Journal of King Saud University – Science 31 (2019) 946–955

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

A modified computational algorithm for solving systems of linear integro-differential equations of fractional order

Osama H. Mohammed *, Adyan M. Malik

Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad, Iraq

ARTICLE INFO

Article history:
Received 24 June 2018
Accepted 10 September 2018
Available online 13 September 2018

Keywords:
Fractional order integro-differential equations (FIDEs)
Caputo fractional order derivative

ABSTRACT

In this paper, a simple algorithm is applied to the systems of linear integro-differential equations of fractional order, the fractional derivative is described in the Caputo sense. The applied algorithm consists of a single series in which the unknown constants are determined by the simple means described in the manuscript. Some illustrative examples are given which confirm and illustrate the theoretical results.

1. Introduction

Fractional calculus is a generalization of classical calculus which provides an excellent instrument to describe memory and hereditary properties of various materials and process (Podlubny, 1999; Mohammed, 2016).

The Field of the fractional differential equations aroused the interest of many researchers in several areas including physics, engineering and finance (Atangana and Aguilar, 2017; Aguilar et al., 2017, 2018; Escamilla et al., 2018; Perez et al., 2018; Martinez and Aguilar, 2018).

FIDEs appeared in formulating processes in applied sciences such as physics, engineering, finance, biology ...etc. a lot of problems in acoustics, electromagnetics, viscoelasticity, hydrology and other types of application can be formulated by fractional order differential equations (Mittal, 2008).

FIDEs have been attacked by many researchers such as Momani, 2000 obtained local and global existence and uniqueness solution of the FIDEs. Rawashdeh, 2005 used the collocation spline method to approximate the solution of FIDEs. Mohammed (Mohammed, 2010) applied a reliable algorithm of homotopy analysis method in order to solve FIDEs.

The numerical solution of linear FIDEs have been investigated by Mohammed (2014) using least squares method shifted chebyshev polynomials.A comparative study among three numerical schemes for solving FIDEs was given in Kumar et al. (2017). While (Wang and Zhu, 2017) used wavelet method in order to solve volterra FIDEs. The Existence result and the approximate solution of quadratic FIDEs have been given in Hendi et al. (2019).

This paper concernes with the numerical solution of system of linear FIDEs.

$$ {}^{c}D_t^{\alpha} y_i(t) = f_i(t) + \int_0^t \sum_{j=1}^{m} k_{ij}(t,s) y_j(s) ds, 0 \leqslant t \leqslant 1, 0 < \alpha \leqslant 1, \quad i = 1, 2, \ldots, m \tag{1} $$

subject to the initial conditions

$$ y_i(0) = c_{i0}, i = 1, 2, 3, \ldots, m. \tag{2} $$

Using the power series method, where $k_{ij}(t,s),\ i,j = 1,2,\ldots,m$ are kernels of integral equations and $y_i(t),\ i = 1,2,\ldots,m$ are unknown solutions to be calculated, $f_i(t)$ are real valued functions, ${}^{c}D_t^{\alpha}$ denotes the Caputo fractional derivative of order α.

In the present paper we apply a modified series algorithm to solve systems of linear FIDEs. The algorithm consist of few steps explained in Section 4 which converges easily to the exact solution.

The main objective of this manuscript is to find the approximate solution of linear FIDEs by the uses of a modified series algorithm.

The setup of this manuscript is as follows: in Section 2, we recall some definitions of the fractional calculus. In Section 3, we introduce the power series method. Section 4, is about the formulation

* Corresponding author.
E-mail address: uhm@sc.nahrainuniv.edu.iq (O.H. Mohammed).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.09.005
1018-3647/

of the applied algorithm. Some illustrative applications are given in Section 5. Finally a conclusion have been drawn.

2. Fractional order derivatives and integrals

This section includes the definitions of Riemann-Liouville (R-L) fractional order integration, Riemann-Liouville (R-L) fractional order derivative and Caputo fractional order derivative.

Definition 1. A real function $f(t)$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p\ g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space C_μ^n iff $f^{(n)} \in C_\mu, n \in N$.

Definition 2. The (R-L) fractional integral operator of order $\alpha > 0$ of function $y \in C_\mu, \mu \geqslant -1$ is defined as:

$$I_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t\ (t-\tau)^{(\alpha-1)}\ y(\tau)\ d\tau, & \alpha > 0,\ \tau > 0 \\ y(t), & \alpha = 0. \end{cases}$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 3. The (R-L) fractional derivative of order $\alpha > 0$ of function $y \in C_{-1}^n$, is defined as:

$${}_0D_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t\ (t-\tau)^{(n-\alpha-1)}\ y(\tau)\ d\tau, & n-1 < \alpha \leqslant n \\ \frac{d^n y(t)}{dt^n}, & \alpha = n. \end{cases}$$

where n is an integer.

Definition 4. The Caputo fractional derivative of order $\alpha > 0$ of function $y \in C_{-1}^n$, is defined as:

$${}^cD_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t\ (t-\tau)^{(n-\alpha-1)}\ y^{(n)}(\tau)\ d\tau, & n-1 < \alpha \leqslant n \\ \frac{d^n y(t)}{dt^n}, & \alpha = n. \end{cases}$$

where n is an integer.

The following are some of the most important properties of the Riemann-Liouville fractional order integral and Caputo fractional order derivative (Podlubny, 1999).

(1) $I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha},\ \ \beta > -1,\ \ \alpha > 0.$
(2) ${}^cD_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha},\ \ \beta > -1,\ \ \alpha > 0.$
(3) ${}^cD_t^\alpha (I_t^\alpha y(t)) = y(t).$
(4) $(I_t^\alpha\ {}^cD_t^\alpha y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0^+)}{k!} t^k,\ \ t \geqslant 0,\ \ n-1 < \alpha \leqslant n.$

3. The method of power series

The power series method is the most transparent method of solution of fractional differential and integral equations. The idea of this method is to look for the solution in the form of a power series; the coefficients of the series must be determined (Podlubny, 1999).

Sometimes it is possible to find the general expression for the coefficients, at other times it is only possible to find the recurrence relation for the coefficients.

The solution in both cases could be computed approximately as partial sum of the series.

This of course illustrate the cause of why the power series method is often used for handling or solving applied problems.

