q^\alpha y_i(t) = D_q^\alpha [y_i(t) - \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{b_{ij}}{\Gamma_q(j+1)} t^j] = D_q^\alpha I_q^\alpha f_i(t, y_1(t), \ldots, y_n(t)) = f_i(t, y_1(t), \ldots, y_n(t)), \tag{18}$$

for all $t \in [0,1]$. For $t \in (0,1]$ and $j \in \{0,1,\ldots,\lceil\alpha\rceil-1\}$, we get

$$\begin{aligned} D_q^j y_i(0^+) &= \lim_{j\to\infty} D_q^k y_i(tq^j) \\ &= b_{ij} + \lim_{j\to\infty} I_q^{\alpha-j} f_i(tq^j, y_1(tq^j),\ldots,y_n(tq^j)) \\ &= b_{ij} + \lim_{j\to\infty}\frac{1}{\Gamma_q(\alpha-j)} \\ &\quad\times \int_0^{tq^j}(t-qs)^{(\alpha-1-j)} f_i(s,y_1(s),\ldots,y_n(s))d_qs \\ &= b_{ij} + \lim_{j\to\infty}\frac{1}{\Gamma_q(\alpha-j)}\sum_{r=j}^{\infty} q^r \\ &\quad\times \frac{(q^{\alpha-j};q)_r}{(q;q)_r} f_i(tq^j, y_1(tq^j),\ldots,y_n(tq^j)), \end{aligned} \tag{19}$$

where the limit on the most rigth hand side of (19) vanishes because of $f_i(t,y_1(t),\ldots,y_n(t)) \in \mathbf{C}_\gamma[0,1]$.

Therefore, every solution of The q-Caputo fractional order system (1)–(2) is also a solution of the q-Volterra integral system (15) and vice versa.

Similarly, one can show that the initial value problem of α_order Caputo type q-fractional Eqs. (3)–(4) is equivalent with the following q-integral equation:

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t^j + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s,y(s))d_qs. \tag{20}$$

So, every solution of q-fractional initial value problem (3)–(4) is also a solution of the q-Volterra integral Eq.(20)vice versa.

4. Outline of the approximation method

In this section, an algorithm for the approximate solution of the two q-fractional models (1)–(2) and (3)–(4) will be derived. We assume that $0 = t_0 < t_1 < \ldots < t_N = 1$ be a nonuniform mesh on $[0,1]$ with $t_i = q^{N-i} \in T_q$, for $i = 0,1,\ldots,N$.

problem (i): (see Eqs. (1), (2)) In this case by using Definition 2 and Eq. (15), for $i = 1,\ldots,n$, we get

$$\begin{aligned} y_i(t) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_{ij}}{\Gamma_q(j+1)}t^j \\ &\quad+\frac{(1-q)}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} tq^k(t-tq^{(k+1)})^{(\alpha-1)} f_i(tq^k, y_1(tq^k),\ldots,y_n(tq^k))d_qs. \end{aligned} \tag{21}$$

Collocating Eq. (21) at the nodes t_l yeilds

$$\begin{aligned} y_i(t_l) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_{ij}}{\Gamma_q(j+1)}t_l^j + \frac{(1-q)}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} t_l q^k(t_l - t_l q^{(k+1)})^{(\alpha-1)} \\ &\quad f_i(t_l q^k, y_1(t_l q^k),\ldots,y_n(t_l q^k))d_qs. \end{aligned} \tag{22}$$

We suppose that M_1 be a positive integer, $y_{il} = y_i(t_l)$ and

$$y_i(t) \approx y_i^N(t) = \sum_{\rho=0}^{N}\lambda_{i\rho}\varphi_\rho(t), \tag{23}$$

where $\varphi_\rho(t)$ and $\lambda_{i\rho}(\rho = 0,1,\ldots,N; i = 1,\ldots,n)$ are the radial basis functions and unknown RBF coefficients, respectively. Since $y_i^N(t)$ is the approximation of $y_i(t)$, we can write

$$\begin{aligned} y_i^N(t_l) &\approx \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_{ij}}{\Gamma_q(j+1)}t_l^j \\ &\quad+\frac{(1-q)}{\Gamma_q(\alpha)}\sum_{k=0}^{M_1} t_l q^k(t_l - t_l q^{(k+1)})^{(\alpha-1)} f_i(t_l q^k, y_1^N(t_l q^k),\ldots,y_n^N(t_l q^k))d_qs. \end{aligned} \tag{24}$$

So, we have a nonlinear algebraic system with $nN+n$ equations and $nN+n$ unknowns $\lambda_{i\rho}(\rho = 0,1,\ldots,N; i = 1,\ldots,n)$. The algebraic system (24) can be solved by any suitable direct or iterative method. In this paper, we will use fsolve command, which its default is the NewtonRaphson iterative algorithm, in MATLAB software.

