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# Homotopy perturbation method for special nonlinear partial differential equations

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**Abstract** In this article, homotopy perturbation method is applied to solve nonlinear parabolic–hyperbolic partial differential equations. Examples of one-dimensional and two-dimensional are presented to show the ability of the method for such equations.


## 1. Introduction

Homotopy perturbation method has been used by many mathematicians and engineers to solve various functional equations (Yıldırım and Ozis, 2007; Biazar et al., 2007; Noor and Mohyud-Din, 2008; Ozis and Yıldırım, 2007; Odibat and Momani, 2008; Siddiqui et al., 2008; Ghori et al., 2007). This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities (He, 2004), nonlinear wave equations (He, 2005a), boundary value problem (He, 2006), limit cycle and bifurcation of nonlinear problems (He, 2005b), and many other subjects (He, 1999, 2000, 2003, 2004). It can be said that homotopy perturbation method is a universal one, is able to solve various kinds of nonlinear functional equations.

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## 2. Basic idea of method

For the purpose of applications illustration of the methodology of the proposed method, using homotopy perturbation method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where $A$ is a general differential operator, $f(r)$ is a known analytic function, $B$ is a boundary condition and $\Gamma$ is the boundary of the domain $\Omega$.

The operator $A$ can be generally divided into two operators, $L$ and $N$, where $L$ is a linear, while $N$ is a nonlinear operator. Eq. (1) can be, therefore, written as follows

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Using the homotopy technique, we construct a homotopy $U(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(U,p) = (1-p)[L(U) - L(u_0))] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{4}$$

or

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(r)] = 0, \tag{5}$$

where $p \in [0,1]$, is called homotopy parameter, and $u_0$ is an initial approximation for the solution of Eq. (1), which satisfies the boundary conditions. Obviously from Eqs. (4) and (5) we will have

$$H(U,0) = L(U) - L(u_0) = 0, \tag{6}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{7}$$

we can assume that the solution of (4) or (5) can be expressed as a series in $p$, as follows

$$U = U_0 + pU_1 + p^2 U_2 + \cdots. \tag{8}$$

Setting $p = 1$, results in the approximate solution of Eq. (1)

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + \cdots. \tag{9}$$

In this paper, we consider Cauchy problem for the nonlinear parabolic–hyperbolic equation of the following type

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\partial^2}{\partial t^2} - \Delta\right)u = F(u),$$

with initial conditions

$$\frac{\partial^k u}{\partial t^k}(0, X) = \varphi_k(X), \quad X = (x_1, x_2, \ldots, x_i), \quad k = 0, 1, 2.$$

where the nonlinear term is represented by $F(u)$, and $\Delta$ is the Laplace operator in $R^n$.

## 3. Examples

**Example 1.** Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = -\left(\frac{1}{3}\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{1}{6}\frac{\partial^2 u}{\partial t^2}\right)^3 - 16u, \tag{10}$$

subject to the following initial conditions

$$u(0,x) = -x^4, \quad \frac{\partial u}{\partial t}(0,x) = 0, \quad \frac{\partial^2 u}{\partial t^2}(0,x) = 0. \tag{11}$$

With the exact solution

$$u(t,x) = -x^4 + 4t^3.$$

To solve Eq. (10) by homotopy perturbation method, we construct the following homotopy

$$(1-p)\left(\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3}\right) + p\left(\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 U}{\partial t \partial x^2} - \frac{\partial^4 U}{\partial x^2 \partial t^2} + \frac{\partial^4 U}{\partial x^4} + \left(\frac{1}{3}\frac{\partial^2 U}{\partial x^2}\right)^2 - \left(\frac{1}{6}\frac{\partial^2 U}{\partial t^2}\right)^3 + 16U\right) = 0,$$

or

$$\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3} + p\left(\frac{\partial^3 u_0}{\partial t^3} - \frac{\partial^3 U}{\partial t \partial x^2} - \frac{\partial^4 U}{\partial x^2 \partial t^2} + \frac{\partial^4 U}{\partial x^4} + \left(\frac{1}{3}\frac{\partial^2 U}{\partial x^2}\right)^2 - \left(\frac{1}{6}\frac{\partial^2 U}{\partial t^2}\right)^3 + 16U\right) = 0. \tag{12}$$

