Journal of King Saud University – Science (2011) **23**, 341–347

King Saud University
Journal of King Saud University – Science
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ORIGINAL ARTICLE

A note on the unsteady torsional sinusoidal flow of fractional viscoelastic fluid in an annular cylinder

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Received 2 July 2010; accepted 24 July 2010
Available online 23 March 2011

KEYWORDS
Second grade fluid;
Shear stress;
Torsional oscillatory flow;
Fractional calculus

Abstract In this note, the velocity field and the associated shear stress corresponding to the torsional oscillatory flow of a generalized second grade fluid, between two infinite coaxial circular cylinders, are determined by means of Laplace and Hankel transforms. Initially both cylinders and fluid are at rest and then the two cylinders suddenly start torsional oscillations around their common axis with simple harmonic motions having angular frequencies ω_1 and ω_2. The solutions that have been obtained are presented under integral and series forms in terms of the generalized G and R functions and satisfy the governing differential equation and all imposed initial and boundary conditions. The respective solutions for the motion between the cylinders, when one of them is at rest, can be obtained from our general solutions. Furthermore, the corresponding solutions for the similar flow of ordinary second grade fluid and Newtonian fluid are also obtained as limiting cases of our general solutions. At the end, flows corresponding to the Newtonian, second grade and generalized second grade fluids are shown graphically by plotting velocity profiles.

1. Introduction

Flows in the neighborhood of spinning or oscillating bodies are of interest to both academic workers sand industry. Among them, the flows between oscillating cylinders are some of the most important and interesting problems of motion. As early as 1886, Stokes established an exact solution for the rotational oscillations of an infinite rod immersed in a classical linearly viscous fluid. Casarella and Laura (1969) obtained an exact solution for the motion of the same fluid due to both longitudinal and torsional oscillations of the rod. Later,

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Peer review under responsibility of King Saud University.
doi:10.1016/j.jksus.2010.07.022

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Rajagopal (1983) found two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal and Bhatnagar (1995). Others interesting results have been recently obtained by Khan et al. (2005), Fetecau and Fetecau (2006) Mahmood et al. (2009), Vieru et al. (2007), Fetecau et al. (2008), Massoudi and Phuoc (2008), Khan et al. (2009), and Mahmood et al. (2010).

Recently, the fractional calculus has encountered much success in the description of viscoelasticity. Specifically, rheological constitutive equations with fractional derivatives play an important role in the description of the properties of polymer solutions and melts. The starting point of the fractional derivative model of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by so-called Riemann–Liouville fractional differential operator. This generalization allows us to define precisely non-integer order integrals or derivatives (Podlubny, 1999).

It is important to mention here that a number of research papers in the literature (Fetecau et al., 2008; Massoudi and Phuoc, 2008; Khan et al., 2009; Mahmood et al., 2010) are devoted to the study of the flow of different viscoelastic fluids between two cylinders, when only one of them is oscillating and other is at rest. On the other hand, the exact solutions corresponding to the flow of these fluids between two cylinders, when both of them are oscillating along or around their common axis simultaneously, are very rare in literature. Recently, Mahmood et al. (2009, 2010) have studied the flow of fractional Maxwell and second grade fluids between two cylinders, when both of them are oscillating around, respectively, along their common axis. As far as the knowledge of authors is concerned, in the literature, no attempt has been made to study the flows of fractional second grade fluid due to torsional oscillations of two cylinders. Therefore, in this paper, we are interested into the torsional oscillatory motion of a generalized second grade fluid between two infinite coaxial circular cylinders when both of them are oscillating around their common axis with given constant angular frequencies ω_1 and ω_2. Velocity field and associated tangential stress of the motion are determined by using Laplace and Hankel transforms and are presented under integral and series forms in terms of the generalized G and R functions. It is worthy to point out that the solutions that have been obtained satisfy the governing differential equation and all imposed initial and boundary conditions as well. The solutions corresponding to the ordinary second grade fluid and those for Newtonian fluid, performing the same motion, are also determined as special cases of our general solutions. Furthermore, the respective solutions for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions.