problem (ii): (see Eqs. (3), (4)) In this case, we consider two following states:

(a) $\mathbf{f(t,y(t))} = \sum_{\mathbf{i=1}}^{\mathbf{n}}\mu_\mathbf{i}\mathbf{t}^{\vartheta_\mathbf{i}}$ **with constant** $\mu_\mathbf{i}$ **and** $\vartheta_\mathbf{i}$**.**

Theorem 2. *Let $f(t,y(t)) = \sum_{i=1}^n \mu_i t^{\vartheta_i}$ with constant μ_i and ϑ_i. Then the solution of problem (3)–(4) is*

$$\begin{aligned} y(t) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t^j \\ &\quad+\frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{\infty} q^r\left(\Pi_{k=0}^{\infty}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}\right)\sum_{i=1}^{n}\mu_i(tq^r)^{\vartheta_i}. \end{aligned} \tag{25}$$

Proof. We know that problem (3)–(4) is equivalent to Eq. (20), on the other hand, from Definition 2, we get

$$\int_0^t (t-qs)^{(\alpha-1)} f(s,y(s))d_qs = (1-q)\sum_{r=0}^{\infty} tq^r(t-tq^{r+1})^{(\alpha-1)}\sum_{i=1}^{n}\mu_i(tq^r)^{\vartheta_i}. \tag{26}$$

Using Definition 1 implies that

$$\begin{aligned} y(t) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t^j \\ &\quad+\frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{\infty} q^r\left(\Pi_{k=0}^{\infty}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}\right)\sum_{i=1}^{n}\mu_i(tq^r)^{\vartheta_i}. \end{aligned} \tag{27}$$

Let M_2 and M_3 be two positive integers, so, in this case $y(t)$ can be easily estimated with the following

$$\begin{aligned} y(t) &\approx \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t^j \\ &\quad+\frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{M_2} q^r\left(\Pi_{k=0}^{M_3}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}\right)\sum_{i=1}^{n}\mu_i(tq^r)^{\vartheta_i}. \end{aligned} \tag{28}$$

(b) $\mathbf{f(t,y(t)) = g(t) + h(t)F(y(t))}$ **where** $\mathbf{h(t) \cong 0}$**.**

In this case, by substituting (5) and (7) in Eq. (20), we obtain

$$\begin{aligned} y(t) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t^j + \frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{\infty} q^r(\Pi_{k=0}^{\infty}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}) \\ &\quad\times(g(tq^r) + h(tq^r)F(y(tq^r))). \end{aligned} \tag{29}$$

After collocate Eq. (29) at $N+1$ points t_i, for $i = 0,1,\ldots,N$, we have

$$\begin{aligned} y(t_i) &= \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t_i^j + \frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{\infty} q^r(\Pi_{k=0}^{\infty}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}) \\ &\quad\times(g(t_i q^r) + h(t_i q^r)F(y(t_i q^r))). \end{aligned} \tag{30}$$

Now, let M_2 and M_3 be positive integers, $y_i = y(t_i)$ and

$$y(t) \approx y^N(t) = \sum_{j=0}^{N}\lambda_j\varphi_j(t), \tag{31}$$

where $\varphi_j(t)$ and $\lambda_j(j = 0,1,\ldots,N)$ are the radial basis functions and unknown RBF coefficients, respectively. So, we obtain the following system of nonlinear algebric equation

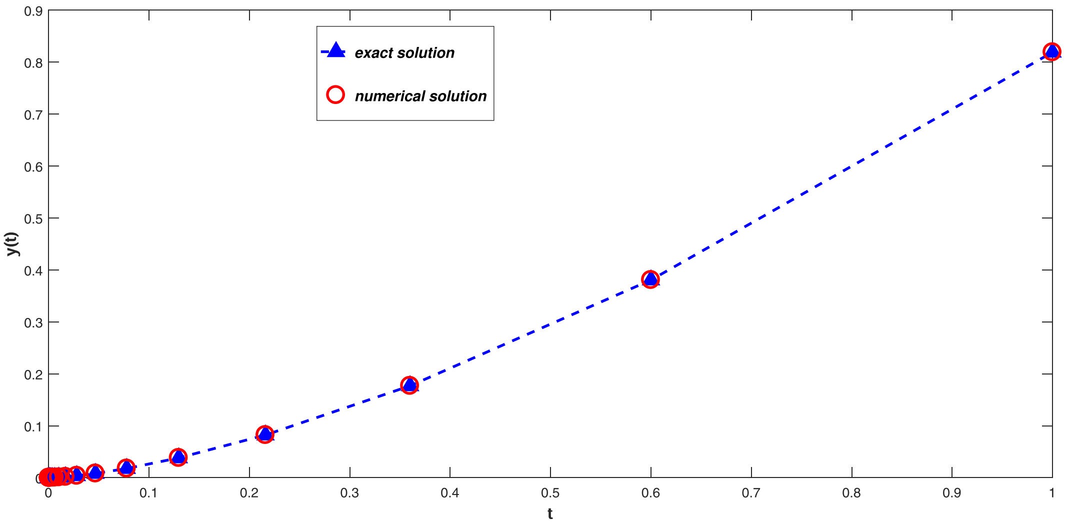

Fig. 1. Numerical solution and exact solution for Example 1 $N = 20, q = 3/5, \alpha = 1/2$).