Suppose the solution of Eq. (12) has the following form

$$U = U_0 + pU_1 + p^2 U_2 + \cdots. \tag{13}$$

Substituting (13) into (12) and equating the coefficients of the terms with the identical powers of $p$ leads to

$$p^0: \frac{\partial^3 U_0}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3} = 0,$$

$$p^1: \frac{\partial^3 U_1}{\partial t^3} + \frac{\partial^3 u_0}{\partial t^3} - \frac{\partial^3 U_0}{\partial t \partial x^2} - \frac{\partial^4 U_0}{\partial x^2 \partial t^2} + \frac{\partial^4 U_0}{\partial x^4} + \left(\frac{1}{3}\frac{\partial^2 U_0}{\partial x^2}\right)^2 - \left(\frac{1}{6}\frac{\partial^2 U_0}{\partial t^2}\right)^3 + 16U_0 = 0,$$

$$p^2: \frac{\partial^3 U_2}{\partial t^3} - \frac{\partial^3 U_1}{\partial t \partial x^2} - \frac{\partial^4 U_1}{\partial x^2 \partial t^2} + \frac{\partial^4 U_1}{\partial x^4} + \frac{2}{9}\frac{\partial^2 U_0}{\partial x^2}\frac{\partial^2 U_1}{\partial x^2} - \frac{1}{72}\frac{\partial^2 U_1}{\partial t^2}\frac{\partial^2 U_0}{\partial t^2}\frac{\partial^2 U_0}{\partial t^2} + 16U_1 = 0,$$

$$\vdots$$

$$p^j: \frac{\partial^3 U_j}{\partial t^3} - \frac{\partial^3 U_{j-1}}{\partial t \partial x^2} - \frac{\partial^4 U_{j-1}}{\partial x^2 \partial t^2} + \frac{\partial^4 U_{j-1}}{\partial x^4} + \frac{1}{9}\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial x^2}\frac{\partial^2 U_{j-1-k}}{\partial x^2} - \frac{1}{216}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{\partial U_i}{\partial t}\frac{\partial U_k}{\partial t}\frac{\partial U_{j-k-i-1}}{\partial t} + 16U_{j-1} = 0,$$

$$\vdots$$

For simplicity we take $U_0 = u_0 = -x^4$. So we derive the following recurrent relation for $j = 1, 2, 3, \ldots$.

$$U_j = -\int_0^t \int_0^t \times \int_0^t \left(\begin{array}{c} -\frac{\partial^3 U_{j-1}}{\partial x^2 \partial \xi_1} - \frac{\partial^4 U_{j-1}}{\partial x^2 \partial \xi_1^2} + \frac{\partial^4 U_{j-1}}{\partial x^4} + \frac{1}{9}\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial x^2}\frac{\partial^2 U_{j-1-k}}{\partial x^2} \\ -\frac{1}{216}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{\partial U_i}{\partial t}\frac{\partial U_k}{\partial t}\frac{\partial U_{j-k-i-1}}{\partial t} + 16U_{j-1} \end{array}\right) d\xi_1 d\xi_2 dt. \tag{14}$$

The solution reads

$$U_1 = 0,$$
$$U_2 = 0,$$
$$U_3 = 4t^3,$$
$$U_4 = 0,$$

and by repeating this approach we obtain, $U_5 = U_6 = \cdots = 0$. Therefore, the approximate solution of Example 1 can be readily obtained by

$$u = \sum_{i=0}^{\infty} U_i = -x^4 + t^3,$$

and hence, $u = -x^4 + 4t^3$, which is an exact solution.

