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ORIGINAL ARTICLE

Generalization of the integral transform method to nonlinear heat-conduction problems in multilayered spherical media

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Abstract A method of solving heat conduction problems in multilayered spherical media is presented; it is based on producing an alternative formulation in which the inner layer of the composite sphere is lumped and the transient temperature distribution in the outer layer is obtained, in this, the nonlinear boundary conditions are treated as a source. The temperature distribution of the alternative formulation is then solved analytically. Different parametric studies are worked out and plotted to compare the two formulations for different values of Biot number and different thermal characteristic ratio with those obtained from a numerical solution developed using an explicit finite difference method, and to find the limiting criterion where breakdown of the approximation occurs.

1. Introduction

The conduction of heat in solids has numerous applications in various branches of science and engineering (Holaman, 1984; Ozisik, 1993; Carslaw and Jaeger, 1959; Liukov, 1968; Mikhailov and Ozisik, 1984; Siegel and Howell, 1980; Monteiro et al., 2009). Solving the problem of transient heat diffusion in two-layer composite is mathematically difficult (Mulhholland and Cobble, 1972; Salt, 1983; Frankel et al., 1987). Because of its difficulty, these problems are mostly treated numerically with some exception where analytic solutions are presented. In the field of linear heat transfer, the transient heat diffusion equation is linearized by considering the thermal properties to be independent of temperature, furthermore, the boundary conditions are also assumed to be linear. This class of linear transient heat diffusion has been treated in detail (Holaman, 1984; Ozisik, 1993; Carslaw and Jaeger, 1959; Liukov, 1968; Mikhailov and Ozisik, 1984; Siegel and Howell, 1980) with exact, approximate, and purely numerical methods. However, when the thermophysical properties and/or the volumetric heat source become temperature dependent, the field equation becomes nonlinear. In addition, if the temperature

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Nomenclature

C	specific heat
h_∞	heat transfer coefficient
k	thermal conductivity
N	normalization integral
r	radius
R_{in}	inner radius
R_{out}	outer radius
T	temperature
t	time
T_r	reference temperature
T_∞	temperature at external environment

Greek symbols

ϑ	dimensionless temperature
ϑ_∞	dimensionless temperature at external environment
ϑ_r	dimensionless reference temperature
Ω	nonlinear term
α	thermal diffusivity
β	Biot number
χ	dimensionless
δ	Dirac-delta function
ε	surface emissivity
ϕ	integral transform of temperature
γ	thermal diffusivity ratio
κ	thermal conductivity ratio
λ	eigenvalue
ρ	density
σ	Stefan–Boltzmann constant
τ	dimensionless time
ξ	ratio between inner radius and outer radius
ψ	eigenfunction

Subscripts

0	property estimate at initial temperature
∞	external environment
i	1, 2
in	inner surface
j	$1, 2, 3, \ldots$
n	$1, 2, 3, \ldots$
out	outer surface
r	reference

level becomes high, radiation and/or change of phase may occur, and, as a result, the boundary conditions become nonlinear (Davies, 1988). The problems of heat diffusion with nonlinear boundary conditions appear in combustion systems (Carslaw and Jaeger, 1959; Liukov, 1968), wherein the pre-ignition heating, the particle entering a furnace and traveling toward a flame front receives heat uniformly by thermal radiation from the furnace walls and losses heat uniformly by convection to the surrounding gases.

This work presents an analysis of transient heat diffusion in the two-region composite medium which simplifies the mathematical analysis of the problem and produces a governing equation with some auxiliary conditions. However, these problems were presented in (Abdel-Hamid and Frankel, 1991; Helal, 2003; Abd-El-Malek and Helal, 2006). The paper extends this work for solving nonlinear heat problems in spherical domains. In this work, one layer is lumped by assuming a uniform temperature all over at any given instance and treating the nonlinear term in the boundary condition as a source to obtain the associated linear homogeneous problem. However, the temperature is allowed to vary with time.

The validity of the proposed formulation is examined by comparing the temperature distribution in the first layer obtained from the approximation with the temperature distribution in the same layer obtained from the exact analytic solution. In order to quantify or find the limiting criterion where stop working of the approximation occurs, the two formulations are compared for different values of Biot number and different thermal characteristics ratio.

The advantage of using the integral transform techniques is that it provides a systematic and straight-forward approach to the solution of a certain class of heat equations (Monteiro et al., 2009; Cotta and Mikhailov, 1993; Cotta, 1994; Serfaty and Cotta, 1990; Naveira et al., 2007). The method is particularly suitable for the solution of both homogeneous and nonhomogeneous boundary value problems of heat conduction. The proposed method (Abdel-Hamid and Frankel, 1991; Helal, 2003; Abd-El-Malek and Helal, 2006) will be extended to solve the heat diffusion problem in a composite spherical finite region subject to nonlinear boundary conditions due to radiation exchange at the interface according to the fourth power law. The proposed method provides a straightforward methodology for heat equation problems subject to nonlinear boundary conditions and gives an algorithm that is more efficient and simpler than the other classical schemes, so the proposed method becomes applicable to solve a wider variety of nonlinear problems over the other methods. However, it is more transparent and requires less effort to arrive at the final results.

The utilities of the given method are summarized in the following sentence; this method presents an analysis of transient heat diffusion in the two-region composite medium which simplifies the mathematical analysis of the problem and produces a governing equation with some auxiliary conditions. The proposed method reduces the two-layer problem to a one-region problem with a new set of boundary conditions which compensates the effect of the outer layer. Solving the problem of transient heat diffusion subject to nonlinear boundary conditions due to radiation exchange at the interface according to the fourth power law. Treating the nonlinear term in the boundary condition as a source is employed to obtain the associated homogeneous problem. Then the problem can be solved by any conventional method.

2. Mathematical formulation

A two-layer sphere contains an inner region $0 \leqslant r \leqslant R_{in}$ and an outer region $R_{in} \leqslant r \leqslant R_{out}$ which are in perfect thermal

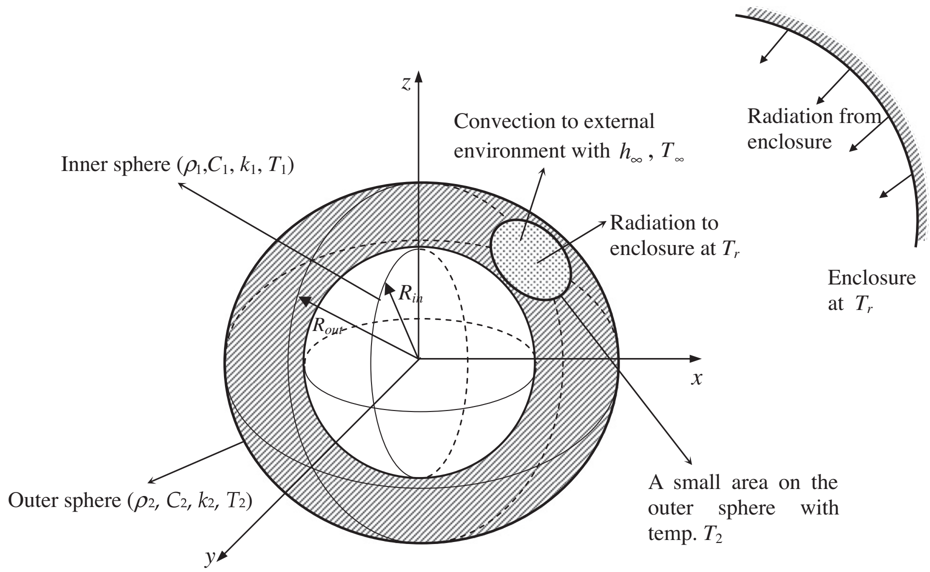

Figure 1 The exchanges of convection and radiation between a small area of solid sphere with the enclosure.