2. Torsional oscillations between two cylinders

Among the many constitutive assumptions that have been employed to study non-Newtonian fluid behavior, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin–Ericksen fluids of second grade. The Cauchy stress tensor $\mathbf{T}$ for such fluids is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p$ is the pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^T,$$
$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_1. \tag{2}$$

In the above relations, $\mathbf{v}$ is the velocity, d/dt denotes the material time derivative and grad the gradient operator. Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$div\,\mathbf{v} = tr\,\mathbf{A}_1 = 0. \tag{3}$$

If this model is required to be compatible with thermodynamics, then the material moduli must meet the following restrictions:

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{4}$$

The sign of the material moduli α_1 and α_2 has been the subject of much controversy. A comprehensive discussion on the restrictions given in (4), as well as a critical review on the fluids of differential type, can be found in the extensive work of Dunn and Rajagopal (1995).

Generally, the constitutive equation of the generalized second grade fluids has the same form as (1), but $\mathbf{A}_2$ is defined by

$$\mathbf{A}_2 = D_t^\beta\mathbf{A}_1 + \mathbf{A}_1(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A}_1, \tag{5}$$

where D_t^β is the Riemann–Liouville fractional calculus operator of order β with respect to t defined by

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta}f(\tau)d\tau, \quad 0 < \beta \leqslant 1, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function. When $\beta = 1$, Eq. (5) may be simplified as $(2)_2$ while for $\alpha_1 = 0$ the constitutive relationship (1) describes the Rainer–Rivlin viscous fluid.

2.1. Mathematical formulation of the problem and governing equation

Suppose that an incompressible generalized second grade fluid is situated in the annular region between two infinite straight circular cylinders of radii R_1 and $R_2(> R_1)$ as shown in Fig. 1. At time $t = 0$, the fluid and cylinders are at rest. At time $t = 0^+$, inner and outer cylinders suddenly begin to oscillate around their common axis ($r = 0$) with the velocities $W_1\sin(\omega_1 t)$ and $W_2\sin(\omega_2 t)$, where ω_1 is the frequency of velocity of inner cylinder and ω_2 is that of outer cylinder. Owing to the shear, the fluid between the cylinders is gradually moved, its velocity being of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{e}_\theta, \tag{7}$$

where $\mathbf{e}_\theta$ is the unit vector along θ-direction. For such flows the constraint of incompressibility is automatically satisfied.

Introducing (7) into the constitutive equation, we find that

$$\tau(r,t) = (\mu + \alpha_1 D_t^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)v(r,t), \tag{8}$$

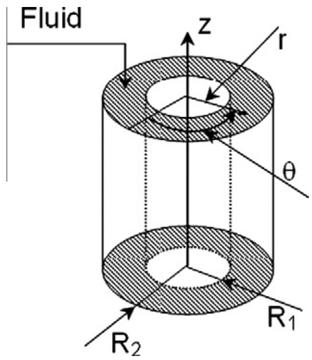

Figure 1 Flow geometry.

where $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress, which is different from zero. In the absence of body forces and assuming no pressure gradient in the flow direction, the balance of the linear momentum leads to the relevant equation

$$\rho\frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r,t). \tag{9}$$

Eliminating $\tau(r,t)$ between Eqs. (8) and (9) we get the governing equation of our problem

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha D_t^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)v(r,t);$$
$$r \in (R_1, R_2),\ \ t > 0, \tag{10}$$

where $\alpha = \alpha_1/\rho$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid (ρ being its constant density).

The appropriate initial and boundary conditions are

$$v(r,0) = 0; \quad r \in (R_1, R_2), \tag{11}$$

$$v(R_1,t) = W_1\sin(\omega_1 t), \quad v(R_2,t) = W_2\sin(\omega_2 t) \text{ for } t > 0. \tag{12}$$

2.2. Calculation of the velocity field

Applying the Laplace transform to Eqs. (10)–(12) and using the Laplace transform formula for sequential fractional derivatives (Podlubny, 1999), we obtain the ordinary differential equation

$$\frac{\partial^2\bar{v}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}(r,q)}{\partial r} - \frac{1}{r^2}\bar{v}(r,q) - \frac{q}{\alpha q^{\beta} + \nu}\bar{v}(r,q) = 0;$$
$$r \in (R_1, R_2), \tag{13}$$

where the image function $\bar{v}(r,q)$ of $v(r,t)$ has to satisfy the conditions

$$\bar{v}(R_1,q) = \frac{W_1\omega_1}{q^2 + \omega_1^2}, \quad \bar{v}(R_2,q) = \frac{W_2\omega_2}{q^2 + \omega_2^2}. \tag{14}$$