Table 1
Numerical solution and absolute value error for Example 1 ($N = 20, q = 3/5, \alpha = 1/2$).

t	$y(t)$	$y^N(t)$	Absolute error	Method in Zhang and Tang (2019)
0	0	0	0	0
$(3/5)^{1}9$	0.000000389570411	0.000000389570411	$1.58819e-22$	$4.44849e-08$
$(3/5)^{1}7$	0.000001803566719	0.000001803566719	$8.47033e-22$	$2.29471e-08$
$(3/5)^{1}5$	0.000008349845921	0.000008349845921	$3.38813e-21$	$1.20204e-07$
$(3/5)^{1}3$	0.000038656694080	0.000038656694080	$1.35525e-20$	$5.72027e-08$
$(3/5)^{1}1$	0.000178966176295	0.000178966176295	$1.35525e-20$	$3.23198e-08$
$(3/5)^9$	0.000828547112477	0.000828547112477	$3.25261e-19$	$1.90138e-08$
$(3/5)^7$	0.003835866261468	0.003835866261468	$1.30104e-18$	$1.13302e-08$
$(3/5)^5$	0.017758640099387	0.017758640099387	$6.93889e-18$	$6.78151e-09$
$(3/5)^3$	0.082215926386051	0.082215926386051	$2.77556e-17$	$4.06534e-09$
$(3/5)$	0.380629288824312	0.380629288824311	$1.11022e-16$	$1.22486e-07$
1	0.818983831497407	0.818983831497406	$3.33067e-16$	$5.42886e-08$

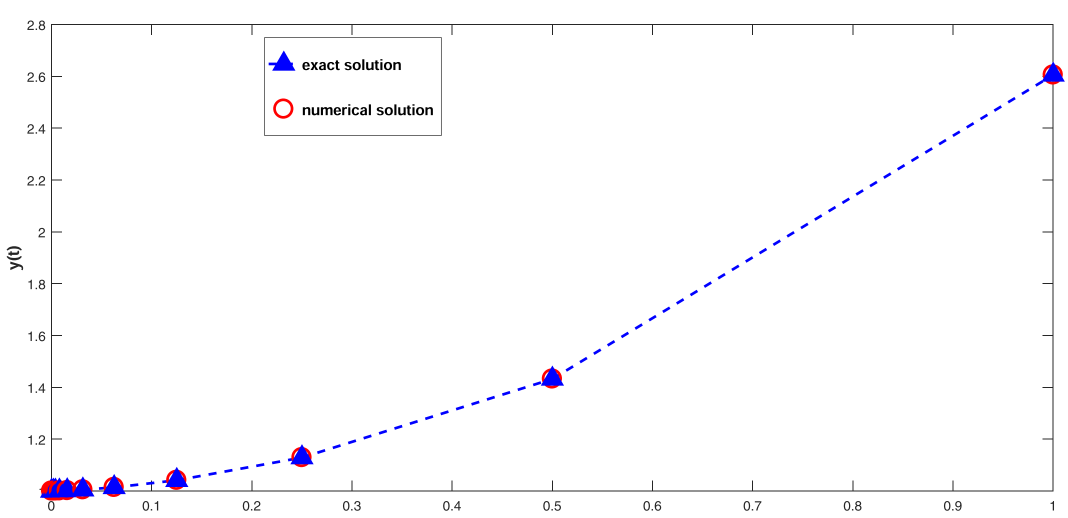

Fig. 2. Numerical solution and exact solution for Example 2 $N = 10, q = 1/2, \alpha = 1/2$).