contact, k_1 and k_2 are the thermal conductivities. The outer surface transfers heat to a convecting medium maintained at T_∞ and having a heat transfer coefficient h_∞. Heat transfer from

the surface takes place by radiation to the enclosure. Fig. 1 illustrates a small area of solid sphere with emissivity ε and exchanges energy by radiation with the enclosure at temperature T_r. The transient heat conduction equation in the ith layer of the composite may be written as (Ozisik, 1993):

$$\frac{\partial^2 T_i(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial T_i(r,t)}{\partial r} = \frac{1}{\alpha_i}\frac{\partial T_i(r,t)}{\partial t}, \quad r_{i-1} \leqslant r \leqslant r_i, \ i = 1,2, \quad r_0 = 0, \ r_1 = R_{in}, \ r_2 = R_{out}, \ t > 0 \tag{2.1}$$

subject to the following boundary conditions:

$$T_1(r,t) = finite, \quad r = 0, \ t > 0 \tag{2.2}$$

$$T_1(r,t) = T_2(r,t), \quad r = R_{in}, \ t > 0 \tag{2.3}$$

$$k_1\frac{\partial T_1(r,t)}{\partial r} = k_2\frac{\partial T_2(r,t)}{\partial r}, \quad r = R_{in}, \ t > 0 \tag{2.4}$$

$$k_2\frac{\partial T_2(r,t)}{\partial r} + h_\infty(T_2(r,t) - T_\infty) = -\sigma\varepsilon\left[T_2^4(r,t) - T_r^4\right], \quad r = R_{out}, \ t > 0 \tag{2.5}$$

and the initial conditions:

$$T_i(r,t) = T_0, \quad r_{i-1} \leqslant r \leqslant r_i, \ i = 1,2, \ t = 0 \tag{2.6}$$

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where

$$\alpha_i = \frac{k_i}{\rho_i C_i} \quad i = 1,2.$$

is the thermal diffusivity including specific heat C_i and density ρ_i of the ith layer and σ is the Stefan–Boltzmann constant $\{\sigma = 5.6697 \times 10^{-8}\ \mathrm{W/(m^2\,K^4)}\}$

3. Nondimensionalization

For convenience, we recast the above system of the governing equations and auxiliary conditions into a dimensionless form. Redefining the variables as follows

$$\chi = \frac{r}{R_{out}}, \ \tau = \frac{\alpha_2 t}{R_{out}^2}, \ \vartheta_i(\chi,\tau) = \frac{T_i(r,t)}{T_0}, \ \xi = \frac{R_{in}}{R_{out}}, \ \kappa_i = \frac{k_i}{k_2}$$

$$\gamma_i = \frac{\alpha_i}{\alpha_2}, \ \beta_{out} = \frac{h_\infty R_{out}}{k_2}, \ \Omega = \frac{\sigma\varepsilon R_{out} T_0^3}{k_2}, \ \vartheta_r = \frac{T_r}{T_0}, \ \vartheta_\infty = \frac{T_\infty}{T_0}$$

$$f(\chi,\tau,\vartheta_2) = \beta_{out}\vartheta_\infty - \Omega\left(\vartheta_2^4(\chi,\tau) - \vartheta_r^4\right)$$

Introducing these new (dimensionless) variables into the governing equations and the auxiliary to obtain the problem in the more concise form:

$$\frac{\partial^2\vartheta_i(\chi,\tau)}{\partial\chi^2} + \frac{2}{\chi}\frac{\partial\vartheta_i(\chi,\tau)}{\partial\chi} = \frac{1}{\gamma_i}\frac{\partial\vartheta_i(\chi,\tau)}{\partial\tau}, \quad \chi_{i-1} \leqslant \chi \leqslant \chi_i, \ i = 1,2, \quad \chi_0 = 0, \ \chi_1 = \xi, \ \chi_2 = 1, \ \tau > 0 \tag{3.1}$$

subject to the following boundary conditions:

$$\vartheta_1(\chi,\tau) = finite, \quad \chi = 0, \ \tau > 0 \tag{3.2}$$

$$\vartheta_1(\chi,\tau) = \vartheta_2(\chi,\tau), \quad \chi = \xi, \ \tau > 0 \tag{3.3}$$

$$\kappa_1\frac{\partial\vartheta_1(\chi,\tau)}{\partial\chi} = \frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}, \quad \chi = \xi, \ \tau > 0 \tag{3.4}$$

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi} + \beta_{out}\vartheta_2(\chi,\tau) = f(\chi,\tau,\vartheta_2), \quad \chi = 1, \ \tau > 0 \tag{3.5}$$

and the initial conditions:

$$\vartheta_i(\chi,\tau) = 1, \quad \chi_{i-1} \leqslant \chi \leqslant \chi_i, \ i = 1,2, \ \tau = 0 \tag{3.6}$$

4. The alternative method

The first approach, a new formulation based on the work in (Abdel-Hamid and Frankel, 1991; Helal, 2003) is presented. The problem was formulated by lumping the inner layer and treating it as thin film. The inner layer of the composite sphere is lumped by assuming a uniform temperature throughout. The temperature, however, is allowed to vary with time. This assumption reduces the two-layer problem to a one-region problem with a new set of boundary conditions which compensates the effects of the second layer. Theses boundary conditions will be derived if we assume a radially averaged value of the temperature such that the quantity of heat due to the averaged temperature is equal to the quantity of heat into the inner sphere at any given instance. The equation of the temperature throughout the inner sphere is

$$\frac{1}{\chi^2}\frac{d}{d\chi}\left(\chi^2\frac{\partial\vartheta_1(\chi,\tau)}{\partial\chi}\right)=\frac{1}{\gamma_1}\frac{\partial\vartheta_1(\chi,\tau)}{\partial\tau},\quad 0\leqslant\chi\leqslant\xi,\ \tau>0 \tag{4.1}$$

Multiply both sides of Eq. (4.1) by $\chi^2 d\chi$ and integrating over χ, we obtain

$$\xi^2\frac{\partial\vartheta_1(\chi,\tau)}{\partial\chi}=\frac{1}{\gamma_1}\frac{\partial}{\partial\tau}\int_0^\xi\chi^2\vartheta_1(\chi,\tau)d\chi\quad 0\leqslant\chi\leqslant\xi,\tau>0 \tag{4.2}$$

To reduce the right hand side, we define a radially averaged value of the temperature denoted by T_1^* such that the quantity of heat due to the averaged temperature is equal to the quantity of heat into the inner sphere at any given instance i.e.:

$$\frac{4}{3}\pi C_1\rho_1\xi^3\vartheta_1^*(\tau)=\int_0^\xi 4\pi C_1\rho_1\chi^2\vartheta_1 d\chi, \tag{4.3}$$

or

$$\int_0^\xi\chi^2\vartheta_1 d\chi=\frac{\xi^3}{3}\vartheta_1^*(\tau). \tag{4.4}$$

Since we assumed the temperature is uniform at any distance throughout the inner sphere, i.e.,

$$\vartheta_1^*(\tau)=\vartheta_1(\xi,\tau)=\vartheta_2(1,\tau). \tag{4.5}$$

Substituting Eq. (4.4) into Eq. (4.2) and by using the above assumption with the aid of conditions (2.3) and (2.4) gives

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}=\frac{1}{3}\xi\frac{\kappa_1}{\gamma_1}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau},\quad\chi=\xi,\ \tau>0. \tag{4.6}$$

The last equation is the general boundary condition of the outer sphere and is considered the alternative condition after the inner sphere is lumped. The governing field equation of the lumped layer problem may be written as

$$\frac{\partial^2\vartheta_2(\chi,\tau)}{\partial\chi^2}+\frac{2}{\chi}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}=\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau},\quad\xi\leqslant\chi\leqslant 1,\ \tau>0, \tag{4.7}$$

subject to the following boundary conditions:

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}=\frac{1}{3}\xi\frac{\kappa_1}{\gamma_1}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau},\quad\chi=\xi,\tau>0, \tag{4.8}$$