In the following, let us denote by

$$\bar{v}_n(q) = \int_{R_1}^{R_2} r\bar{v}(r,q)B_1(rr_n)dr; \quad n = 1,2,3,\ldots, \tag{15}$$

the finite Hankel transforms of $\bar{v}(r,q)$, where r_n are the positive roots of the transcendental equation $B_1(R_1 r) = 0$ and

$$B_1(rr_n) = J_1(rr_n)Y_1(R_2 r_n) - J_1(R_2 r_n)Y_1(rr_n). \tag{16}$$

In the above relation, $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of order *one* of the first and second kind. Applying the finite Hankel transform to Eq. (13) and taking into account the conditions (14), we find that

$$\frac{2W_2\omega_2}{\pi(q^2+\omega_2^2)} - \frac{2W_1\omega_1}{\pi(q^2+\omega_1^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)} - r_n^2\bar{v}_n(q)$$
$$-\frac{q}{\alpha q^{\beta}+\nu}\bar{v}_n(q) = 0, \tag{17}$$

or equivalently,

$$\bar{v}_n(q) = \frac{2W_2\omega_2(\alpha q^{\beta}+\nu)}{\pi(q^2+\omega_2^2)\left(\alpha r_n^2 q^{\beta} + q + \nu r_n^2\right)}$$
$$-\frac{2W_1\omega_1(\alpha q^{\beta}+\nu)}{\pi(q^2+\omega_1^2)\left(\alpha r_n^2 q^{\beta} + q + \nu r_n^2\right)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)}. \tag{18}$$

In order to determine $\bar{v}(r,q)$, we firstly write $\bar{v}_n(q)$ under the suitable form

$$\bar{v}_n(q) = \frac{2W_2\omega_2}{\pi r_n^2(q^2+\omega_2^2)} - \frac{2W_1\omega_1}{\pi r_n^2(q^2+\omega_1^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)}$$
$$-\frac{2W_2\omega_2 q}{\pi r_n^2(q^2+\omega_2^2)(\alpha r_n^2 q^{\beta}+q+\nu r_n^2)}$$
$$+\frac{2W_1\omega_1 q}{\pi r_n^2(q^2+\omega_1^2)(\alpha r_n^2 q^{\beta}+q+\nu r_n^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)} \tag{19}$$

and use the inverse Hankel transform formula (Sneddon, 1955)

$$\bar{v}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}\bar{v}_n(q) \tag{20}$$

and (A1) from Appendix A. Furthermore, in order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inversion Laplace transform method, writing

$$\frac{1}{\alpha r_n^2 q^{\beta} + q + \nu r_n^2} = \frac{1}{q^{\beta}\left[\nu r_n^2 q^{-\beta} + (q^{1-\beta} + \alpha r_n^2)\right]}$$
$$= \sum_{k=0}^{\infty}\frac{\left(-\nu r_n^2\right)^k q^{-\beta k-\beta}}{\left(q^{1-\beta}+\alpha r_n^2\right)^{k+1}} \tag{21}$$

and use Eq. (A2), where (Lorenzo and Hartley, 1999)

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty}\frac{(c)_j d^j t^{(j+c)a-b-1}}{\Gamma(j+1)\Gamma[(j+c)a-b]} \tag{22}$$

is the generalized G function and $(c)_j$ is the Pochhammer polynomial (Lorenzo and Hartley, 1999).

Finally, Eqs. (19)–(21), (A5) and (22) give the velocity field

$$v(r,t) = \frac{W_1R_1(R_2^2-r^2)\sin(\omega_1 t) + W_2R_2(r^2-R_1^2)\sin(\omega_2 t)}{(R_2^2-R_1^2)r}$$
$$-\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{J_1(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$
$$\times\left[W_2\omega_2J_1(R_1r_n)\int_0^t\cos(\omega_2(t-\tau))G_{1-\beta,-\beta k-\beta,k+1}\left(-\alpha r_n^2,\tau\right)d\tau\right.$$
$$\left.-W_1\omega_1J_1(R_2r_n)\int_0^t\cos(\omega_1(t-\tau))G_{1-\beta,-\beta k-\beta,k+1}\left(-\alpha r_n^2,\tau\right)d\tau\right]. \tag{23}$$