Table 2
Numerical solution and absolute value error for Example 2 $N = 10, q = 1/2, \alpha = 1/2$).

t	$Y(t)$	$Y^N(t)$	Absolute value error
0	1.000000000000000	1.000000000000000	0
$(1/2)^9$	1.000072627919360	1.000072627919360	0
$(1/2)^8$	1.000205787657204	1.000205787657204	0
$(1/2)^7$	1.000584119464960	1.000584119464960	0
$(1/2)^6$	1.001663815501096	1.001663815501096	0
$(1/2)^5$	1.004772031242264	1.004772031242264	$2.22045e-16$
$(1/2)^4$	1.013870979799737	1.000828547112477	$2.22045e-16$
$(1/2)^3$	1.041346666660917	1.041346666660917	$2.22044e-16$
$(1/2)^2$	1.128902423708945	1.128902423708945	0
$(1/2)$	1.432226668417044	1.432226668417043	$4.44089e-16$
1	2.605126119625209	2.605126119625208	$1.3323e-15$

Table 4
Numerical solution and absolute error for Example 4 $N = 10, q = 1/2, \alpha = 1$).

t	$y_1(t)$	$y_1^N(t)$	Absolute error
0	0	0.000000000001337	$1.33683e-12$
$(1/2)^9$	0.001953122161687	0.001953122160983	$7.03764e-13$
$(1/2)^8$	0.003906227293564	0.003906227290589	$2.97537e-12$
$(1/2)^7$	0.007812318350801	0.007812318350060	$7.41147e-13$
$(1/2)^6$	0.015623546879650	0.015623546876897	$2.75346e-12$
$(1/2)^5$	0.031238377380371	0.031238377378862	$1.50979e-12$
$(1/2)^4$	0.062407093954124	0.062407093947164	$6.95929e-12$
$(1/2)^3$	0.124259139836641	0.124259139839404	$2.76279e-12$
$(1/2)^2$	0.244148412653959	0.244148412650705	$3.25428e-12$
$(1/2)$	0.455460666624558	0.455460666640393	$1.58348e-11$
1	0.702468139652606	0.702468139646127	$6.47926e-12$

$$\sum_{j=0}^{N}\lambda_j\varphi_j(t_i) \approx \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{b_j}{\Gamma_q(j+1)}t_i^j + \frac{t_i^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{r=0}^{M_2}q^r\left(\Pi_{k=0}^{M_3}\frac{1-q^{r+k+1}}{1-q^{r+\alpha+i}}\right)(g(t_iq^r) + h(t_iq^r)F(\sum_{j=0}^{N}\lambda_j\varphi_j(t_iq^r))). \tag{32}$$

with $N+1$ unknowns $\lambda_j(j = 0, 1, \ldots, N)$. We will use fsolve Matlab's command for solving this system.

5. Examination of the method experimentally

To show the efficiency and accuracy of the earlier described method, we present in this section some significant examples.All of the programs associated with the implementation of the

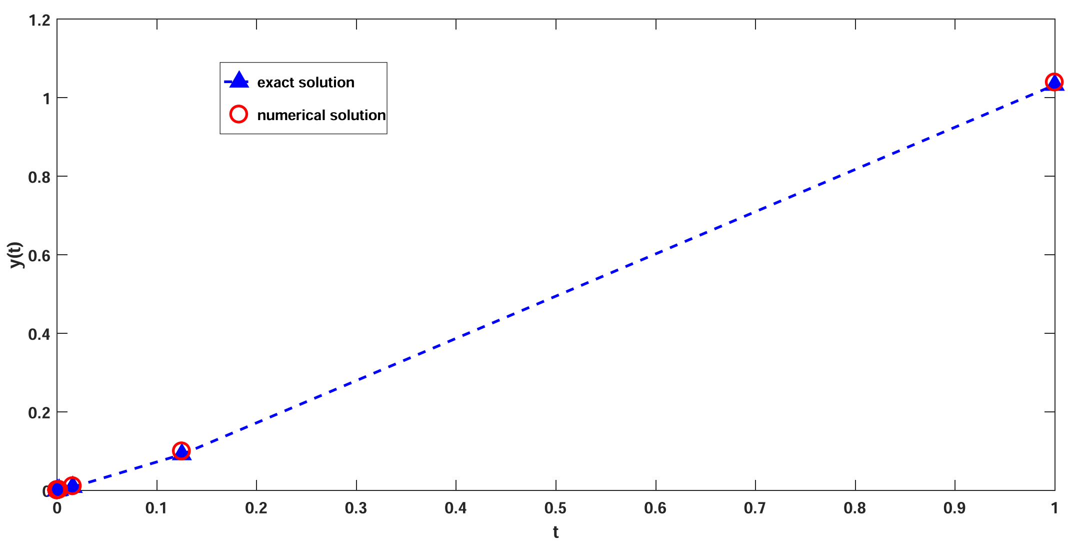

Fig. 3. Numerical solution and exact solution for Example 3 $N = 10, q = 1/8, \alpha = 1/3$).