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}+\beta_{out}\vartheta_2(\chi,\tau)=f(\chi,\tau,\vartheta_2),\quad\chi=1,\ \tau>0 \tag{4.9}$$

and the initial conditions:

$$\vartheta_2(\chi,\tau)=1,\quad\xi\leqslant\chi\leqslant 1,\ \tau=0 \tag{4.10}$$

The second approach is introduced here by solving the problem (4.7)–(4.10) discussed above by treating the nonlinearity term in the boundary condition (4.9) as a source in the differential Eq. (4.7). The nonlinear term must appear only at $\chi = 1$, and hence we make use of the Dirac-delta function to represent the function $f(\chi,\tau,\vartheta_2)$ as a source at $\chi = 1$. Also, the time dependent term in the first boundary condition (4.8) causes difficulties that arise in traditional finite integral transform technique. Treating this term as a source, the problem can be made easier to solve. The Dirac-delta function represents the term $\frac{1}{3}\xi\frac{\kappa_1}{\gamma_1}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau}$ as a source at $\chi = \xi$. The other condition is unchanged; therefore, we may consider the following equivalent problem:

$$\frac{\partial^2\vartheta_2(\chi,\tau)}{\partial\chi^2}+\frac{2}{\chi}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}+\frac{1}{3}\xi\frac{\kappa_1}{\gamma_1}\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau}\delta(\chi-\xi)+f(\chi,\tau,\vartheta_2)\delta(\chi-1)=\frac{\partial\vartheta_2(\chi,\tau)}{\partial\tau},\quad 0\leqslant\chi\leqslant 1,\ \tau>0 \tag{4.11}$$

subject to the following boundary and initial conditions:

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}=0,\quad\chi=\xi,\ \tau>0. \tag{4.12}$$

$$\frac{\partial\vartheta_2(\chi,\tau)}{\partial\chi}+\beta_{out}\vartheta_2(\chi,\tau)=0,\quad\chi=1,\ \tau>0 \tag{4.13}$$

and the initial conditions:

$$\vartheta_2(\chi,\tau)=1,\quad\xi\leqslant\chi\leqslant 1,\ \tau=0 \tag{4.14}$$

if we use the transformation $\Theta(\chi,\tau) = \chi\vartheta(\chi,\tau)$, the transformed system becomes:

$$\frac{\partial^2\Theta_2(\chi,\tau)}{\partial\chi^2}+\frac{1}{3}\xi\frac{\kappa_1}{\gamma_1}\frac{\partial\Theta_2(\chi,\tau)}{\partial\tau}\delta(\chi-\xi)+f(\chi,\tau,\Theta_2)\delta(\chi-1)=\frac{\partial\Theta_2(\chi,\tau)}{\partial\tau},\quad 0\leqslant\chi\leqslant 1,\ \tau>0 \tag{4.15}$$

subject to

$$\chi\frac{\partial\Theta_2(\chi,\tau)}{\partial\chi}-\Theta_2(\chi,\tau)=0,\quad\chi=\xi,\ \tau>0 \tag{4.16}$$

$$\frac{\partial\Theta_2(\chi,\tau)}{\partial\chi}+(\beta_{out}-1)\Theta_2(\chi,\tau)=0,\quad\chi=1,\tau>0 \tag{4.17}$$

$$\Theta_2(\chi,\tau)=1,\quad 0\leqslant\chi\leqslant 1,\tau=0 \tag{4.18}$$

5. Problem solution

The finite integral transform method is applied to determine the temperature distribution $\Theta_2(\chi,\tau)$. In the finite integral technique, the integral transform pair needed for the solution of a given problem is developed by considering representation of an arbitrary function in terms of the eigenfunctions corresponding to the given eigenvalue problem. Obtaining the required eigenvalue problem may be accomplished by considering the homogeneous part of the nonhomogeneous field equation and then employing separation of variables to obtain the following eigenvalue problem (Holaman, 1984; Ozisik, 1993; Carslaw and Jaeger, 1959; Liukov, 1968):

$$\frac{\partial^2\psi(\chi)}{\partial\chi^2}+\lambda_n^2\psi(\chi)=0, \tag{5.1}$$

subject to:

$$\chi \frac{\partial \psi(\chi)}{\partial \chi} - \psi(\chi) = 0, \quad \chi = \xi \tag{5.2}$$

$$\frac{\partial \psi(\chi)}{\partial \chi} + (\beta_{out} - 1)\psi(\chi) = 0, \quad \chi = 1 \tag{5.3}$$

The eigenfunction corresponding to the nth eigenvalue λ_n is given by:

$$\psi_n(\chi) = \frac{\lambda_n \xi \cos(\lambda_n \xi) - \sin(\lambda_n \xi)}{\lambda_n \xi \sin(\lambda_n \xi) + \cos(\lambda_n \xi)} \cos(\lambda_n \chi) + \sin(\lambda_n \chi) \tag{5.4}$$

λ_n's should be the positive roots of the following transcendental equation

$$\begin{aligned}&[\lambda_n \xi \sin(\lambda_n \xi) + \cos(\lambda_n \xi)][-\lambda_n \sin(\lambda_n) + (\beta_{out} - 1) \\ &\quad \times \cos(\lambda_n)] + [\lambda_n \xi \cos(\lambda_n \xi) - \sin(\lambda_n \xi)][\lambda_n \cos(\lambda_n) \\ &\quad + (\beta_{out} - 1) \sin(\lambda_n)] \\ &\quad = 0\end{aligned} \tag{5.5}$$

The orthogonality relation associated with the eigenvalue problem is given by

$$\int_\xi^1 \psi_n(\chi)\psi_m(\chi) d\chi = \begin{bmatrix} 0, & m \neq n \\ N(\lambda_n), & m = n \end{bmatrix} \tag{5.6}$$

where $N(\lambda_n)$ is the normalization integral.

The appropriate integral transform pair can now be defined as

Integral transform:

$$\phi_n(\tau) = \int_\xi^1 \psi_n(\chi)\Theta_1(\chi, \tau) d\chi, \quad n = 1, 2, 3, \ldots \tag{5.7}$$

Inversion formula:

$$\Theta_2(\chi, \tau) = \sum_{n=1}^{\infty} \frac{\psi_n(\chi)}{N(\lambda_n)} \phi_n(\tau) \tag{5.8}$$

Operating on Eq. (4.15) with $\int_\xi^1 \psi_n(\xi) d\xi$, we obtain:

$$\begin{aligned}&\int_\xi^1 \psi_n(\chi) \tfrac{\partial^2 \Theta_2(\chi,\tau)}{\partial \chi^2} d\chi + \tfrac{1}{3} \tfrac{\kappa_1}{\gamma_1} \tfrac{\partial}{\partial \tau} \int_\xi^1 \xi \psi_n(\chi) \Theta_2(\chi, \tau) \delta(\chi - \xi) d\chi \\ &\qquad + \int_\xi^1 \psi_n(\chi) f(\chi, \tau, \Theta_2) \delta(\chi - 1) d\chi \\ &= \tfrac{\partial}{\partial \tau} \int_\xi^1 \psi_n(\chi) \Theta_2(\chi, \tau) d\chi\end{aligned}$$

knowing that:

$$\int_\xi^1 \xi \psi_n(\chi) \Theta_2(\chi, \tau) \delta(\chi - \xi) d\chi = \xi \psi_n(\xi) \Theta_2(\xi, \tau) \tag{5.10}$$

and

$$\int_\xi^1 \psi_n(\xi) f(\chi, \tau, \Theta_2) \delta(\chi - 1) d\xi = \psi_n(1) f(1, \tau, \Theta_2) \tag{5.11}$$