2.3. Calculation of the shear stress

Applying the Laplace transform to Eq. (8), we find that

$$\bar{\tau}(r,q) = (\mu + \alpha_1 q^\beta)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\bar{v}(r,q), \tag{24}$$

where

$$\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\bar{v}(r,q) = \frac{2R_1R_2}{(R_2^2 - R_1^2)r^2}\left(\frac{R_1W_2\omega_2}{q^2+\omega_2^2} - \frac{R_2W_1\omega_1}{q^2+\omega_1^2}\right)$$
$$+\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k \times \frac{J_1(R_1r_n)[(2/r)B_1(rr_n) - r_n\widetilde{B}_1(rr_n)]}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}$$
$$\left[W_2\omega_2J_1(R_1r_n)\frac{q}{q^2+\omega_2^2}\frac{(k+1)_j(-\alpha r_n^2)^j}{j!q^{k+j(1-\beta)+1}}\right.$$
$$\left.-W_1\omega_1J_1(R_2r_n)\frac{q}{q^2+\omega_1^2}\frac{(k+1)_j(-\alpha r_n^2)^j}{j!q^{k+j(1-\beta)+1}}\right] \tag{25}$$

has been obtained from (23) and (A8), where in the above relation

$$\widetilde{B}_1(rr_n) = J_0(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_0(rr_n).$$

Introducing (25) into (24), applying again the discrete inversion Laplace transform method to the obtained result and using (A3) and (A5), where

$$R_{a,b}(c,d,t) = \sum_{j=0}^{\infty}\frac{c^j(t-d)^{(j+1)a-b-1}}{\Gamma[(j+1)a-b]} \tag{26}$$

is R function (Lorenzo and Hartley, 1999), we find for the shear stress the expression

$$\tau(r,t) = \frac{2R_1R_2}{(R_2^2-R_1^2)r^2}\Big[\mu(R_1W_2\sin(\omega_2t) - R_2W_1\sin(\omega_1t))$$
$$+\alpha_1\left\{R_1W_2\omega_2R_{2,\beta}(-\omega_2^2,0,t) - R_2W_1\omega_1R_{2,\beta}(-\omega_1^2,0,t)\right\}\Big]$$
$$+\pi\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{J_1(R_1r_n)[(1/r)B_1(rr_n) - r_n\widetilde{B}_1(rr_n)]}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}$$
$$\times\Big[W_2\omega_2J_1(R_1r_n)\Big\{\mu\int_0^t\cos(\omega_2(t-\tau))$$
$$G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)d\tau + \alpha_1G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,t)$$
$$-\alpha_1\omega_2\int_0^t\sin(\omega_2(t-\tau))G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,\tau)d\tau\Big\}$$
$$-W_1\omega_1J_1(R_2r_n)\Big\{\mu\int_0^t\cos(\omega_1(t-\tau))$$
$$G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2,\tau)d\tau + \alpha_1G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,t)$$
$$-\alpha_1\omega_1\int_0^t\sin(\omega_1(t-\tau))G_{1-\beta,-\beta k-1,k+1}(-\alpha r_n^2,\tau)d\tau\Big\}\Big]. \tag{27}$$

3. Limiting cases

3.1. Solutions for ordinary second grade fluid ($\boldsymbol{\beta} \to 1$ *)*

Making $\beta \to 1$ into Eqs. (23) and (27) and using (A4) and (A7), we obtain the velocity field

$$v_S(r,t) = \frac{W_1R_1(R_2^2-r^2)\sin(\omega_1t) + W_2R_2(r^2-R_1^2)\sin(\omega_2t)}{(R_2^2-R_1^2)r}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}$$
$$\times\left[\frac{W_2\omega_2J_1(R_1r_n)}{\nu^2r_n^4 + \omega_2^2(1+\alpha r_n^2)^2}\left\{\nu r_n^2\left(\cos(\omega_2t) - \exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)\right)\right.\right.$$
$$\left.+\omega_2(1+\alpha r_n^2)\sin((\omega_2t))\right\} - \frac{W_1\omega_1J_1(R_2r_n)}{\nu^2r_n^4 + \omega_1^2(1+\alpha r_n^2)^2}$$
$$\times\left\{\nu r_n^2\left(\cos(\omega_1t) - \exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)\right)\right.$$
$$\left.\left.+\omega_1(1+\alpha r_n^2)\sin(\omega_1t)\right\}\right], \tag{28}$$