Several examples have been solved and treated in Podlubny (1999) and Kilbas et al. (2006) by the aid of the power series method, let us mention a few of them:

$$(i)\quad {}_0D_t^\alpha y(t) = f(t),\ \ t > 0,\ \ 0 < \alpha < 1; \tag{3}$$

subject to

$$y(0) = 0. \tag{4}$$

The solution of problem (3) and (4) is given in Podlubny (1999) as:

$$y(t) = \sum_{n=0}^\infty a_n\ t^{n+\alpha},$$

where

$$a_n = \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)},\ \ n \geqslant 0.$$

$$(ii)\quad {}_0D_t^\alpha y(t) = f(t),\ \ t > 0,\ \ 0 < \alpha < 1; \tag{5}$$

subject to

$${}_0D_t^{\alpha-1} y(t)|_{t=0} = B. \tag{6}$$

where B is constant.

In this case the solution of the problem (5) and (6) has the following form (Podlubny, 1999):

$$y(t) = \sum_{n=0}^\infty a_n\ t^{n+\alpha-1},$$

where

$$a_0 = \frac{B}{\Gamma(\alpha)},$$

and

$$a_{n+1} = \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)},\ \ n \geqslant 0.$$

$$(iii)\quad {}_aD_t^\alpha y(t) + (t+1)^\alpha y(t) = 0,\ \ (a \leqslant -1). \tag{7}$$

The general solution in this case is given by Kilbas et al. (2006) as:

$$y(t) = (t+1)^{\alpha-1} \sum_{n=0}^\infty a_n\ (t+1)^{n\alpha},$$

where

$$a_{2k+1} = 0,\ \ (k \in N)$$

$$a_{2k} = (-1)^k Q(k) a_0$$

with

$$Q(k) = \prod_{j=1}^k \frac{\Gamma(2j\alpha)}{\Gamma((2j+1)\alpha)}.$$

Therefore, the general solution to Eq. (7) has the form:

$$y(t) = a_0 (t+1)^{\alpha-1} \left[1 + \sum_{k=1}^\infty (-1)^k Q(k) (t+1)^{2k\alpha}\right].$$

$$(iv)\quad {}_0D_t^\alpha y(t) = t^\beta g(t),\ \ t > 0,\ \ 0 < \alpha < 1,\ \ \beta > -1; \tag{8}$$

subject to

$$y(0) = 0. \tag{9}$$

The solution of the problem (8) and (9) that we have been looking for may be given in the form (Podlubny, 1999):

$$y(t) = \sum_{n=0}^\infty a_n t^{n+\alpha+\beta},$$

where

$$a_n = \frac{\Gamma(1+n+\beta)g^{(n)}(0)}{\Gamma(1+n+\alpha+\beta)\Gamma(n+1)},\ n \geqslant 0.$$

For more fractional order differential equations that have been solved with the help of the power series method see (Podlubny, 1999; Kilbas et al., 2006).

4. The algorithm

Consider the linear FIDEs given by problem (1) and (2), in this section a numerical algorithm will be applied in order to find the solution of problem (1) and (2) using the power series method.

For the existence of a unique solution of problem (1) and (2) we recommended to see (Heydari et al., 2014).

To start the algorithm, first operating I_t^α on both sides of Eq. (1), yields

$$I_t^{\alpha c}D_t^\alpha y_i(t) = I_t^\alpha f_i(t) + I_t^\alpha\left(\int_0^t \sum_{j=1}^m k_{ij}(t,s)y_j(s)ds\right), \quad (10)$$

$0 \leqslant t \leqslant 1,\ 0 < \alpha \leqslant 1,\ i = 1,2,\ \ldots,\ m.$

And secondly we suppose the solution of problem (1) and (2) to be in the following form:

$$y_i(t) = y_i(0) + \sum_{j=1}^m c_{ij}\ t^{\alpha+j-1},\ i = 1,2,3,\ldots,m. \quad (11)$$

where c_{ij} are constants to be determined.

The coefficients c_{ij} can be determined step by step and as follows: Set $c_{i0} = y_i(0), i = 1,2,\ldots,m$ and also $c_j = c_{ij}, j = 0,1,2,\ldots,m$ and suppose the solution of problem (1) and (2) be

$$y_i(t) = c_0 + c_1 t^\alpha,\ i = 1,2,\ldots,m. \quad (12)$$

where c_1 is unknown constant which can be determined by putting Eq. (12) into Eq. (10) and after simple calculations we obtained

$$(M_1c_1 - q_1)\ t^\alpha + \Phi_1(t) = 0. \quad (13)$$

where M_1 is $m \times m$ constant matrix, q_1 is $m \times 1$ constant vector, $\Phi_1(t)$ is a polynomial of order greater than α. Now neglecting $\Phi_1(t)$ and comparing the coefficients of t^α on both sides of Eq. (13) we obtained:

$$M_1c_1 - q_1 = 0. \quad (14)$$

From Eq. (14) we can find the value of c_1 which is the 1st step. In the next step let the solution of problem (1) and (2) will be given as:

$$y_i(t) = c_0 + c_1t^\alpha + c_2t^{\alpha+1}. \quad (15)$$

where c_0 and c_1 are known and c_2 is unknown constant. Now putting Eq. (15) into Eq. (10), we get

$$(M_2c_2 - q_2)t^{\alpha+1} + \Phi_2(t) = 0. \quad (16)$$

where $\Phi_2(t)$ is a polynomial of order gerater than $\alpha + 1$. Now comparing the coefficients of $t^{\alpha+1}$ and neglecting $\Phi_2(t)$ into Eq. (16), we obtain

$$M_2c_2 - q_2 = 0. \quad (17)$$

The unknown value c_2 in Eq. (17) can be determined easily. If we continue the same procedure for m iterations then we get a series of the following form

$$y_i(t) = c_0 + c_1t^\alpha + c_2t^{\alpha+1} + \ldots + c_mt^{\alpha+m-1},\ i = 1,2,3,\ldots,m. \quad (18)$$

Which gives an approximate solution for the exact solution of problem (1) and (2) in the given interval.

Next we shall prove that the suggested modified series converges absolutely.

4.1. Theorem

For any power series

$$\sum_{j=1}^m c_j\ (t-t_0)^{\alpha+j-1},\ i = 1,2,3,\ldots,m$$

there are only 3 possibilities for the values of t for which the series converges:

(i) The power series (11) converges only when $t = t_0$.
(ii) The power series (11) converges for all t.
(iii) There is a positive number R such that the power series (11) converges

if $|t - t_0| < R$ and diverges if $|t - t_0| > R$.

In case (iii) we say that R is the radius of convergence of the power series.

For convenience, we include the other two cases in this definition by defining $R = 0$ in case (i) and $R = \infty$ in case (ii).

Proof. To test the convergence, we shall use the ratio test, and as follows:

$$\lim_{j\to\infty}\left|\frac{c_{j+1}(t-t_0)^{\alpha+j}}{c_j(t-t_0)^{\alpha+j-1}}\right| = |t-t_0|\lim_{j\to\infty}\left|\frac{c_{j+1}}{c_j}\right| = |t-t_0|L.$$

where $L = \lim_{j\to\infty}\left|\frac{c_{j+1}}{c_j}\right|$.