Following the standard transformation procedures (Monteiro et al., 2009; Cotta and Mikhailov, 1993; Cotta, 1994; Serfaty and Cotta, 1990; Naveira et al., 2007) and with the aid of Eqs. (5.10) and (5.11), Eq. (5.9) reduces to the following first order nonhomogeneous ordinary differential equation:

$$\begin{aligned}&\frac{d\phi_n(\tau)}{d\tau} + \lambda_n^2 \phi_n(\tau) = \frac{1}{3} \frac{\kappa_1}{\gamma_1} \xi \psi_n(\xi) \frac{d\Theta_2(\xi, \tau)}{d\tau} \\ &\quad + \psi_n(1) f(1, \tau, \Theta_2), \quad n = 1, 2, 3, \ldots\end{aligned} \tag{5.12}$$

subject to a transformed initial condition given by introducing the integral transform Eq. (5.9a) into the initial condition (5.2c), therefore;

$$\phi_n(0) = \frac{\sin(\lambda_n \xi)^2 + \cos(\lambda_n \xi)^2 + \lambda_n \xi \sin(1 - \xi)\lambda_n - \cos(1 - \xi)\lambda_n}{\lambda_n[\lambda_n \xi \sin(\lambda_n \xi) + \cos(\lambda_n \xi)]} \tag{5.13}$$

The value of $\Theta^4(1, \tau)$ can be obtained from the inversion formula (5.9b), so $f(1, \tau, \Theta)$ can be evaluated, and hence (5.12) yields:

$$\begin{aligned}&\frac{d\phi_n(\tau)}{d\tau} + \lambda_n^2 \phi_n(\tau) = \frac{1}{3} \frac{\kappa_1}{\gamma_1} \xi \psi_n(\xi) \left[\sum_{i=1}^{\infty} \frac{\psi_i(\xi)}{N(\lambda_i)} \frac{d\phi_i(\tau)}{d\tau} \right] \\ &\quad + \psi_n(1) \left[\beta_{out} \Theta_\infty - \Omega \left[\left(\sum_{i=1}^{\infty} \frac{\psi_i(1)}{N(\lambda_i)} \phi_i(\tau) \right)^4 - \Theta_r^4 \right] \right], \\ &\quad n = 1, 2, 3, \ldots\end{aligned} \tag{5.14}$$

which is a system of nonlinear differential equations whose solution can be obtained using an appropriate numerical integration scheme. Runge–Kutta-4th order method for a finite number (say m) of the differential equations can be performed.

For $j = 0, 1, 2, \ldots$, we have

$$\begin{aligned}F_n(\phi_1^j, \phi_2^j, \ldots, \phi_m^j) &= \frac{d\phi_n^j(\tau)}{d\tau} \\ &= -\lambda_n^2 \phi_n^j(\tau) + \frac{1}{3} \frac{\kappa_1}{\gamma_1} \xi \psi_n(\xi) \left[\sum_{i=1}^{\infty} \frac{\psi_i(\xi)}{N(\lambda_i)} \frac{d\phi_i(\tau)}{d\tau} \right] \\ &\quad + \psi_n(1) \left[\beta \Theta_\infty - \Omega \left[\left(\sum_{i=1}^{m} \frac{\psi_i(1)}{N(\lambda_i)} \phi_n^j(\tau) \right)^4 - \Theta_r^4 \right] \right], \quad n \\ &= 1, 2, \ldots, m\end{aligned} \tag{5.15}$$

subject to the initial conditions

$$\phi_n^0(0) = \frac{\sin(\lambda_n \xi)^2 + \cos(\lambda_n \xi)^2 + \lambda_n \xi \sin(1 - \xi)\lambda_n - \cos(1 - \xi)\lambda_n}{\lambda_n[\lambda_n \xi \sin(\lambda_n \xi) + \cos(\lambda_n \xi)]} \tag{5.16}$$

The procedure of the method is as follows:

Consider, for $n = 1, 2, \ldots, m$, that

$$K_{1,n}^j = \Delta\tau F_n(\phi_1^j, \phi_2^j, \ldots, \phi_m^j) \tag{5.17}$$

$$K_{2,n}^j = \Delta\tau F_n\left(\phi_1^j + 0.5K_{1,1}^j, \phi_2^j + 0.5K_{1,2}^j, \ldots, \phi_m^j + 0.5K_{1,m}^j\right) \tag{5.18}$$

$$K_{3,n}^j = \Delta\tau F_n\left(\phi_1^j + 0.5K_{2,1}^j, \phi_2^j + 0.5K_{2,2}^j, \ldots, \phi_m^j + 0.5K_{2,m}^j\right) \tag{5.19}$$

$$K_{4,n}^j = \Delta\tau F_n\left(\phi_1^j + K_{3,1}^j, \phi_2^j + K_{3,2}^j, \ldots, \phi_m^j + K_{3,m}^j\right) \tag{5.20}$$

where $\Delta\tau$ is the time step and $j = 0, 1, 2, \ldots$.

Assume

$$\begin{aligned}\Delta\phi_n^j &= \frac{1}{6}\left[K_{1,n}^j + 2\left(K_{2,n}^j + K_{3,n}^j\right) + K_{4,n}^j\right], \quad n \\ &= 1, 2, \ldots, m\end{aligned} \tag{5.21}$$

Finally, the integral transforms for $n = 1, 2, \ldots, m$ can be obtained as

$$\phi_n^{j+1} = \phi_n^j + \Delta\phi_n^j \tag{5.22}$$

The calculation is started with $j = 0$ and the integral transforms $\phi_1^1, \phi_2^1, \ldots, \phi_m^1$ are evaluated because the initial integral transforms $\phi_1^0, \phi_2^0, \ldots, \phi_m^0$ are available by Eq. (5.16), knowing $\phi_1^1, \phi_2^1, \ldots, \phi_m^1$ the integral transform $\phi_1^2, \phi_2^2, \ldots, \phi_m^2$ at the end of the second time step are evaluated by setting $j = 1$. The procedure is repeated to calculate the integral transform ϕ_n at the subsequent time steps. Once $\phi_n(\tau)$ is obtained, the temperature distribution can be reconstructed through the use of the inversion formula.

6. Comparison between original and alternative formulations

A comparison between the solutions of the original formulation obtained from a numerical solution developed using an explicit finite difference method (Ozisik, 1993 pp. 436–498) and the alternative formulation obtained by solving the outer sphere after lumping the inner sphere as discussed in the previous section. To determine the parameters needed for a valid comparison, all the dimensionless variables which appeared in both solutions were examined. By inspection of these parameters, it is easy to conclude that the Biot number of the outer sphere (β_{out}), thickness ratio (ξ), thermal conductivity ratio (κ_i), the thermal diffusivity (γ_i) and the nonlinearity term Ω are the dimensionless parameters required for a valid comparison. The effect of each parameter is examined by allowing one parameter at a time to vary, while keeping the remaining parameters fixed. The study is conducted for a wide range of values of each parameter, however, only representative results are displayed in Figs. 3–11. In all figures, solid lines denote the results from the exact formulation (two-region composite) and dotted lines represent the results from the alternative formulation (lumped).

The effect of the thermal diffusivity γ_1 results in little deviation between the original and alternative formulations as shown in Figs. 2 and 3 when ξ, β_{out}, Ω and κ_1 are kept at fixed values. The difference in dimensionless temperature predicted from the two solution methods is within (0.1–0.5%).

The effect of the thermal conductivity ratio κ_1 on the temperature distribution from both the original and alternative formulations is displayed in Figs. 4 and 5. The difference between the original and alternative formulations decreases as κ_2 increases. A maximum difference of about 14% at $\kappa_2 = 0.5$ decreases to 0.02% at $\kappa_2 = 10$.

Figs. 6 and 7 display the effect of thickness ratio ξ on the two formulations. Results show that the difference between the original and alternative formulations increases as ξ increases. A maximum difference of about 5% is calculated when $\xi = 0.6$.

Figs. 8 and 9 display the effect of Biot number β_{in} on the two formulations. The difference between the original and alternative formulations is also increased as β_{out} increases. A maximum difference of 4% occurs when $\beta_{out} = 10$.