Table 3
Numerical solution and absolute value error for Example 3 $N = 10, q = 1/8, \alpha = 1/3$).

t	$y(t)$	$y^N(t)$	Absolute value error	Method in Tang and Zhang (2019)
0	0.000100000000000	0.000100000000000	0	0
$(1/8)^9$	0.000100000339875	0.000235644848742	$1.35645e-04$	$7.10653e-04$
$(1/8)^8$	0.000100003845252	0.000235645502072	$1.35642e-04$	$1.59919e-03$
$(1/8)^7$	0.000100043504059	0.000235650833029	$1.35607e-04$	$3.59173e-03$
$(1/8)^6$	0.000100492192244	0.000235700154354	$1.35208e-04$	$8.06390e-03$
$(1/8)^5$	0.000105568519578	0.000236520354251	$1.30952e-04$	$1.81032e-02$
$(1/8)^4$	0.000163000607271	0.000268923723107	$1.05923e-04$	$4.06404e-02$
$(1/8)^3$	0.000812770505922	0.001174594502890	$3.61824e-03$	$9.12346e-02$
$(1/8)^2$	0.008164077730671	0.010408552873172	$2.24448e-03$	$2.04815e-01$
$(1/8)$	0.091334624757964	0.099095364598783	$7.76074e-03$	$4.59794e-01$
1	1.032301949525868	1.038672793906552	$6.37084e-03$	1.03220

Table 5
Numerical solution and absolute error for Example 4 $N = 10, q = 1/2, \alpha = 1)$

t	$y_2(t)$	$y_2^N(t)$	Absolute error
0	1.000000000000000	0.999999999998487	$1.51257e-12$
$(1/2)^9$	0.999997456871446	0.999997456864538	$6.90803e-12$
$(1/2)^8$	0.999989827521264	0.999989827520871	$3.930194e-13$
$(1/2)^7$	0.999959310652706	0.999959310658919	$6.21358e-12$
$(1/2)^6$	0.999837251692708	0.999837251699265	$6.55653e-12$
$(1/2)^5$	0.999349152046140	0.999349152053606	$7.46603e-12$
$(1/2)^4$	0.997398930360074	0.997398930361860	$1.78668e-12$
$(1/2)^3$	0.989632734120284	0.989632734133493	$1.32099e-11$
$(1/2)^2$	0.959114182538539	0.959114182536891	$1.64735e-12$
$(1/2)$	0.845249015882399	0.845249015891874	$9.47542e-12$
1	0.494014946056096	0.494014946047678	$8.41804e-12$

suggested method for solving the studied examples are written in MATLAB 2016b in a Laptop PC with 4 GB Ram. It should be noted that the default MATLAB's tolerance is used in the fsolve solver. We set $M_1 = M_2 = M_3 = K_1 = K_2 = 100$ in all computations. The Multiquadric radial basis function is used in the approximated solutions (24) and (31), and the trial and error method is applied for choosing shape parameters.

Example 1. First, we consider the following Caputo type q-fractional initial value problem

$$
{}^cD_q^{\frac{1}{2}}y(t) = t, \quad 0 < t \le 1, \qquad y(0) = 0.
$$

The exact solution is $y(t) = \frac{\Gamma_q(2)}{\Gamma_q(\frac{5}{2})}t^{\frac{3}{2}}$. The exact and approximated solution at $N = 10$, are shown in Fig. 1.The obtained results for Example 1 are compared the results in Zhang and Tang (2019), in Table 1. From this table, one can see high accuracy of the proposed method.

Example 2. Consider the Caputo type q-fractional initial value problem

$$
{}^cD_q^{\frac{1}{2}}y(t) = t + t^2, \quad 0 < t \le 1, \qquad y(0) = 1.
$$

The exact solution is $y(t) = 1 + \frac{\Gamma_q(2)}{\Gamma_q(\frac{5}{2})}t^{\frac{3}{2}} + \frac{\Gamma_q(3)}{\Gamma_q(\frac{7}{2})}t^{\frac{5}{2}}$.Fig. 2 shows the exact solution and approximated solution at N = 10. Absolute value