Then the power series converges absolutely if $|t - t_0|L < 1$ and diverges if $|t - t_0|L > 1$.

If $|t - t_0|L = 1$, then the test inconclusive. □

5. Illustrative examples

In this section, we consider a linear systems of FIDEs, then the proposed method is applied in order to obtain the approximate results.

Example 1. Consider the following linear FIDEs.

$$\begin{cases} {}^cD_t^\alpha y_1(t) - 2t^2 - \int_0^t((t-s)y_1 + (t-s)y_2)\ ds = 0;\ 0 < \alpha \leqslant 1 \\ {}^cD_t^\alpha y_2(t) + 3t^2 + \frac{1}{5}t^5 - \int_0^t((t-s)y_1 - (t-s)y_2)\ ds = 0. \end{cases} \quad (19)$$

With the initial conditions

$$y_1(0) = 1,\ y_2(0) = 1. \quad (20)$$

And the exact solutions of problem (19) and (20) are $y_1(t) = 1 + t^3$ and $y_2(t) = 1 - t^3$.

According to Section 4, operating I_t^α on both sides of Eq. (19), we get:

$$\begin{cases} y_1(t) - 1 - 2I_t^\alpha(t^2) - I_t^\alpha(\int_0^t((t-s)y_1 + (t-s)y_2)\ ds) = 0; \\ y_2(t) - 1 + 3I_t^\alpha(t^2) + \frac{1}{5}I_t^\alpha(t^5) - I_t^\alpha(\int_0^t((t-s)y_1 - (t-s)y_2)\ ds) = 0. \end{cases} \quad (21)$$

Since $y_1(0) = 1$ and $y_2(0) = 1$. Therefore $c_{10} = 1,\ \ c_{20} = 1$ and hence $c_0 = \begin{pmatrix} c_{10} \\ c_{20} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha = 1 + c_{11}t^\alpha. \\ y_2(t) = c_{20} + c_{21}t^\alpha = 1 + c_{21}t^\alpha. \end{cases} \tag{22}$$

Substituting Eq. (22) into Eq. (21), we have

$$\begin{cases} c_{11}t^\alpha - 2I_t^\alpha(t^2) - I_t^\alpha(\int_0^t((t-s)(1+c_{11}s^\alpha) + (t-s)(1+c_{21}s^\alpha))ds) = 0; \\ c_{21}t^\alpha + 3I_t^\alpha(t^2) + \frac{1}{5}I_t^\alpha(t^5) - I_t^\alpha(\int_0^t((t-s)(1+c_{11}s^\alpha) - (t-s)(1+c_{21}s^\alpha))ds) = 0. \end{cases} \tag{23}$$

Hence, we have

$$\begin{cases} c_{11}t^\alpha - 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{c_{11}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} - \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} = 0; \\ c_{21}t^\alpha + 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{11}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} + \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} = 0. \end{cases} \tag{24}$$

Or

$$(M_1c_1 - q_1)\ t^\alpha + \Phi_1(t) = 0.$$

where

$$c_1 = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Phi_1(t) = \begin{pmatrix} P_{11}(t) \\ P_{21}(t) \end{pmatrix},$$

where

$$P_{11}(t) = -3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{c_{11}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} - \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}.$$

$$P_{21}(t) = 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{11}}{(\alpha+1)(\alpha+2)} \times \frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} + \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}.$$

Therefore by comparing coefficients of t^α and neglecting $\Phi_1(t)$, we get:

$c_{11} = 0$ and $c_{21} = 0$.

Next step, let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha + c_{12}t^{\alpha+1} = 1 + c_{12}t^{\alpha+1}. \\ y_2(t) = c_{20} + c_{21}t^\alpha + c_{22}t^{\alpha+1} = 1 + c_{22}t^{\alpha+1}. \end{cases} \tag{25}$$

Substituting Eq. (25) into Eq. (21), we have

$$\begin{cases} c_{12}t^{\alpha+1} - 2I_t^\alpha(t^2) - I_t^\alpha(\int_0^t((t-s)(1+c_{12}s^{\alpha+1}) + (t-s)(1+c_{22}s^{\alpha+1})ds) = 0; \\ c_{22}t^{\alpha+1} + 3I_t^\alpha(t^2) + \frac{1}{5}I_t^\alpha(t^5) - I_t^\alpha(\int_0^t((t-s)(1+c_{12}s^{\alpha+1}) - (t-s)(1+c_{22}s^{\alpha+1})ds) = 0. \end{cases} \tag{26}$$

Therefore after some simple calculations, we get:

$$\begin{cases} c_{12}t^{\alpha+1} - 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} - \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} = 0; \\ c_{22}t^{\alpha+1} + 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} + \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} = 0. \end{cases} \tag{27}$$

Or

$$(M_2c_2 - q_2)\ t^{\alpha+1} + \Phi_2(t) = 0;$$

where

$$c_2 = \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Phi_2(t) = \begin{pmatrix} P_{12}(t) \\ P_{22}(t) \end{pmatrix},$$

where

$$P_{12}(t) = -3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} - \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3}.$$

$$P_{22}(t) = 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{12}}{(\alpha+2)(\alpha+3)} \times \frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3} + \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3}.$$

Therefore by comparing coefficients of $t^{\alpha+1}$ and neglecting $\Phi_2(t)$, we have: $c_{12} = 0$ and $c_{22} = 0$.

Next step, let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha + c_{12}t^{\alpha+1} + c_{13}t^{\alpha+2} = 1 + c_{13}t^{\alpha+2}. \\ y_2(t) = c_{20} + c_{21}t^\alpha + c_{22}t^{\alpha+1} + c_{23}t^{\alpha+2} = 1 + c_{23}t^{\alpha+2}. \end{cases} \tag{28}$$

Substituting Eq. (28) into Eq. (21),we have

$$\begin{cases} c_{13}t^{\alpha+2} - 2I_t^\alpha(t^2) - I_t^\alpha(\int_0^t((t-s)(1+c_{13}s^{\alpha+2}) + (t-s)(1+c_{23}s^{\alpha+2}))ds) = 0; \\ c_{23}t^{\alpha+2} + 3I_t^\alpha(t^2) + \frac{1}{5}I_t^\alpha(t^5) - I_t^\alpha(\int_0^t((t-s)(1+c_{13}s^{\alpha+2}) - (t-s)(1+c_{23}s^{\alpha+2}))ds) = 0. \end{cases} \tag{29}$$