**Example 2.** Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 2u^2, \tag{15}$$

subject to the initial conditions

$$u(0,x)=e^x,\quad \frac{\partial u}{\partial t}(0,x)=e^x,\quad \frac{\partial^2 u}{\partial t^2}(0,x)=e^x. \tag{16}$$

With the exact solution

$$u(t,x)=e^{x+t}.$$

To solve Eq. (15) by homotopy perturbation method, we construct the following homotopy

$$(1-p)\left(\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}\right)+p\left(\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 U}{\partial x\partial t^2}-\frac{\partial^4 U}{\partial x^2\partial t^2}+\frac{\partial^4 U}{\partial x^4}-\left(\frac{\partial^2 U}{\partial t^2}\right)^2+\left(\frac{\partial^2 U}{\partial x^2}\right)^2+2U^2\right)=0,$$

or

$$\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}+p\left(\frac{\partial^3 u_0}{\partial t^3}-\frac{\partial^3 U}{\partial x\partial t^2}-\frac{\partial^4 U}{\partial x^2\partial t^2}+\frac{\partial^4 U}{\partial x^4}-\left(\frac{\partial^2 U}{\partial t^2}\right)^2+\left(\frac{\partial^2 U}{\partial x^2}\right)^2+2U^2\right)=0. \tag{17}$$

Suppose the solution of Eq. (17) has the following form

$$U=U_0+pU_1+p^2U_2+\cdots. \tag{18}$$

Substituting (18) into (12) and equating the coefficients of the terms with the identical powers of $p$ leads to

$$p^0:\frac{\partial^3 U_0}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}=0,$$

$$p^1:\frac{\partial^3 U_1}{\partial t^3}+\frac{\partial^3 u_0}{\partial t^3}-\frac{\partial^3 U_0}{\partial t\partial x^2}-\frac{\partial^4 U_0}{\partial x^2\partial t^2}+\frac{\partial^4 U_0}{\partial x^4}-\left(\frac{\partial^2 U_0}{\partial t^2}\right)^2+\left(\frac{\partial^2 U_0}{\partial x^2}\right)^2+2U_0^2=0,$$

$$p^2:\frac{\partial^3 U_2}{\partial t^3}-\frac{\partial^3 U_1}{\partial t\partial x^2}-\frac{\partial^4 U_1}{\partial x^2\partial t^2}+\frac{\partial^4 U_1}{\partial x^4}-2\frac{\partial^2 U_1}{\partial t^2}\frac{\partial^2 U_0}{\partial t^2}+2\frac{\partial^2 U_0}{\partial x^2}\frac{\partial^2 U_1}{\partial x^2}+4U_0U_1=0,$$

$$\vdots$$

$$p^j:\frac{\partial^3 U_j}{\partial t^3}-\frac{\partial^3 U_{j-1}}{\partial t\partial x^2}-\frac{\partial^4 U_{j-1}}{\partial x^2\partial t^2}+\frac{\partial^4 U_{j-1}}{\partial x^4}-\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial t^2}\frac{\partial^2 U_{j-1-k}}{\partial t^2}+\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial x^2}\frac{\partial^2 U_{j-1-k}}{\partial x^2}+2\sum_{k=0}^{j-1}U_kU_{j-1-k}=0,$$

$$\vdots$$

starting with

$$U_0=u_0=\left(1+t+\frac{t^2}{2}\right)e^x. \tag{19}$$

We have the following recurrent equations for $j=1,2,3,\ldots$.

$$U_j=-\int_0^t\int_0^t\times\int_0^t\begin{pmatrix}-\frac{\partial^3 U_{j-1}}{\partial x^2\partial\xi_1}-\frac{\partial^4 U_{j-1}}{\partial x^2\partial\xi_1^2}+\frac{\partial^4 U_{j-1}}{\partial x^4}-\sum_{k=0}^{j-1}\frac{\partial U_k}{\partial\xi_1}\frac{\partial^2 U_{j-1-k}}{\partial\xi_1^2}\\+\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial x^2}\frac{\partial^2 U_{j-1-k}}{\partial x^2}+2\sum_{k=0}^{j-1}U_kU_{j-1-k}\end{pmatrix}d\xi_1d\xi_2dt. \tag{20}$$