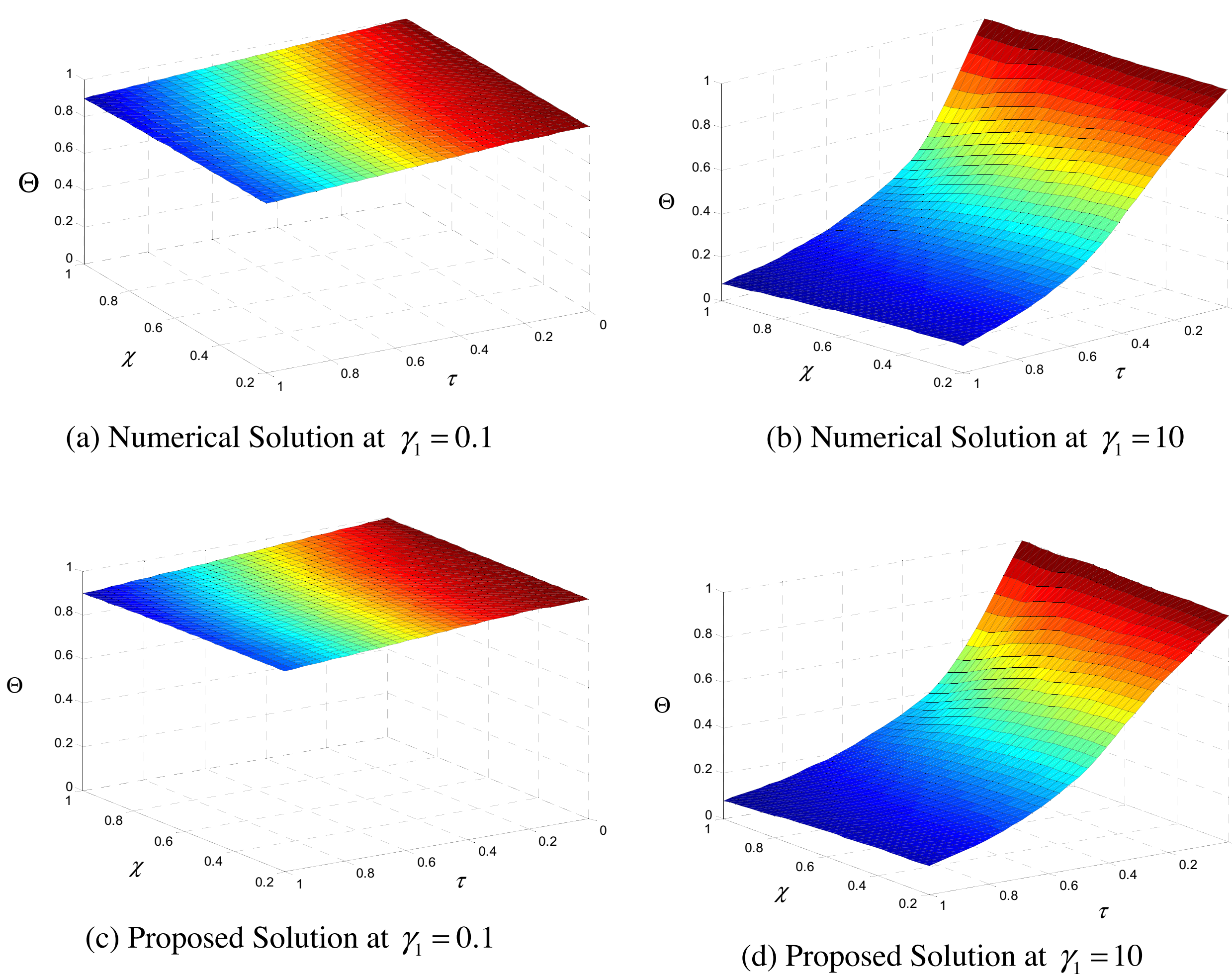
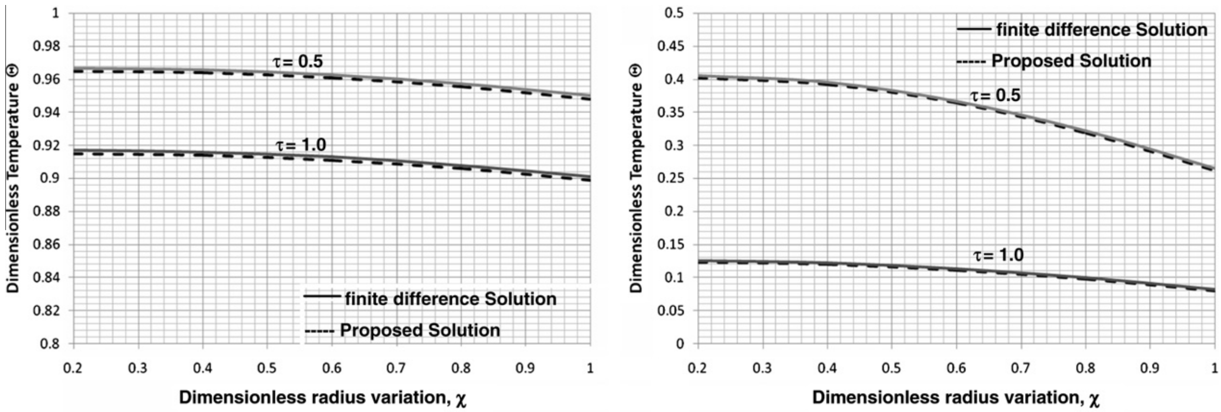
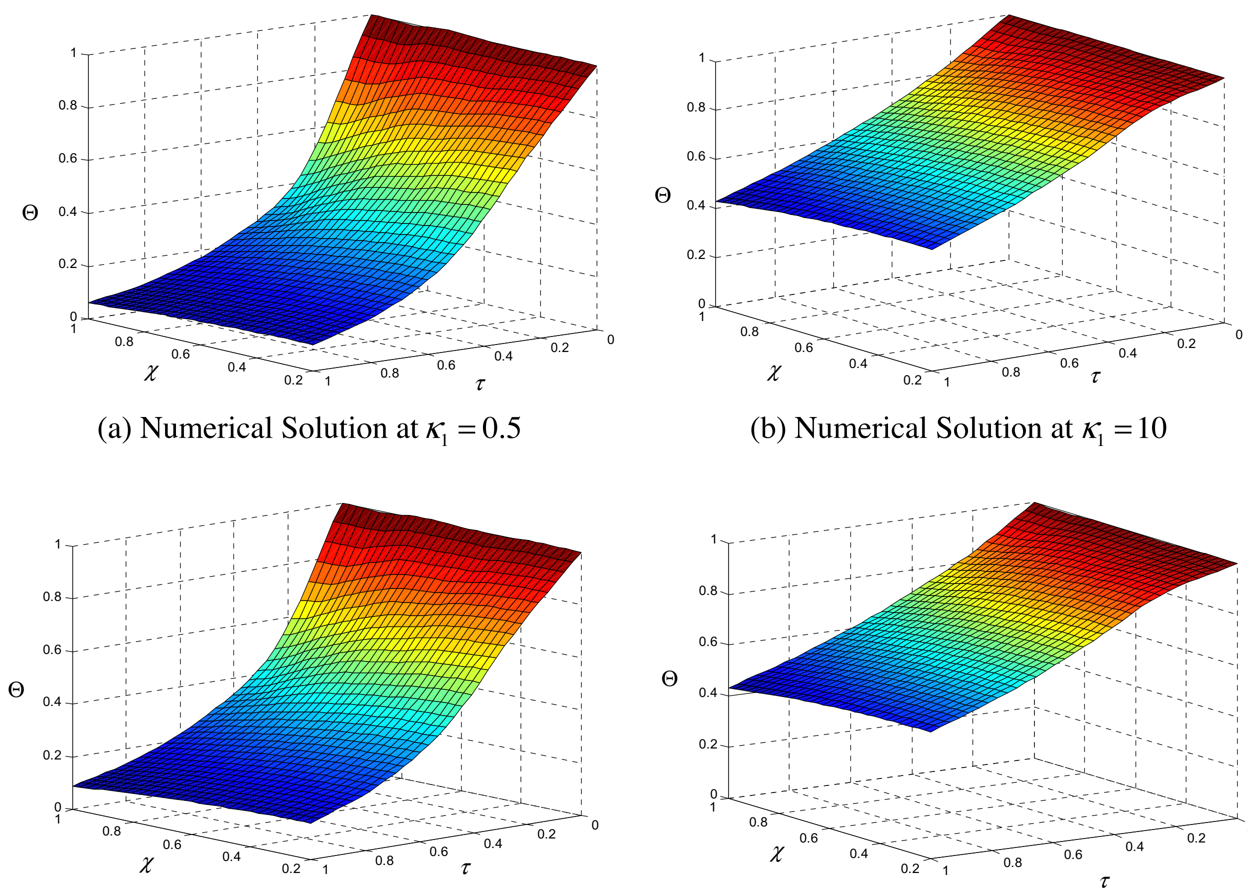


Figure 2 Dimensionless temperature as a function of the radius variation for different thermal diffusivity ratio values and for (β = 1, $\Omega = 0$, $\kappa_1 = 10$ and $\xi = 0.2$).