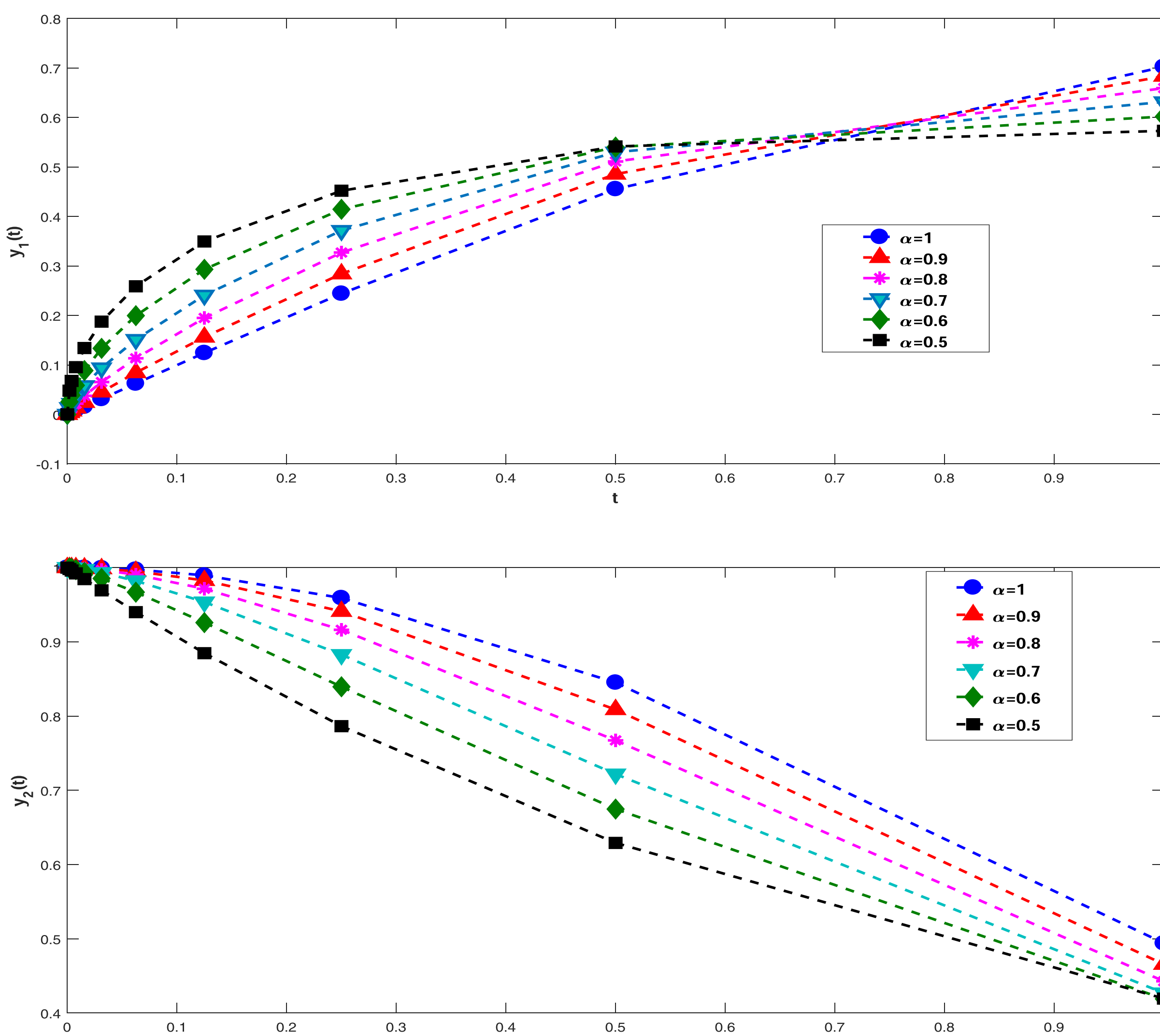

Fig. 4. Numerical solution for different value of α for Example 4 $N = 10, q = 1/2)$.

Table 6
Numerical results for Example 5 $N = 10, q = 1/3, \alpha = 1$).

t	$y_1(t)$	$y_1^N(t)$	$\|y_1 - y_1^N\|$
0	0	8.399500539546523$e-11$	8.3995$e-11$
$(1/3)^9$	5.080526335720037$e-05$	5.080527530022111$e-05$	1.1943$e-11$
$(1/3)^8$	1.524157884374290$e-04$	1.524157231143283$e-04$	6.5323$e-11$
$(1/3)^7$	4.572473211896416$e-04$	4.572472881195014$e-04$	3.3070$e-11$
$(1/3)^6$	0.001371740772258	0.001371740956903	1.8465$e-10$
$(1/3)^5$	0.004115190151394	0.004115190393954	2.4256$e-10$
$(1/3)^4$	0.012344701995526	0.012344701915369	8.0156$e-11$
$(1/3)^3$	0.037010659197795	0.037010658981692	2.1610$e-10$
$(1/3)^2$	0.110399301846614	0.110399301891554	4.4940$e-11$
$(1/3)$	0.314209777667648	0.314209777829496	1.6185$e-10$
1	0.506694834907219	0.506694834174431	7.3279$e-10$

error is reported in Table 2. It is obvious from these Figure and Table that, the suggested scheme is very accurate.