Therefore, we have

$$\begin{cases} c_{13}t^{\alpha+2} - 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} - \frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} - \frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} = 0; \\ c_{23}t^{\alpha+2} + 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} + \frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} = 0. \end{cases} \tag{30}$$

Or

$$(M_3c_3 - q_3)\ t^{\alpha+2} + \Phi_3(t) = 0;$$

where

$$c_3 = \begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_3 = \begin{pmatrix} 3\frac{\Gamma(3)}{\Gamma(\alpha+3)} \\ -3\frac{\Gamma(3)}{\Gamma(\alpha+3)} \end{pmatrix} \text{ and } \Phi_3(t) = \begin{pmatrix} P_{13}(t) \\ P_{23}(t) \end{pmatrix},$$

where

$$P_{13}(t) = -\frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} - \frac{c_{23}}{(\alpha+3)(\alpha+4)} \times \frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}.$$

$$P_{23}(t) = \frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5} - \frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4} + \frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}.$$

Therefore by comparing coefficients of $t^{\alpha+2}$ and neglecting $\Phi_3(t)$, we have: $c_{13} = 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}$ and $c_{23} = -3\frac{\Gamma(3)}{\Gamma(\alpha+3)}$.

Next step, let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha + c_{12}t^{\alpha+1} + c_{13}t^{\alpha+2} + c_{14}t^{\alpha+3} = 1 + 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + c_{14}t^{\alpha+3}. \\ y_2(t) = c_{20} + c_{21}t^\alpha + c_{22}t^{\alpha+1} + c_{23}t^{\alpha+2} + c_{24}t^{\alpha+3} = 1 - 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2} + c_{24}t^{\alpha+3}. \end{cases} \tag{31}$$

Substituting Eq. (31) into Eq. (21), we have

$$\begin{cases} 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}+c_{14}t^{\alpha+3}-2I_t^{\alpha}(t^2)-I_t^{\alpha}(\int_0^t((t-s)(1+3\frac{\Gamma(3)}{\Gamma(\alpha+3)}s^{\alpha+2}+c_{14}s^{\alpha+3})+(t-s)(1-3\frac{\Gamma(3)}{\Gamma(\alpha+3)}s^{\alpha+2}+c_{24}s^{\alpha+3}))ds)=0; \\ -3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}+c_{24}t^{\alpha+3}+3I_t^{\alpha}(t^2)+\frac{1}{5}I_t^{\alpha}(t^5)-I_t^{\alpha}(\int_0^t((t-s)(1+3\frac{\Gamma(3)}{\Gamma(\alpha+3)}s^{\alpha+2}+c_{14}s^{\alpha+3})-(t-s)(1-3\frac{\Gamma(3)}{\Gamma(\alpha+3)}s^{\alpha+2}+c_{24}s^{\alpha+3}))ds)=0. \end{cases} \tag{32}$$

Hence, we have

$$\begin{cases} 3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}+c_{14}t^{\alpha+3}-3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}-\frac{c_{14}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}-\frac{c_{24}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}=0; \\ -3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}+c_{24}t^{\alpha+3}+3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}+\frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5}-\frac{c_{14}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}-\frac{6}{(\alpha+3)(\alpha+4)}\frac{\Gamma(3)}{\Gamma(\alpha+3)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}+\frac{c_{24}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}=0. \end{cases} \tag{33}$$

Or

$$(M_4c_4-q_4)\ t^{\alpha+3}+\Phi_4(t)=0;$$

where

$$c_4=\begin{pmatrix} c_{14} \\ c_{24} \end{pmatrix},\quad M_4=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_4=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \Phi_4(t)=\begin{pmatrix} P_{14}(t) \\ P_{24}(t) \end{pmatrix},$$

where

$$P_{14}(t)=-\frac{c_{14}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}-\frac{c_{24}}{(\alpha+4)(\alpha+5)}\times\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}.$$

$$P_{24}(t)=\frac{1}{5}\frac{\Gamma(6)}{\Gamma(\alpha+6)}t^{\alpha+5}-\frac{c_{14}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}-\frac{6}{(\alpha+3)(\alpha+4)}$$

$$\frac{\Gamma(3)}{\Gamma(\alpha+3)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}+\frac{c_{24}}{(\alpha+4)(\alpha+5)}\frac{\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+5}.$$

Therefore by comparing coefficients of $t^{\alpha+3}$ and neglecting $\Phi_4(t)$, we have: $c_{14}=0$ and $c_{24}=0$.

Therefore the approximate solution of problem (19) and (20) for m = 4 becomes as:

$$y_1(t)=c_{10}+c_{11}t^{\alpha}+c_{12}t^{\alpha+1}+c_{13}t^{\alpha+2}+c_{14}t^{\alpha+3} \\ =1+3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}.$$

$$y_2(t)=c_{20}+c_{21}t^{\alpha}+c_{22}t^{\alpha+1}+c_{23}t^{\alpha+2}+c_{24}t^{\alpha+3} \\ =1-3\frac{\Gamma(3)}{\Gamma(\alpha+3)}t^{\alpha+2}.$$

Tables 1 and 2 represent the approximate solution of problem (19) and (20) for different values of α using the proposed algorithm compared with variational iteration method (VIM) (Nawaz, 2011), homotopy analysis method (HAM) (Zhang et al., 2011) and the exact solution when $\alpha = 1$.

Figs. 1 and 2 illustrates the approximate solution of problem (19) and (20) using the proposed algorithm for m = 4 and for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

Example 2. Consider the following linear FIDEs.

$$\begin{cases} {}^{c}D_t^{\alpha}y_1(t)=1+t-\frac{t^3}{3}+\int_0^t((t-s)y_1(s)+(t-s)y_2(s))\ ds;\ 0<\alpha\leqslant 1 \\ {}^{c}D_t^{\alpha}y_2(t)=1-t-\frac{t^4}{12}+\int_0^t((t-s)y_1(s)-(t-s)y_2(s))\ ds. \end{cases} \tag{34}$$

With the initial conditions

$$y_1(0)=0,\ y_2(0)=0. \tag{35}$$

And the exact solutions of problem (34) and (35) are $y_1(t)=t+\frac{t^2}{2}$ and $y_2(t)=t-\frac{t^2}{2}$.