and associated shear stress

$$\tau_S(r,t) = \frac{2R_1R_2}{(R_2^2-R_1^2)r^2}[\mu(R_1W_2\sin(\omega_2t) - R_2W_1\sin(\omega_1t))$$
$$+\alpha_1(R_1W_2\omega_2\cos(\omega_2t) - R_2W_1\omega_1\cos(\omega_1t))]$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)[(1/r)B_1(rr_n) - r_n\widetilde{B}_1(rr_n)]}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}$$
$$\times(J_1(R_1r_n)g_2 - J_1(R_2r_n)g_1), \tag{29}$$

where in above Eq. (29)

$$g_m = W_m\omega_m\left[\frac{\mu}{\nu^2r_n^4 + \omega_m^2(1+\alpha r_n^2)^2}\left\{\nu r_n^2\left(\cos(\omega_mt) - \exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)\right)\right.\right.$$
$$\left.+\omega_m(1+\alpha r_n^2)\sin(\omega_mt)\right\} + \frac{\alpha_1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2t}{1+\alpha r_n^2}\right)$$
$$-\frac{\alpha_1\omega_m}{\nu^2r_n^4 + \omega_m^2(1+\alpha r_n^2)^2}\times\left\{\omega_m\left(1+\alpha r_n^2\right)\left(\exp(-\frac{\nu r_n^2t}{1+\alpha r_n^2}) - \cos(\omega_mt)\right)\right.$$
$$\left.+\nu r_n^2\sin(\omega_mt)\right\}$$

corresponding to the ordinary or classical second grade fluid, performing the same motion. The velocity field (28) and shear stress field (29) are same as obtained in the Ref. Mahmood et al. (in press).

Eqs. (28) and (29) correspond to the transient solutions of the velocity field and shear stress, respectively, of the second grade fluid. As time goes to infinity, these terms vanish and we are left with the corresponding steady-state solutions.

3.2. Solutions for Newtonian fluid

Making $\alpha \to 0$ (equivalently $\alpha_1 \to 0$) into Eqs. (28) and (29), velocity field and associated shear stress for Newtonian fluid, performing the same motion, can be obtained. For instance, the velocity field is

$$v_N(r,t) = \frac{W_1R_1(R_2^2-r^2)\sin(\omega_1t) + W_2R_2(r^2-R_1^2)\sin(\omega_2t)}{(R_2^2-R_1^2)r}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n) - J_1^2(R_2r_n)}\left[\frac{W_2\omega_2J_1(R_1r_n)}{\nu^2r_n^4+\omega_2^2}\left\{\nu r_n^2(\cos(\omega_2t)\right.\right.$$
$$\left.-\exp\left(-\nu r_n^2t\right)) + \omega_2\sin((\omega_2t))\right\}$$
$$-\frac{W_1\omega_1J_1(R_2r_n)}{\nu^2r_n^4+\omega_1^2}\left\{\nu r_n^2(\cos(\omega_1t) - \exp(-\nu r_n^2t))\right.$$
$$\left.\left.+\omega_1\sin(\omega_1t)\right\}\right]. \tag{30}$$

4. Concluding remarks

Our purpose in this paper was to establish exact analytic solutions for the velocity field and shear stress corresponding to the flow of a generalized second grade fluid between two infinite coaxial circular cylinders, by using Laplace and Hankel transforms. The motion of fluid was due to the simple harmonic sine oscillations of both cylinders around their common axis, with different angular frequencies ω_1 and ω_2 of their velocities. It is important to point out that the velocity field and the shear stress for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions by making $W_1 = 0$, $W_2 = W$ and $\omega_2 = \omega$ (when inner cylinder is at rest) or $W_1 = W$, $W_2 = 0$ and $\omega_1 = \omega$ (when outer cylinder is at rest). For instance, the velocity field for the flow of generalized second grade fluid, when inner cylinder is at rest and the outer cylinder is oscillating, is given by (from Eq. (23))

$$v(r,t) = \frac{WR_2(r^2 - R_1^2)\sin(\omega t)}{\left(R_2^2 - R_1^2\right)r} - \pi W\omega \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k \times \frac{J_1^2(R_1 r_n)B_1(r r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \int_0^t \cos(\omega(t-\tau)) \times G_{1-\beta,-\beta k-\beta,k+1}(-\alpha r_n^2, \tau)d\tau. \tag{31}$$