(a) $\gamma_1 = 0.1$ (b) $\gamma_1 = 10$

Figure 3 Effect of thermal diffusivity ratio on the accuracy of the proposed solution ($\beta = 1$, $\Omega = 0$, $\kappa_1 = 10$ and $\xi = 0.2$).



(a) Numerical Solution at $\kappa_1 = 0.5$ (b) Numerical Solution at $\kappa_1 = 10$

(c) Proposed Solution at $\kappa_1 = 0.5$ (d) Proposed Solution at $\kappa_1 = 10$

Figure 4 Dimensionless temperature as a function of the radius variation for different thermal conductivity ratio values and for ($\beta = 1$, $\Omega = 0$, $\gamma_1 = 1$ and $\xi = 0.2$).

The effect of stronger nonlinearity Ω is considered by taking $\Omega = 2$ as in Figs. 10 and 11. A difference within (0.1% to 0.4%) between the two solutions was noticed.

By the aid of the all cases studied, we can find that an acceptable factor is always achieved for using the lumped layer approximation. An acceptable approximation occurs when:

$$\frac{\beta_{out}\xi}{\kappa_1}\Omega < 0.1 \tag{6.1}$$

It can be shown that the ratio given by Eq. (6.1) reduces to another form as:

$$\frac{h_\infty R_{in}}{k_1}\Omega < 0.1 \tag{6.2}$$

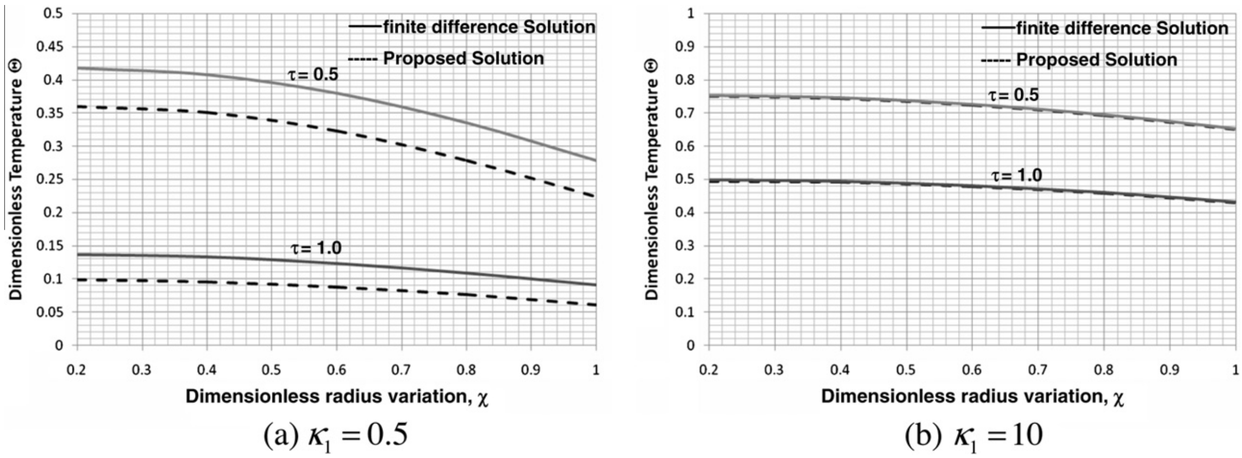


Figure 5 Effect of thermal conductivity ratio on the accuracy of the proposed solution (β = 1, Ω = 0, γ_1 = 1 and ξ = 0.2).