Operating I_t^{α} on both sides of Eq. (34), we get

$$\begin{cases} y_1(t)=I_t^{\alpha}(1)+I_t^{\alpha}(t)-\frac{1}{3}I_t^{\alpha}(t^3)+I_t^{\alpha}(\int_0^t((t-s)y_1(s)+(t-s)\,y_2(s))\ ds); \\ y_2(t)=I_t^{\alpha}(1)-I_t^{\alpha}(t)-\frac{1}{12}I_t^{\alpha}(t^4)+I_t^{\alpha}(\int_0^t((t-s)y_1(s)-(t-s)y_2(s))\ ds). \end{cases} \tag{36}$$

Since $y_1(0)=0$ and $y_2(0)=0$. Therefore $c_{10}=0$, $c_{20}=0$ and hence $c_0=\begin{pmatrix} c_{10} \\ c_{20} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let

$$\begin{cases} y_1(t)=c_{10}+c_{11}t^{\alpha}=c_{11}t^{\alpha}. \\ y_2(t)=c_{20}+c_{21}t^{\alpha}=c_{21}t^{\alpha}. \end{cases} \tag{37}$$

Substituting Eq. (37) into Eq. (36), we have

$$\begin{cases} c_{11}t^{\alpha}=I_t^{\alpha}(1)+I_t^{\alpha}(t)-\frac{1}{3}I_t^{\alpha}(t^3)+I_t^{\alpha}(\int_0^t((t-s)(c_{11}s^{\alpha})+(t-s)(c_{21}s^{\alpha}))ds); \\ c_{21}t^{\alpha}=I_t^{\alpha}(1)-I_t^{\alpha}(t)-\frac{1}{12}I_t^{\alpha}(t^4)+I_t^{\alpha}(\int_0^t((t-s)(c_{11}s^{\alpha})-(t-s)(c_{21}s^{\alpha}))ds). \end{cases} \tag{38}$$

Table 1
Comparison between the approximate solution of $y_1(t)$ of problem (19) and (20) using the proposed algorithm for m = 4 and for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

t	$\alpha = 0.25$ m = 4	$\alpha = 0.5$ m = 4	$\alpha = 0.75$ m = 4	$\alpha = 1$ m = 4	(VIM) $\alpha = 1$	(HAM) $\alpha = 1$	The exact solution $\alpha = 1$
0	1	1	1	1	1	1	1
0.1	1.006	1.003	1.002	1.001	1.001	1.001	1.001
0.2	1.027	1.018	1.012	1.008	1.008	1.008	1.008
0.3	1.067	1.049	1.036	1.027	1.029	1.027	1.027
0.4	1.127	1.101	1.08	1.064	1.07	1.064	1.064
0.5	1.21	1.177	1.149	1.125	1.141	1.125	1.125
0.6	1.317	1.279	1.245	1.216	1.248	1.216	1.216
0.7	1.448	1.41	1.375	1.343	1.403	1.343	1.343
0.8	1.605	1.572	1.541	1.512	1.614	1.512	1.512
0.9	1.789	1.768	1.748	1.729	1.893	1.729	1.729
1	2	2	2	2	2.25	2	2

Table 2
Comparison between the approximate solution of $y_2(t)$ of problem (19) and (20) using the proposed algorithm for m = 4 for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

t	$\alpha = 0.25$ m = 4	$\alpha = 0.5$ m = 4	$\alpha = 0.75$ m = 4	$\alpha = 1$ m = 4	(VIM) $\alpha = 1$	(HAM) $\alpha = 1$	The exact solution $\alpha = 1$
0	1	1	1	1	1	1	1
0.1	0.994	0.997	0.998	0.999	0.999	0.999	0.999
0.2	0.973	0.982	0.988	0.992	0.992	0.992	0.992
0.3	0.933	0.951	0.964	0.973	0.973	0.973	0.973
0.4	0.873	0.899	0.92	0.936	0.936	0.936	0.936
0.5	0.79	0.823	0.851	0.875	0.874	0.874	0.875
0.6	0.683	0.721	0.755	0.784	0.782	0.782	0.784
0.7	0.552	0.59	0.625	0.657	0.653	0.653	0.657
0.8	0.395	0.428	0.459	0.488	0.479	0.479	0.488
0.9	0.211	0.232	0.252	0.271	0.253	0.253	0.271
1	0	0	0	0	-0.033	-0.033	0

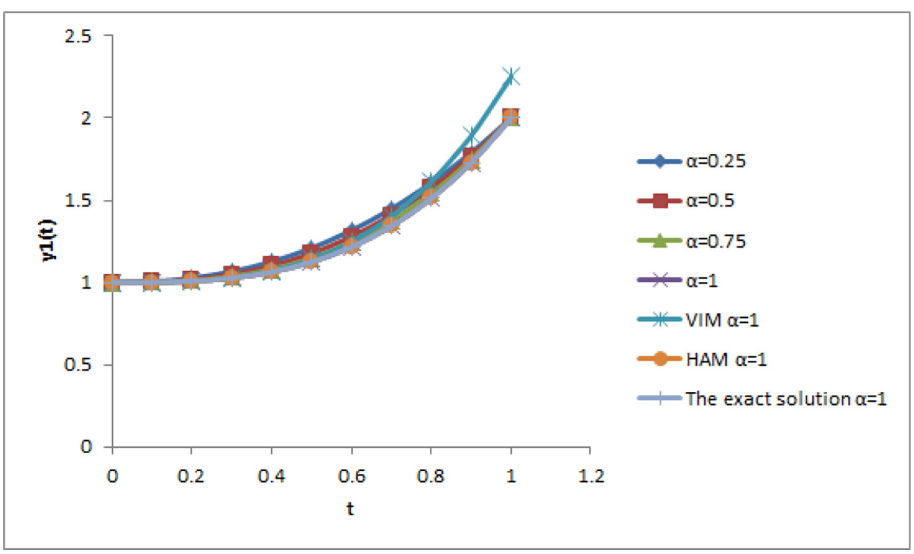

Fig. 1. The approximate solution of $y_1(t)$ using the proposed algorithm for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

Hence, we have

$$\begin{cases} c_{11}t^{\alpha} = \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1} - \frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)}t^{\alpha+3} + \frac{c_{11}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} + \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}; \\ c_{21}t^{\alpha} = \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1} - \frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)}t^{\alpha+4} + \frac{c_{11}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2} - \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}. \end{cases} \quad (39)$$