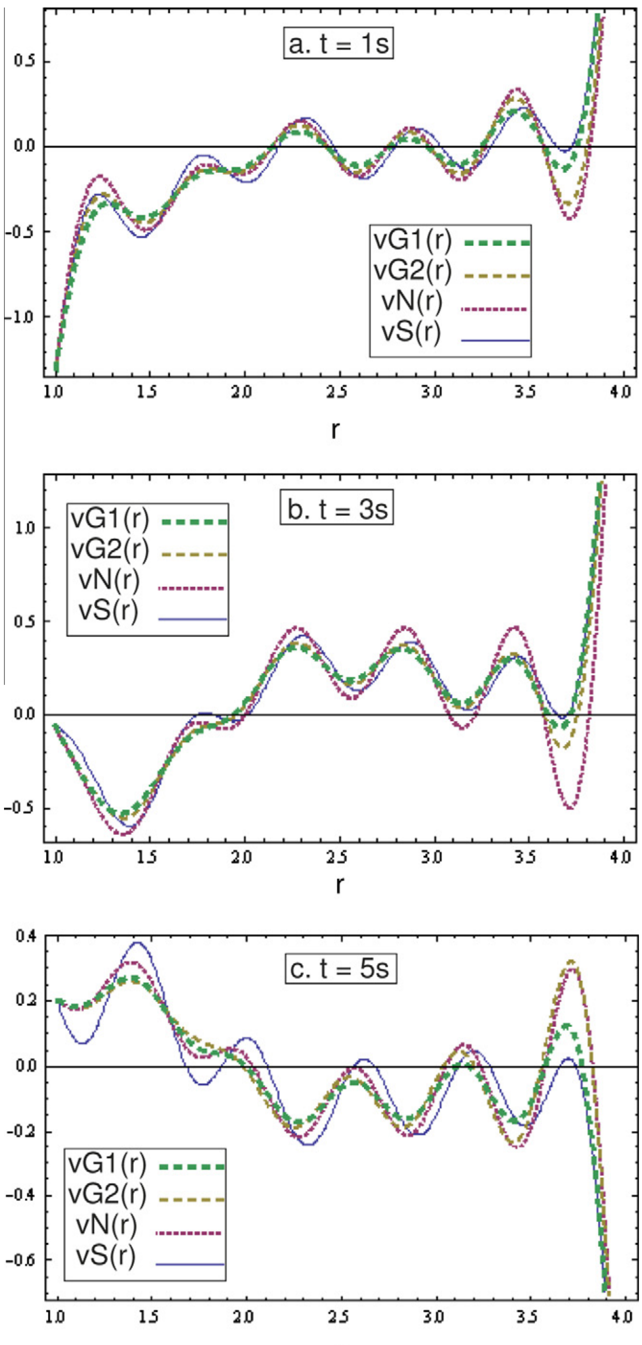

Figure 2 Velocity profiles for different values of time.