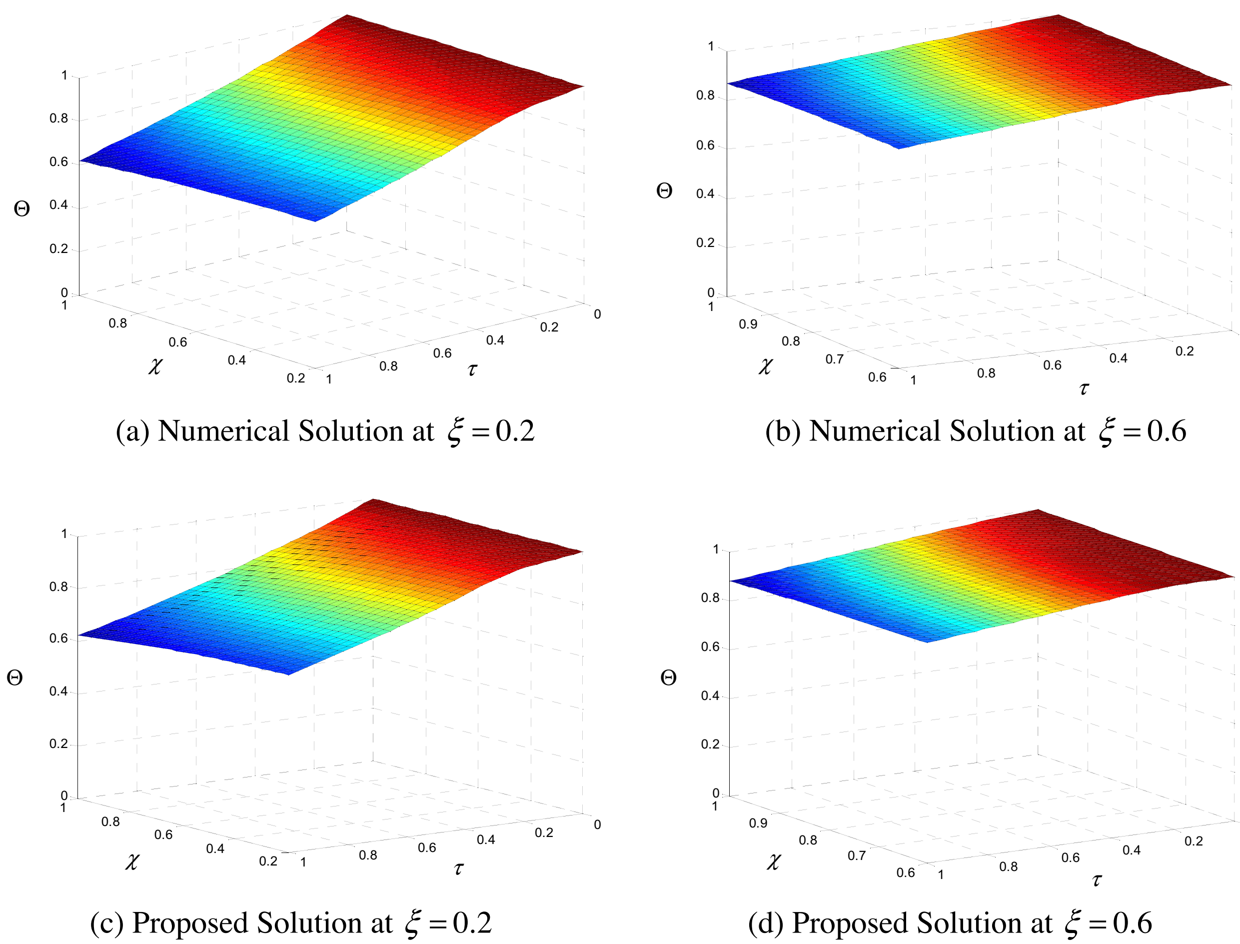


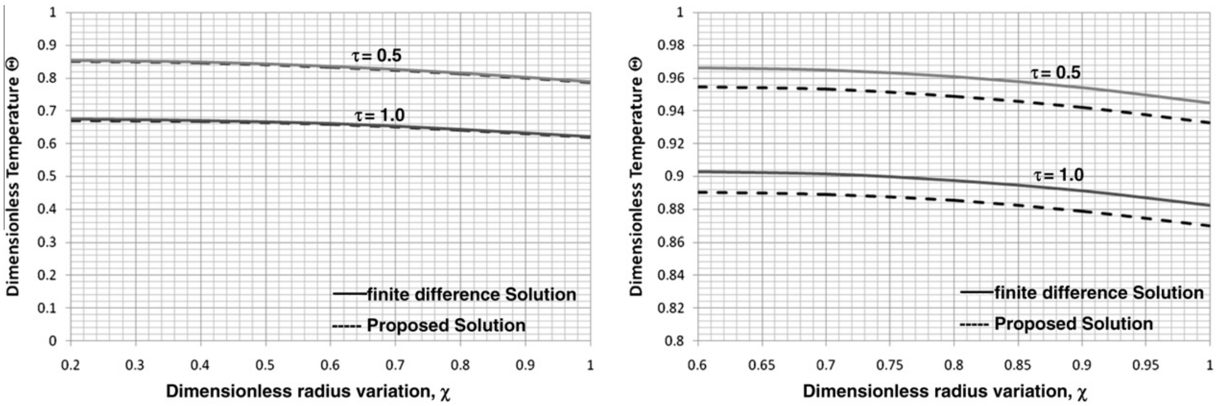
Figure 6 Dimensionless temperature as a function of the radius variation for different radius variation ratio values and for (β = 1, Ω = 0, γ_1 = 0.5, and κ_1 = 10).

It comes as no surprise that the left side of inequality (6.2) is the Biot number of the inner sphere determined based on the characteristics of the inner layer multiplying by the nonlinear term i.e.,

$$\beta_{in}\Omega < 0.1 \tag{6.3}$$

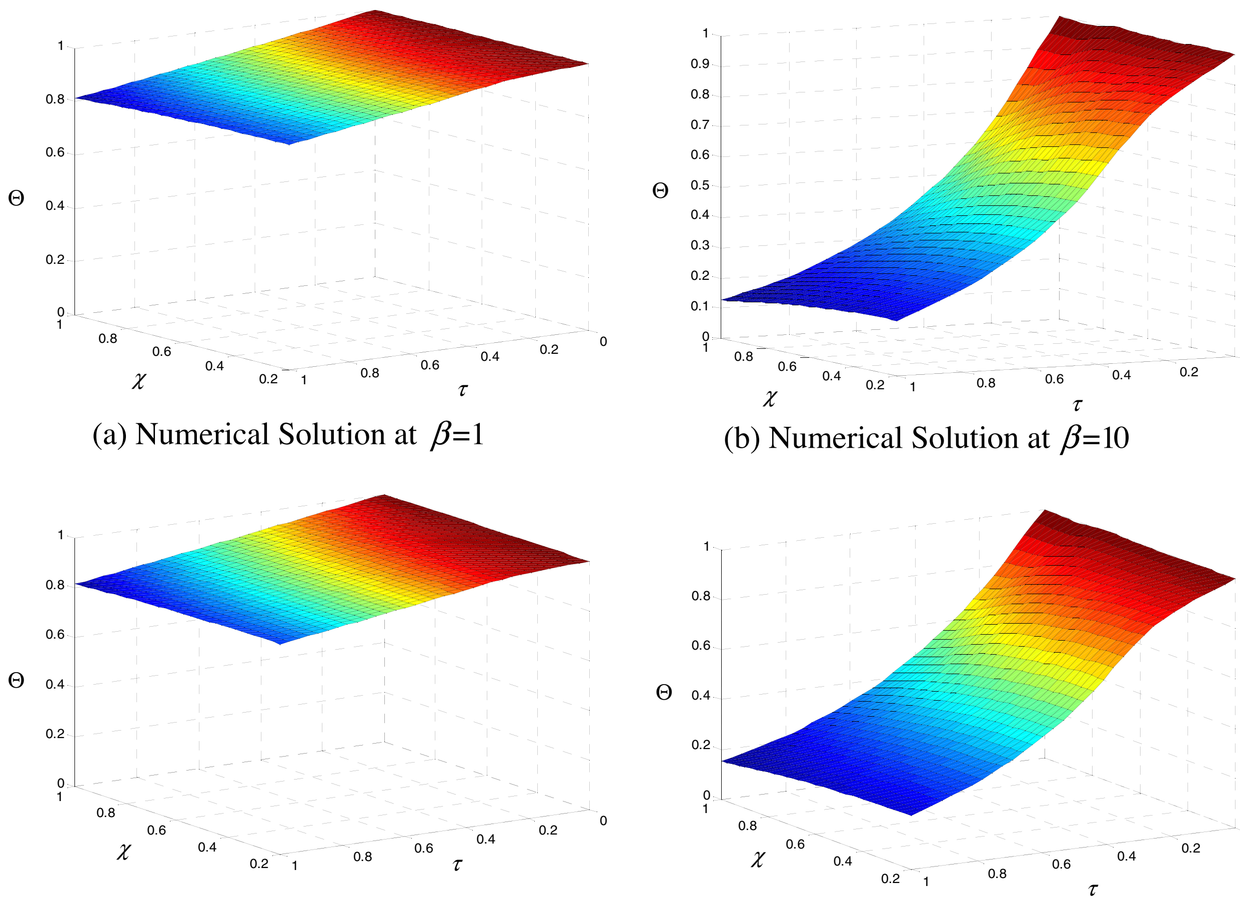
7. Conclusion

New formulation for the transient heat conduction problem in a two-layer composite sphere subject to a nonlinear boundary condition due to coupled convection–radiation heat exchange was considered. The problem was formulated by



(a) $\xi = 0.2a$ (b) $\xi = 0.6$

Figure 7 Effect of radius variation ratio on the accuracy of the proposed solution ($\beta = 1$, $\Omega = 0$, $\gamma_1 = 0.5$, and $\kappa_1 = 10$).



(a) Numerical Solution at β=1 (b) Numerical Solution at β=10

(c) Proposed Solution at $\beta = 1$ (d) Proposed Solution at $\beta = 10$

Figure 8 Dimensionless temperature as a function of the radius variation for different values of Biot number and for ($\Omega = 0$, $\gamma_1 = 0.1$, $\kappa_1 = 5$ and $\xi = 0.2$).

lumping the inner layer and treating it as thin film. The validity of the alternative formulation was examined by comparing the temperature distribution obtained from the alternative formulation to that obtained from the original formulation. The obtained results from the proposed approach are consistent with the results that were obtained from the explicit finite difference method. It was found from the numerical results that the thermal diffusivity ratio, has little effect on the approximation. Results show that the difference between the solutions of original and alternative formulations increases as thickness ratio increases. The difference between the solutions of original and alternative formulations also