Or

$$(M_1 c_1 - q_1)\, t^{\alpha} + \Phi_1(t) = 0.$$

where

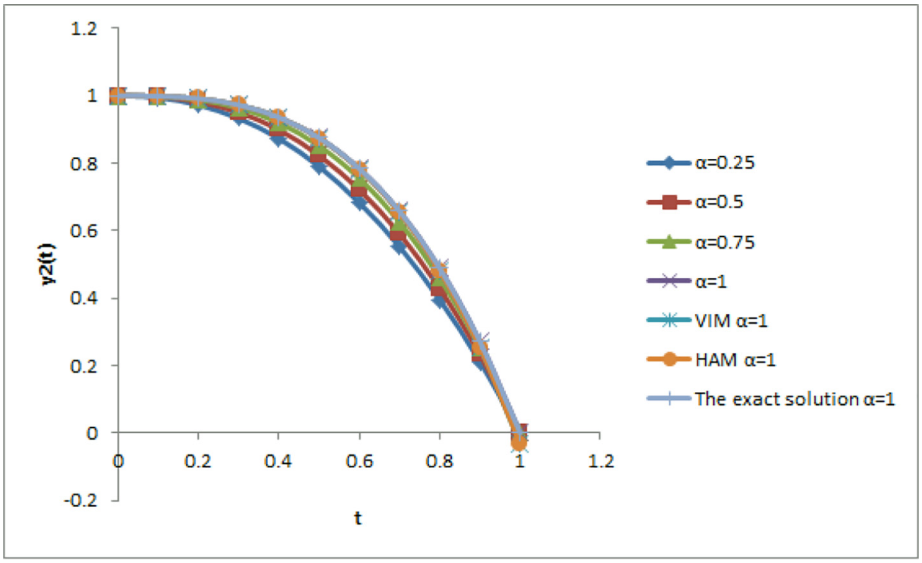

Fig. 2. The approximate solution of $y_2(t)$ using the proposed algorithm for different values of α compared with the (VIM), (HAM) and the exact solution when $\alpha = 1$.

$$c_1 = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_1 = \begin{pmatrix} \frac{\Gamma(1)}{\Gamma(\alpha+1)} \\ \frac{\Gamma(1)}{\Gamma(\alpha+1)} \end{pmatrix} \text{ and } \Phi_1(t) = \begin{pmatrix} P_{11}(t) \\ P_{21}(t) \end{pmatrix},$$

where

$$P_{11}(t) = \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)} t^{\alpha+3} + \frac{c_{11}}{(\alpha+1)(\alpha+2)} \times \frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2} + \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2}.$$

$$P_{21}(t) = -\frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)} t^{\alpha+4} + \frac{c_{11}}{(\alpha+1)(\alpha+2)} \times \frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2} - \frac{c_{21}}{(\alpha+1)(\alpha+2)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2}.$$

Therefore by comparing coefficients of t^α and neglecting $\Phi_1(t)$, we get: $c_{11} = \frac{\Gamma(1)}{\Gamma(\alpha+1)}$ and $c_{21} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha$.

Next step, let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha + c_{12}t^{\alpha+1} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{12}t^{\alpha+1}. \\ y_2(t) = c_{20} + c_{11}t^\alpha + c_{22}t^{\alpha+1} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{22}t^{\alpha+1}. \end{cases} \quad (40)$$

Substituting Eq. (40) into Eq. (36), we have

$$\begin{cases} \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{12}t^{\alpha+1} = I_t^\alpha(1) + I_t^\alpha(t) - \frac{1}{3} I_t^\alpha(t^3) + I_t^\alpha\left(\int_0^t \left((t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)} s^\alpha + c_{12}s^{\alpha+1}\right) + (t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)} s^\alpha + c_{22}s^{\alpha+1}\right)\right) ds\right); \\ \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{22}t^{\alpha+1} = I_t^\alpha(1) - I_t^\alpha(t) - \frac{1}{12} I_t^\alpha(t^{\alpha+4}) + I_t^\alpha\left(\int_0^t \left((t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)} s^\alpha + c_{12}s^{\alpha+1}\right) - (t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)} s^\alpha + c_{22}s^{\alpha+1}\right)\right) ds\right). \end{cases} \quad (41)$$

Therefore after some simple calculations, we get:

$$\begin{cases} \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{12}t^{\alpha+1} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)} t^{\alpha+3} + \frac{2}{(\alpha+1)(\alpha+2)} \\ \quad \frac{\Gamma(1)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2} + \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3} + \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3}; \\ \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + c_{22}t^{\alpha+1} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha - \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)} t^{\alpha+4} + \frac{c_{12}}{(\alpha+2)(\alpha+3)} \\ \quad \frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3} - \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3}. \end{cases} \quad (42)$$

Or

$$(M_2c_2 - q_2)\ t^{\alpha+1} + \Phi_2(t) = 0;$$

where

$$c_2 = \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q_2 = \begin{pmatrix} \frac{\Gamma(2)}{\Gamma(\alpha+2)} \\ -\frac{\Gamma(2)}{\Gamma(\alpha+2)} \end{pmatrix} \text{ and } \Phi_2(t) = \begin{pmatrix} P_{12}(t) \\ P_{22}(t) \end{pmatrix},$$

where

$$P_{12}(t) = -\frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)} t^{\alpha+3} - \frac{2}{(\alpha+1)(\alpha+2)}\frac{\Gamma(1)}{\Gamma(\alpha+1)} \times \frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)} t^{2\alpha+2} + \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3} + \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3}.$$

$$P_{22}(t) = -\frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)} t^{\alpha+4} + \frac{c_{12}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3} - \frac{c_{22}}{(\alpha+2)(\alpha+3)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)} t^{2\alpha+3}.$$

Therefore by comparing coefficients of $t^{\alpha+1}$ and neglecting $\Phi_2(t)$, we have: $c_{12} = \frac{\Gamma(2)}{\Gamma(\alpha+2)}$ and $c_{22} = -\frac{\Gamma(2)}{\Gamma(\alpha+2)}$.

Next step, let

$$\begin{cases} y_1(t) = c_{10} + c_{11}t^\alpha + c_{12}t^{\alpha+1} + c_{13}t^{\alpha+2} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha + \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} + c_{13}t^{\alpha+2}. \\ y_2(t) = c_{20} + c_{11}t^\alpha + c_{22}t^{\alpha+1} + c_{23}t^{\alpha+2} = \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha - \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} + c_{23}t^{\alpha+2}. \end{cases} \quad (43)$$

Substituting Eq. (43) into Eq. (36), we have

$$\begin{cases} \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+c_{13}t^{\alpha+2}=I_t^{\alpha}(1)+I_t^{\alpha}(t)-\frac{1}{3}I_t^{\alpha}(t^3)+I_t^{\alpha}\left(\int_0^t\left((t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)}s^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}s^{\alpha+1}+c_{23}s^{\alpha+2}\right)+(t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)}s^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}s^{\alpha+1}+c_{23}s^{\alpha+2}\right)\right)ds\right); \\ \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+c_{23}t^{\alpha+2}=I_t^{\alpha}(1)-I_t^{\alpha}(t)-\frac{1}{12}I_t^{\alpha}(t^{\alpha+4})+I_t^{\alpha}\left(\int_0^t\left((t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)}s^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}s^{\alpha+1}+c_{23}s^{\alpha+2}\right)-(t-s)\left(\frac{\Gamma(1)}{\Gamma(\alpha+1)}s^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}s^{\alpha+1}+c_{23}s^{\alpha+2}\right)\right)ds\right). \end{cases} \quad (44)$$