We obtain the following results

$$U_1=\tfrac{t^3}{6}e^x,$$

$$U_2=\tfrac{t^4}{24}e^x,$$

$$U_3=\tfrac{t^5}{120}e^x, vdots$$

Solution of Eq. (15) will be derived by adding these terms, so

$$u=\sum_{i=0}^{\infty}U_i=e^{x+t}.$$

**Example 3.** Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=u\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x}, \tag{21}$$

with initial condition,

$$u(0,x)=\cos x,\quad \frac{\partial u}{\partial t}(0,x)=-\sin x,\quad \frac{\partial^2 u}{\partial t^2}(0,x)=-\cos x.$$

To solve Eq. (21) by homotopy perturbation method, we construct the following homotopy

$$(1-p)\left(\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}\right)+p\left(\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 U}{\partial x\partial t^2}-\frac{\partial^4 U}{\partial x^2\partial t^2}+\frac{\partial^4 U}{\partial x^4}-U\frac{\partial U}{\partial t}-\frac{\partial^2 U}{\partial t^2}\frac{\partial U}{\partial x}\right)=0,$$

or

$$\frac{\partial^3 U}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}+p\left(\frac{\partial^3 u_0}{\partial t^3}-\frac{\partial^3 U}{\partial x\partial t^2}-\frac{\partial^4 U}{\partial x^2\partial t^2}+\frac{\partial^4 U}{\partial x^4}-U\frac{\partial U}{\partial t}-\frac{\partial^2 U}{\partial t^2}\frac{\partial U}{\partial x}\right)=0. \tag{22}$$

Suppose the solution of Eq. (22) has the form (8), substituting (8) into (22), and comparing the terms with identical powers of $p$, leads to

$$p^0:\frac{\partial^3 U_0}{\partial t^3}-\frac{\partial^3 u_0}{\partial t^3}=0,$$

$$p^1:\frac{\partial^3 U_1}{\partial t^3}+\frac{\partial^3 u_0}{\partial t^3}-\frac{\partial^3 U_0}{\partial t\partial x^2}-\frac{\partial^4 U_0}{\partial x^2\partial t^2}+\frac{\partial^4 U_0}{\partial x^4}-U_0\frac{\partial U_0}{\partial t}-\frac{\partial^2 U_0}{\partial t^2}\frac{\partial U_0}{\partial x}=0,$$

$$p^2:\frac{\partial^3 U_2}{\partial t^3}-\frac{\partial^3 U_1}{\partial t\partial x^2}-\frac{\partial^4 U_1}{\partial x^2\partial t^2}+\frac{\partial^4 U_1}{\partial x^4}-U_0\frac{\partial U_1}{\partial t}-U_1\frac{\partial U_0}{\partial t}-\frac{\partial^2 U_1}{\partial t^2}\frac{\partial U_0}{\partial x}-\frac{\partial^2 U_0}{\partial t^2}\frac{\partial U_1}{\partial x}=0,$$

$$\vdots$$

$$p^j:\frac{\partial^3 U_j}{\partial t^3}-\frac{\partial^3 U_{j-1}}{\partial t\partial x^2}-\frac{\partial^4 U_{j-1}}{\partial x^2\partial t^2}+\frac{\partial^4 U_{j-1}}{\partial x^4}-\sum_{k=0}^{j-1}U_k\frac{\partial U_{j-1-k}}{\partial t}-\sum_{k=0}^{j-1}\frac{\partial^2 U_k}{\partial t^2}\frac{\partial U_{j-1-k}}{\partial x}=0,$$

$$\vdots$$

We take

$$U_0 = u_0 = \cos x - t\sin x - \frac{t^2}{2}\cos x. \tag{23}$$

We have the following recurrent equations for $j = 1, 2, 3, \ldots$

$$U_j = -\int_0^t \int_0^t \times \int_0^t \left( \begin{array}{c} -\frac{\partial^3 U_{j-1}}{\partial x^2 \partial \xi_1} - \frac{\partial^4 U_{j-1}}{\partial x^2 \partial \xi_1^2} + \frac{\partial^4 U_{j-1}}{\partial x^4} - \sum_{k=0}^{j-1} \frac{\partial U_k}{\partial \xi_1} \frac{\partial^2 U_{j-1-k}}{\partial \xi_1^2} \\ + \sum_{k=0}^{j-1} \frac{\partial^2 U_k}{\partial x^2} \frac{\partial^2 U_{j-1-k}}{\partial x^2} + 2\sum_{k=0}^{j-1} U_k U_{j-1-k} \end{array} \right) d\xi_1 d\xi_2 dt. \tag{24}$$