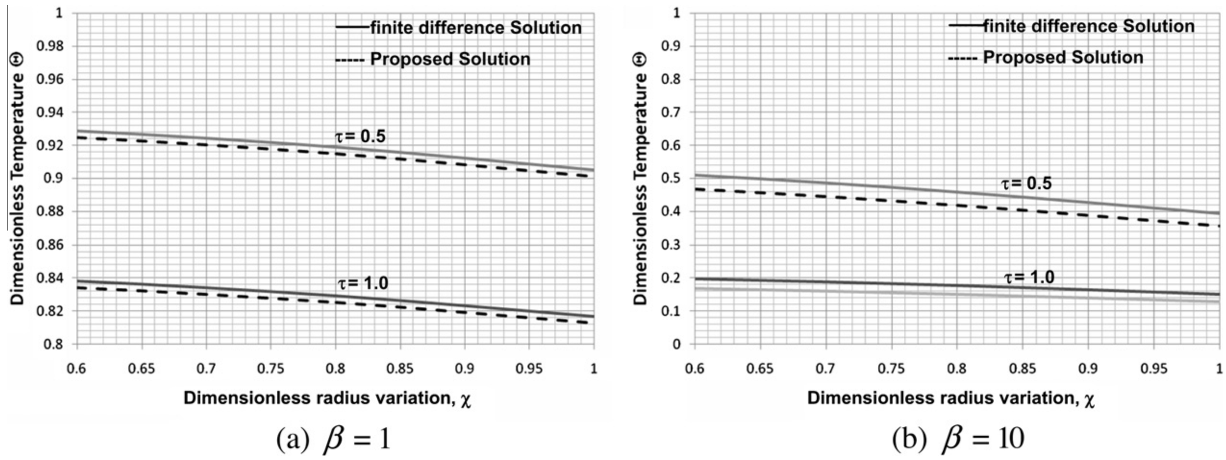


Figure 9 Effect of Biot number on the accuracy of the proposed solution ($\Omega = 0$, $\gamma_1 = 0.1$, $\kappa_1 = 5$ and $\xi = 0.2$).

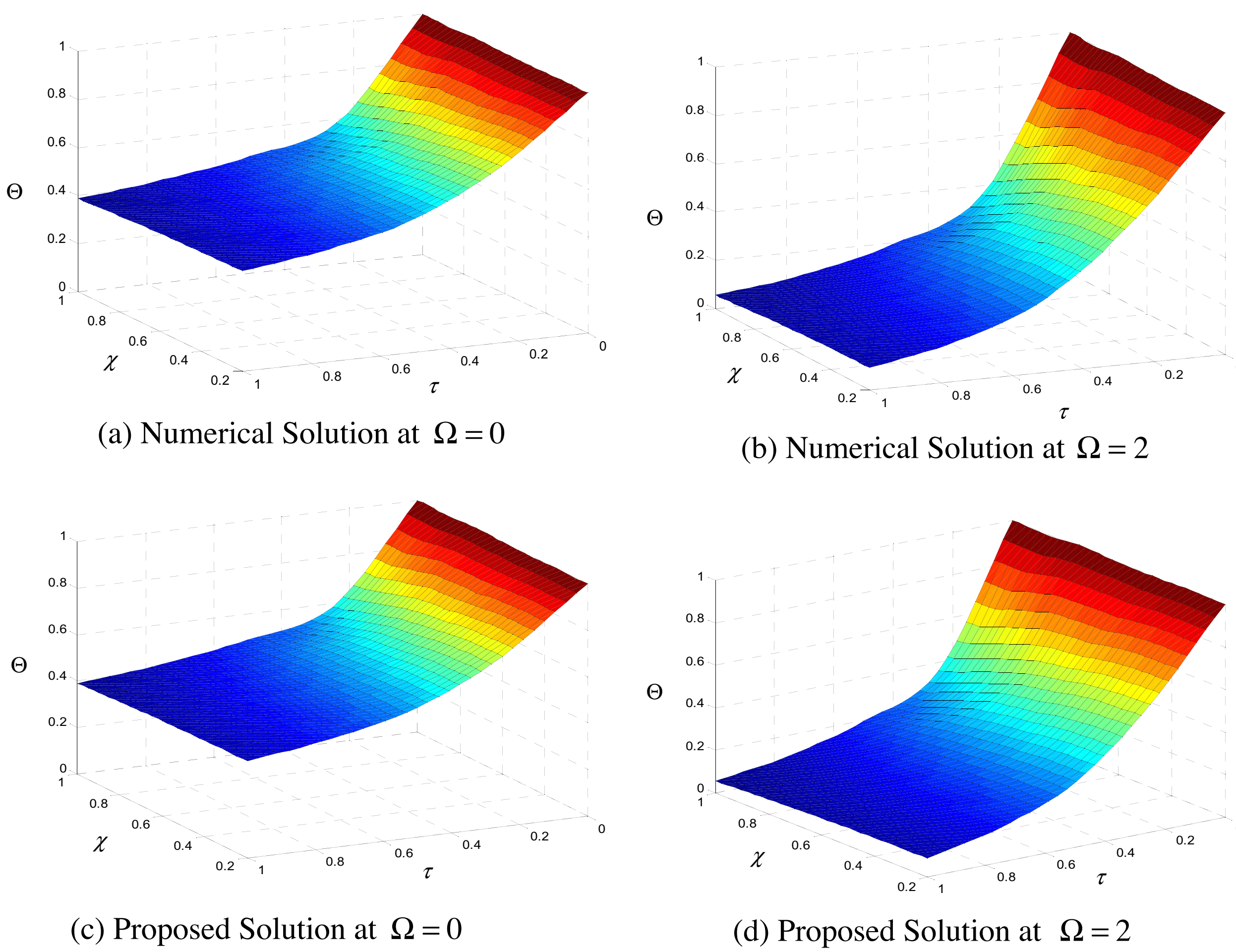
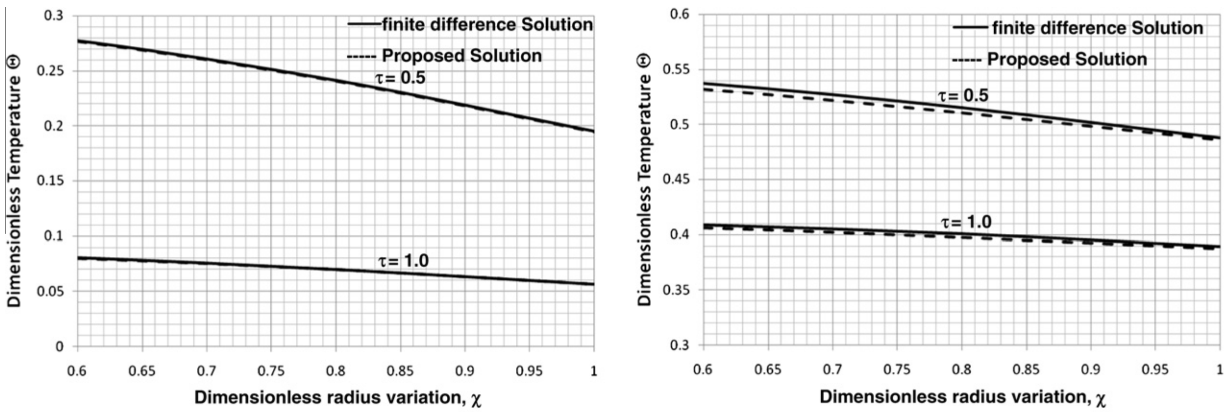


Figure 10 Dimensionless temperature as a function of the radius variation at the existence of nonlinear term Ω and for ($\beta = 0.1$, $\gamma_1 = 1$, $\kappa_1 = 1$ and $\xi = 0.2$).

increases as the Biot number increases. The effect of stronger nonlinearity Ω is considered by taking $\Omega = 2$. A difference within (0.1–0.4%) between the two solutions was noticed. It was also shown that a reasonable approximation can be achieved if the Biot number of the inner layer multiplying by the nonlinear term is less than (0.1).



(a) $\Omega = 0$ (b) $\Omega = 0$

Figure 11 Effect of the nonlinear term Ω on the accuracy of the proposed solution ($\beta = 0.1$, $\gamma_1 = 1$, $\kappa_1 = 1$ and $\xi = 0.2$).

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