Therefore, we get:

$$\begin{cases} \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+c_{13}t^{\alpha+2}=\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}-\frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)}t^{\alpha+3}+\frac{2}{(\alpha+1)(\alpha+2)}\frac{\Gamma(1)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}+\frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}+\frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}; \\ \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}+c_{23}t^{\alpha+2}=\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}-\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}-\frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)}t^{\alpha+4}+\frac{1}{(\alpha+2)(\alpha+3)}\frac{\Gamma(2)}{\Gamma(\alpha+2)}\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3}+\frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}-\frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}. \end{cases} \quad (45)$$

Or

$$(M_3c_3-q_3)\ t^{\alpha+2}+\Phi_3(t)=0;$$

where

$$c_3=\begin{pmatrix}c_{13}\\c_{23}\end{pmatrix},M_3=\begin{pmatrix}1&0\\0&1\end{pmatrix},q_3=\begin{pmatrix}0\\0\end{pmatrix}\text{ and }\Phi_3(t)=\begin{pmatrix}P_{13}(t)\\P_{23}(t)\end{pmatrix},$$

where

$$P_{13}(t)=-\frac{1}{3}\frac{\Gamma(4)}{\Gamma(\alpha+4)}t^{\alpha+3}+\frac{2}{(\alpha+1)(\alpha+2)}\frac{\Gamma(1)}{\Gamma(\alpha+1)}\times\frac{\Gamma(\alpha+3)}{\Gamma(2\alpha+3)}t^{2\alpha+2}+\frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}+\frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}.$$

$$P_{23}(t)=-\frac{1}{12}\frac{\Gamma(5)}{\Gamma(\alpha+5)}t^{\alpha+4}+\frac{1}{(\alpha+2)(\alpha+3)}\frac{\Gamma(2)}{\Gamma(\alpha+2)}\times\frac{\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}t^{2\alpha+3}+\frac{c_{13}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}-\frac{c_{23}}{(\alpha+3)(\alpha+4)}\frac{\Gamma(\alpha+5)}{\Gamma(2\alpha+5)}t^{2\alpha+4}.$$

Therefore by comparing coefficients of $t^{\alpha+2}$ and neglecting $\Phi_3(t)$, we have: $c_{13}=0$ and $c_{23}=0$.

Therefore the approximate solution of problem (34) and (35) for m = 3 becomes as:

$$y_1(t)=c_{10}+c_{11}t^{\alpha}+c_{12}t^{\alpha+1}+c_{13}t^{\alpha+2}=\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}.$$

Table 3
Comparison between the approximate solution of $y_1(t)$ of problem (34) and (35) using the proposed algorithm for m = 3 and for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha=1$.

t	$\alpha=0.25$ m = 3	$\alpha=0.5$ m = 3	$\alpha=0.75$ m = 3	$\alpha=1$ m = 3	(VIM) $\alpha=1$	(HAM) $\alpha=1$	The exact solution $\alpha=1$
0	0	0	0	0	0	0	0
0.1	0.59	0.332	0.187	0.105	0.105	0.105	0.105
0.2	0.736	0.492	0.329	0.22	0.22	0.22	0.22
0.3	0.851	0.63	0.466	0.345	0.345	0.344	0.345
0.4	0.954	0.759	0.604	0.48	0.48	0.478	0.48
0.5	1.051	0.884	0.743	0.625	0.625	0.62	0.625
0.6	1.144	1.007	0.886	0.78	0.78	0.769	0.78
0.7	1.235	1.129	1.033	0.945	0.945	0.945	0.945
0.8	1.324	1.252	1.184	1.12	1.12	1.086	1.12
0.9	1.412	1.376	1.34	1.305	1.305	1.25	1.305
1	1.5	1.5	1.5	1.5	1.5	1.417	1.5

Table 4
Comparison between the approximate solution of $y_2(t)$ of problem (34) and (35) using the proposed algorithm for m = 3 and for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha=1$.

t	$\alpha=0.25$ m = 3	$\alpha=0.5$ m = 3	$\alpha=0.75$ m = 3	$\alpha=1$ m = 3	(VIM) $\alpha=1$	(HAM) $\alpha=1$	The exact solution $\alpha=1$
0	0	0	0	0	0	0	0
0.1	0.534	0.3	0.169	0.095	0.095	0.095	0.095
0.2	0.602	0.402	0.269	0.18	0.18	0.18	0.18
0.3	0.629	0.466	0.345	0.255	0.255	0.255	0.255
0.4	0.636	0.506	0.402	0.32	0.32	0.32	0.32
0.5	0.631	0.53	0.446	0.375	0.374	0.374	0.375
0.6	0.616	0.542	0.477	0.42	0.419	0.419	0.42
0.7	0.595	0.544	0.497	0.455	0.452	0.452	0.455
0.8	0.567	0.537	0.508	0.48	0.475	0.475	0.48
0.9	0.536	0.522	0.508	0.495	0.485	0.485	0.495
1	0.5	0.5	0.5	0.5	0.483	0.483	0.5

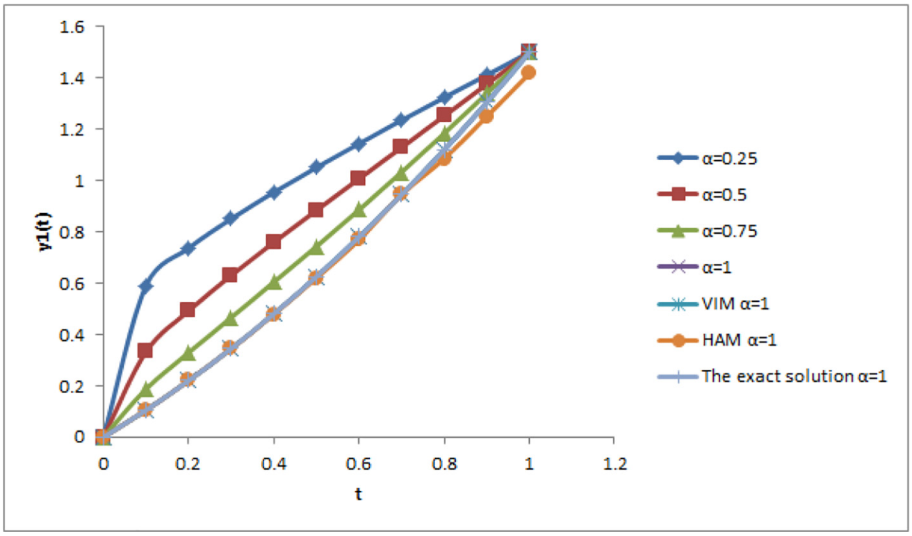

Fig. 3. The approximate solution of $y_1(t)$ using the proposed algorithm for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