Example 3. Consider the following nonlinear Caputo type q-fractional initial value problem

$$ {}^cD_q^{\frac{1}{3}}y(t) = t^{\frac{-3}{2}}y^2(t), \quad 0 < t \le 1, \qquad y(0) = 0.0001. $$

The exact solution of this problem is $y(t) = 0.0001 + \frac{\Gamma_q(\frac{13}{6})}{\Gamma_q(\frac{11}{6})}t^{\frac{7}{6}}$. The numerical results, for $N = 10, q = \frac{1}{8}$ and $\alpha = \frac{1}{3}$, are presented in Fig. 3 and Table 3. A simple comparison between of the proposed scheme and the numerical results of Thang and Zhang (2019) (i.e., iterative scheme) confirms the accuracy of our proposed method.

Example 4. Consider the q-Caputo fractional order system

$$ {}^cD_q^{\alpha}y_1(t) = y_2(t), \qquad {}^cD_q^{\alpha}y_2(t) = -y_1(t) \qquad 0 < t \le 1, $$
$$ y_1(0) = 0 \qquad y_2(0) = 1. $$

The exact solution is $y_1(t) = t^{\alpha}\sum_{j=0}^{\mathscr{M}}(-1)^j \frac{t^{2\alpha j}}{\Gamma_q(2\alpha j+\alpha+1)}$ and $y_2(t) = \sum_{j=0}^{\mathscr{M}}(-1)^j \frac{t^{2\alpha j}}{\Gamma_q(2\alpha j+1)}$, that $\mathscr{M}\longrightarrow\infty$. We set $\mathscr{M} = 10$ in the reported exact solution in Tables 4 and 5. For $\alpha = 1$, we obtain $y_1(t) = sin_q t(1-q)$ and $y_2(t) = cos_q t(1-q)$ $(|t(1-q)| < 1)$. In Fig. 4, we have plotted the approximated solutions behaviour for different values of fractional and integer- order derivatives. Also, in Tables 4 and 5 we illustrate the errors associated to the presented method. From this tables, one can see high accuracy of the proposed method.

Example 5. Consider the q-Caputo fractional order system

$$ {}^cD_q^{\alpha}y_1(t) = y_2(t), \qquad {}^cD_q^{\alpha}y_2(t) = -q^{-\alpha}y_1(qt) \qquad 0 < t \le 1, $$
$$ y_1(0) = 0 \qquad y_2(0) = 1. $$

The exact solution is $y_1(t) = t^{\alpha}\sum_{j=0}^{\mathscr{M}}(-1)^j q^{j(j-1)\alpha}\frac{t^{2\alpha j}}{\Gamma_q(2\alpha j+\alpha+1)}$ and $y_2(t) = \sum_{j=0}^{\mathscr{M}}(-1)^j q^{j(j-1)\alpha}\frac{t^{2\alpha j}}{\Gamma_q(2\alpha j+1)}$, that $\mathscr{M}\longrightarrow\infty$. We set $\mathscr{M} = 10$ in the reported exact solution in Tables 6 and 7.

In Fig. 5, we have plotted the approximated solutions behaviour for different values of fractional and integer-order derivatives. Also, in Tables 6 and 7 we illustrate the errors associated to the presented method. From this tables, one can see high accuracy of the proposed method.

6. Conclusions and future works

In this article, a high accurate global numerical framework (i.e., RBF collocation approach) has been proposed for solving system of nonlinear qFDEs of Caputo type. This type of problems was investigated by analytical methods extensively, but from the numerical point of view, only two recent research works (i.e. FDMs) were suggested to solve them approximately. As far as we know, this is the first paper that solve these equations numerically and globally. Applying a short number of collocation points give us accurate results as fast as possible. Superior numerical results with respect to two recent numerical schemes motive us to implement the method for solving other types of q-fractional models and test problems. As our future research work, we will investigate numerical treatment of q-version of time-fractional diffusion (Ghanbari and Atangana, 2020), Burgers (Akram et al., 2020), foam drainage (Al-Mdallal et al., 2020; Shi et al., 2020), coupled kaup-kupershmidt (Wang et al., 2019), Klein–Gordon (Akram et al., 2020), Shrodinger (Ain, YYYY), Telegraph (Akram et al., 2020), Levi (Feng, 2020) and Ginzburg–Landau equations (Al-Ghafri, 2020).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Table 7
Numerical results for Example 5 $N = 10, q = 1/3, \alpha = 1$).

t	$y_2(t)$	$y_2^N(t)$	$\|y_2 - y_2^N\|$
0	1	0.999999999929966	7.0034$e-11$
$(1/3)^9$	0.999999998064119	0.999999998048430	1.5689$e-11$
$(1/3)^8$	0.999999982577070	0.999999982503621	7.3449$e-11$
$(1/3)^7$	0.999999843193633	0.999999843202192	8.5587$e-12$
$(1/3)^6$	0.999998588742821	0.999998588786891	4.4071$e-11$
$(1/3)^5$	0.999987298695312	0.999987298717936	2.2623$e-11$
$(1/3)^4$	0.999885689061944	0.999885689020719	4.1225$e-11$
$(1/3)^3$	0.998971266691906	0.998971266679367	1.2539$e-11$
$(1/3)^2$	0.990746675759062	0.990746675981385	2.2232$e-10$
$(1/3)$	0.917147141194653	0.917147141235329	4.0676$e-11$
1	0.288727585859356	0.288727585896525	3.7169$e-11$