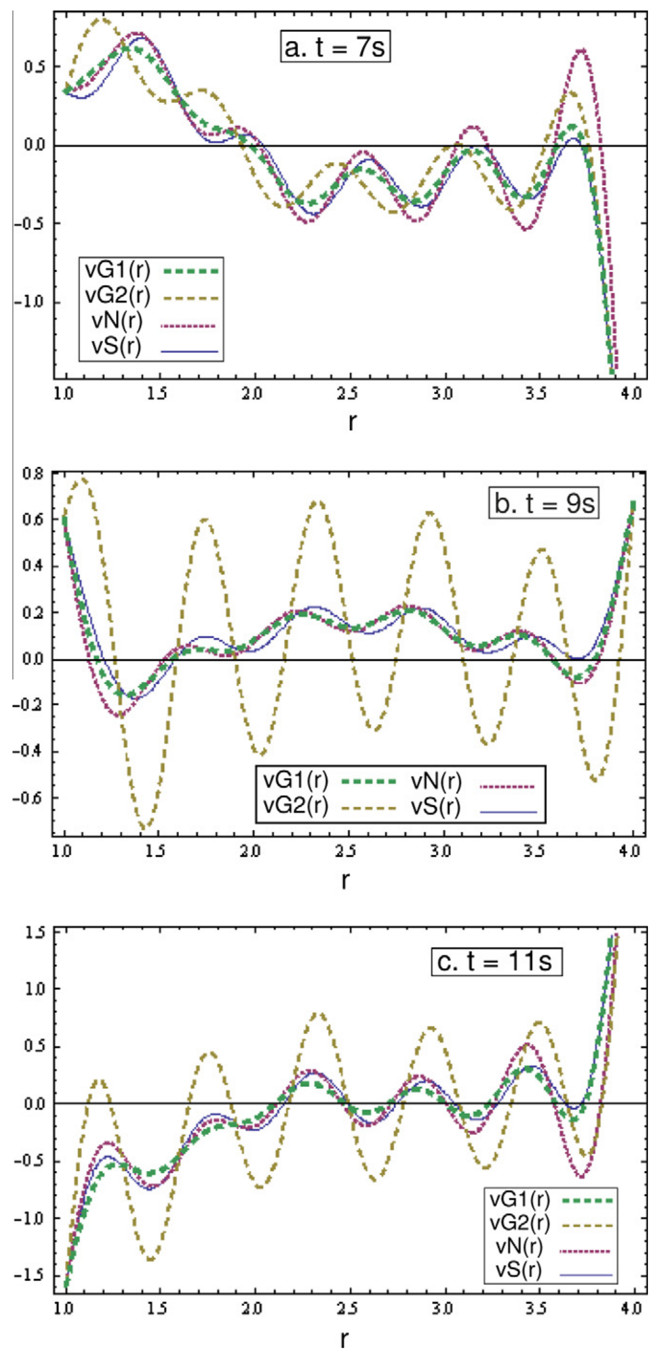

Figure 3 Velocity profiles for different values of time.

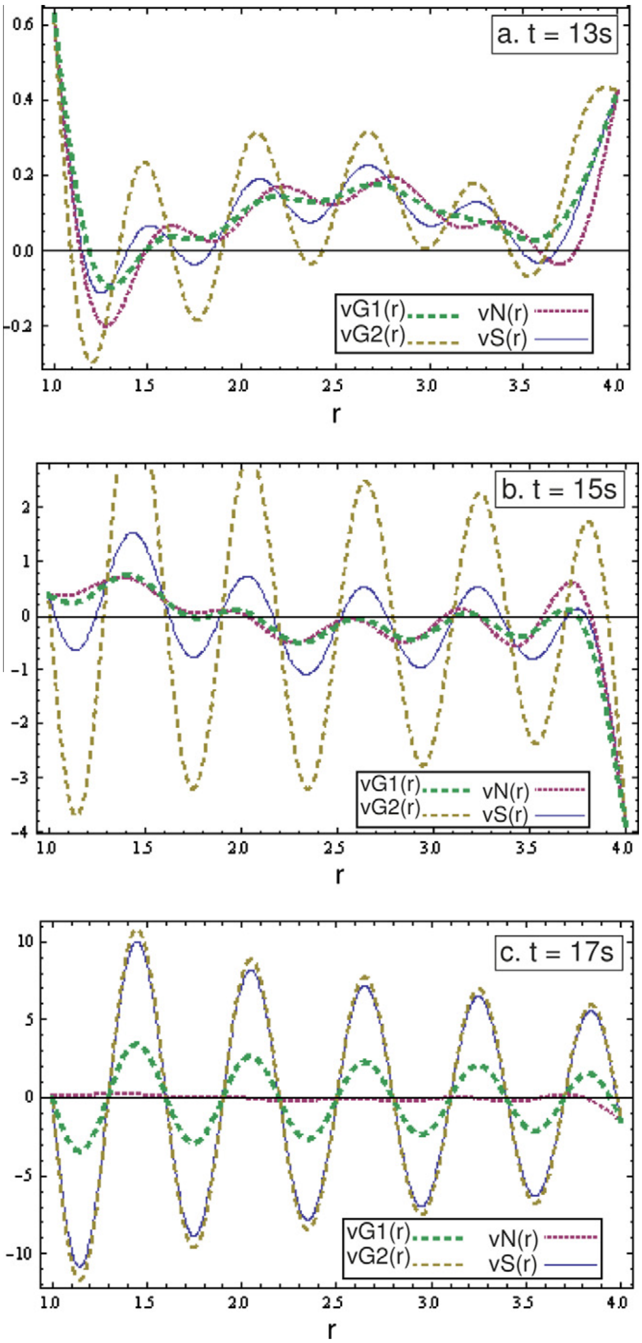

Figure 4 Velocity profiles for different values of time.