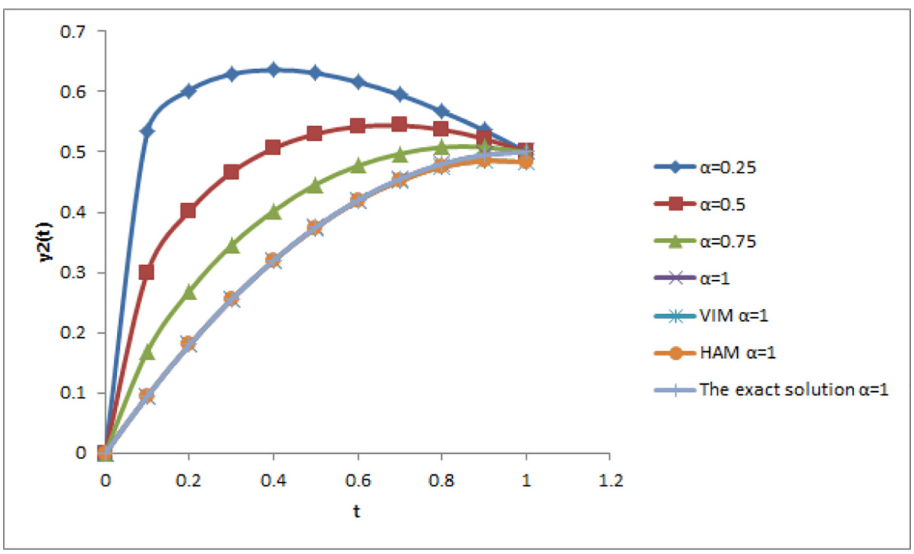

Fig. 4. The approximate solution of $y_2(t)$ using the proposed algorithm for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

$$y_2(t) = c_{20} + c_{21}t^{\alpha} + c_{22}t^{\alpha+1} + c_{23}t^{\alpha+2} = \frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}.$$

Tables 3 and 4 represent the solution of problem (34) and (35) for different values of α using the proposed algorithm compared with variational iteration method (VIM) (Nawaz, 2011), homotopy analysis method (HAM) (Zhang et al., 2011) and the exact solution when $\alpha = 1$.

Figs. 3 and 4 illustrates the approximate solution of problem (34) and (35) using the proposed algorithm for different values of α compared with (VIM), (HAM) and the exact solution when $\alpha = 1$.

6. Conclusions

In this paper, we have applied a modified computational algorithm for solving the fractional order linear integro-differential equations which gives an accurate solution. The algorithm has great potential to solve systems of linear problems of fractional order in short as well as in broad intervals.The beauty of the technique is less calculation, less use of computer memory, economical in terms of computer power, and involve no tedious calculations.

References

Aguilar, C.J.Z., Ugalde, H.M.R., Aguilar, J.F.G., Jimenez, R.F.E., Rodrigunez, M.V., 2017. Solving fractional differential equations of variable-order involving operators with Mittag-Leffler kernel using artificial neural networks. Chaos, Solitons Fractals 103, 382–403.

Aguilar, C.J.Z., Escamilla, A.C., Aguilar, J.F.G., Martinez, V.M.A., Ugalde, H.M.R., 2018. New numerical approximation for solving fractional delay differential equations of variable order using artificial neural networks. Eur. Phys. J. Plus 133 (2), 75.

Atangana, A., Aguilar, J.F.G., 2017. Numerical approximation of Riemann-Li-ouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu. Numer. Methods Partial Diff. Eqs.

Escamilla, A.C., Aguilar, J.F.G., Torres, L., Jimenez, R.F.E., 2018. A numerical solution for a variable-order reaction-diffusion model by using fractional derivatives with non-local and non-singular kernel. Phys. A: Stat. Mechanics Appl. 491, 406–424.

Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F., Cattani, C., 2014. Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations. Commun. Nonlin. Sci. Numer. Simul. https://doi.org/10.1016/j.cnsns.2013.04.026.

Hendi, F.A., Shammakh, W., Al-badrani, H., 2019. Existence result and approximate solutions for quadratic integro-differential equations of fractional order. J. King Saud Univ. Sci. 31, 314–321. https://doi.org/10.1016/j. jksus.2018.05.008.
Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory and Applications of Fractional Differential Equations. Elsevier, San Diego.
Kumar, K., Pandey, R.K., Sharma, S., 2017. Comparative study of three numerical schemes for fractional integro-differential equation. J. Comput. Appl. Math. 315, 287–302.
Martinez, H.Y., Aguilar, J.F.G., 2018. Numerical and analytical solutions of nonlinear differential equations involving fractional operators with power and Mittag-Leffler kernel. Math. Model. Nat. Phen. 13 (1), 1–13.
Mittal, R.C., 2008. Solution of fractional integro-differential equations by Adomain decompositions method. Int. J. Appl. Math. Mech. 4 (2), 87–94.
Momani, S.M., 2000. Local and global existence theorems on fractional integro-differential equations. J. Fract. Calculus 18, 81–86.
Mohammed, O.H., 2010. Solution of fractional integro-differential equations by homotopy analysis method. J. Al-Nahrain Univ. 13 (13), 149–155.
Mohammed, O.H., 2016. A direct method for solving fractional order variational problems by hat basis functions. Ain Shams Eng. J. https://doi.org/10.1016/j.asej.2016.11.006.
Mohammed, D.S., 2014. Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial. Math. Probl. Eng.
Nawaz, Y., 2011. Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. Comput. Math. Appl. 61, 2330–2341.
Perez, J.E.S., Aguilar, J.F.G., Baleanu, D., Tchier, F., 2018. Chaotic attractors with fractional conformable derivatives in the Liouville-Captuo sense and its dynamical behaviors. Entropy 20 (5), 384.
Podlubny, I., 1999. Fractional Differential Equations. Mathematics in Science and Engineering, Vol. 198. Academic Press, San Diego.
Rawashdeh, E.A., 2005. Numerical solution of fractional integro-differential equations by collocation method. Appl. Math. Comput. 176 (1), 1–6.
Wang, Y., Zhu, L., 2017. Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method. Adv. Diff. Eqs. 1, 27.
Zhang, X., Tang, B., He, Y., 2011. Homotopy analysis method for higher-order fractional integro-differential equations. Comput. Math. Appl. 62, 3194–3203.