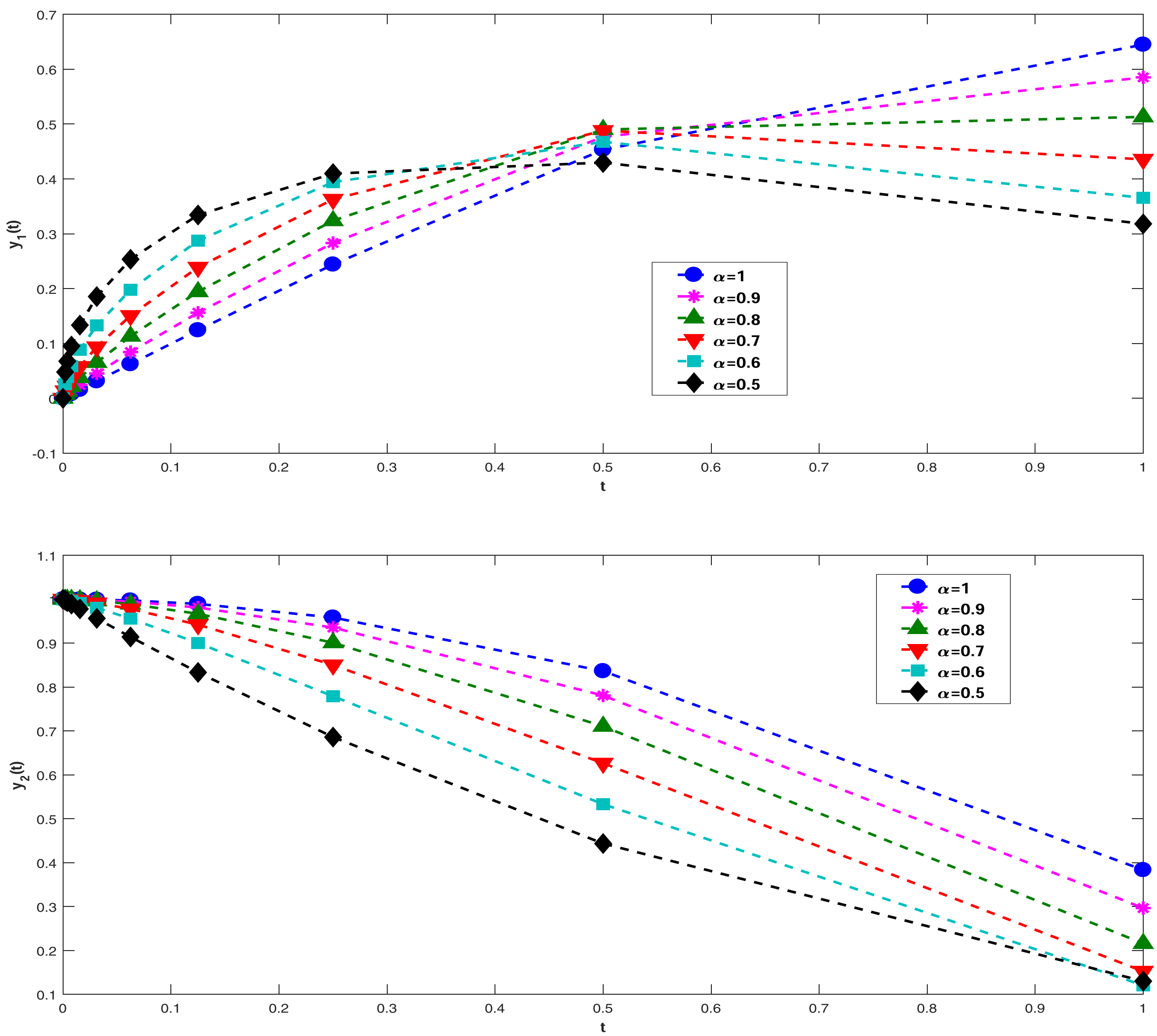

Fig. 5. Numerical solution for different value of α for Example 5 $N = 10, q = 1/2$).

Appendix A

In this appendix a simple Matlab program of nonlinear Example 3 is furnished to ease up the understanding of the proposed method.

```
function F=equation(landa)
global N m t x0  q alpha GAMMA shape B
    o = ones(1,length(t));
      rx(:,:,1) = t*o - (t*o)';   % signed distance matrix
```

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttps://doi.org/10.1016/j.jksus.2020.101288.

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