The solutions that have been obtained, presented under integral and series forms in terms of the generalized G and R functions, satisfy the governing equation and all imposed initial and boundary conditions and for $\beta \to 1$ reduce to the similar solutions for the second grade fluid. Finally, the solutions for the flow of Newtonian fluid for the similar flow between cylinders have been also recovered as a special case of our general solutions, when $\beta \to 1$ and $\alpha \to 0$.

Finally the graphical illustrations, Figs. 2–4, are given to show the comparison between the flow of generalized second grade (curves $vG1(r)$ and $vG2(r)$ for $\beta = 0.9$ and 0.6, respectively), second grade (curve $vS(r)$ for $\beta = 1$) and that of Newtonian fluid (curve $vN(r)$ for $\beta = 1$ and $\alpha \to 0$). These graphs also show the influence of the fractional coefficient β on the velocity $v(r,t)$. In all figures we consider $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$, $\alpha = 9 \times 10^{-3}$ and $\nu = 1.1746 \times 10^{-3}$ while SI units of parameters are used.

Appendix A. Some results used in the text:

The finite Hankel transform of the function

$$a(r) = \frac{AR_1(R_2^2 - r^2) + BR_2(r^2 - R_1^2)}{(R^2 - R_1^2)r} \tag{A1}$$

satisfying $a(R_1) = A$ and $a(R_2) = B$ is

$$a_n = \int_{R_1}^{R_2} ra(r)B_1(rr_n)dr = \frac{2B}{\pi r_n^2} - \frac{2A}{\pi r_n^2}\frac{J_1(R_2 r_n)}{J_1(R_1 r_n)}.$$

$$L^{-1}\left\{\frac{q^b}{(q^a - d)^c}\right\} = G_{a,b,c}(d,t);\ Re(ac - b) > 0,$$

$$Re(q) > 0, \quad \left|\frac{d}{q^a}\right| < 1. \tag{A2}$$

$$L^{-1}\left\{\frac{e^{-dq}q^b}{q^a - c}\right\} = R_{a,b}(c,d,t) = \sum_{j=0}^{\infty}\frac{c^j(t-d)^{(j+1)a-b-1}}{\Gamma[(j+1)a - b]}; \tag{A3}$$

$$d \geqslant 0, \quad Re((j+1)a - b) > 0, \quad Re(q) > 0.$$

$$R_{2,1}(-a^2, 0, t) = cos(at). \tag{A4}$$

If $u_1(t) = L^{-1}\{\bar{u}_1(q)\}$ and $u_2(t) = L^{-1}\{\bar{u}_2(q)\}$ then (A5)

$$L^{-1}\{\bar{u}_1(q)\bar{u}_2(q)\} = (u_1 * u_2)(t) = \int_0^t u_1(t-s)u_2(s)ds$$

$$= \int_0^t u_1(s)u_2(t-s)ds.$$

$$\frac{1}{z+a} = \sum_{k=1}^{\infty}(-1)^k\frac{z^k}{a^{k+1}}. \tag{A6}$$

$$\sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{0,-k-1,k+1}(-\alpha r_n^2, t)$$

$$= \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k \sum_{j=0}^{\infty}\frac{(k+1)_j(-\alpha r_n^2)^j}{j!}\frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k \frac{1}{(1+\alpha r_n^2)^{k+1}}\frac{t^k}{k!}$$

$$= \frac{1}{1+\alpha r_n^2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)^k$$

$$= \frac{1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right). \tag{A7}$$

$$\frac{d}{dr}[B_1(rr_n)] = r_n[J_0(rr_n)Y_1(R_2 r_n) - J_1(R_2 r_n)Y_0(rr_n)]$$

$$-\frac{1}{r}B_1(rr_n). \tag{A8}$$

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