With the aid of the initial approximation given by Eq. (23) and the iteration formula (24) we get the other of component as follows

$$U_1 = \frac{t^3}{3!}\sin x + \frac{t^4}{4!}\cos x,$$

$$U_2 = -\frac{t^5}{5!}\sin x - \frac{t^6}{6!}\cos x,$$

$$U_3 = \frac{t^7}{7!}\sin x + \frac{t^8}{8!}\cos x,$$

$$\vdots$$

The solution in a series form is

$$\begin{aligned} u &= U_0 + U_1 + U_2 + \cdots \\ &= \cos x\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} + \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) \\ &= \cos(x + t), \end{aligned}$$

which is an exact solution.

**Example 4.** Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)u = \frac{\partial u}{\partial t} - 2u, \tag{25}$$

subject to the initial condition,

$$\begin{aligned} &u(0, x_1, x_2) = \sinh(x_1 + x_2), \\ &\frac{\partial u}{\partial t}(0, x_1, x_2) = 2\sinh(x_1 + x_2), \\ &\frac{\partial^2 u}{\partial t^2}(0, x_1, x_2) = 4\sinh(x_1 + x_2). \end{aligned} \tag{26}$$

With the exact solution

$$u(t, x_1, x_2) = \sinh(x_1 + x_2)e^{2t}.$$

To solve Eq. (25) by homotopy perturbation method, we construct the following homotopy

$$\begin{aligned} &(1-p)\left(\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3}\right) + p\left(\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 U}{\partial t \partial x_1^2} - \frac{\partial^3 U}{\partial t \partial x_2^2}\right. \\ &\quad - \frac{\partial^4 U}{\partial x_1^2 \partial t^2} + \frac{\partial^4 U}{\partial x_1^4} + \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U}{\partial x_2^2 \partial t^2} + \frac{\partial^4 U}{\partial x_2^2 \partial x_1^2} \\ &\quad \left. + \frac{\partial^4 U}{\partial x_2^4} - \frac{\partial U}{\partial t} + 2U\right) = 0, \end{aligned}$$

or

$$\begin{aligned} &\frac{\partial^3 U}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3} + p\left(\frac{\partial^3 u_0}{\partial t^3} - \frac{\partial^3 U}{\partial t \partial x_1^2} - \frac{\partial^3 U}{\partial t \partial x_2^2} - \frac{\partial^4 U}{\partial x_1^2 \partial t^2} + \frac{\partial^4 U}{\partial x_1^4}\right. \\ &\quad \left. + \frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U}{\partial x_2^2 \partial t^2} + \frac{\partial^4 U}{\partial x_2^2 \partial x_1^2} + \frac{\partial^4 U}{\partial x_2^4} - \frac{\partial U}{\partial t} + 2U\right) = 0. \end{aligned} \tag{27}$$

Suppose the solution of Eq. (27) has the following form (8), substituting (8) into (27) and equating the coefficients of the terms with the identical powers of $p$ leads to

$$\begin{aligned} &p^0 : \frac{\partial^3 U_0}{\partial t^3} - \frac{\partial^3 u_0}{\partial t^3} = 0, \\ &p^1 : \frac{\partial^3 U_1}{\partial t^3} + \frac{\partial^3 u_0}{\partial t^3} - \frac{\partial^3 U_0}{\partial t \partial x_1^2} - \frac{\partial^3 U_0}{\partial t \partial x_2^2} - \frac{\partial^4 U_0}{\partial x_1^2 \partial t^2} \\ &\quad + \frac{\partial^4 U_0}{\partial x_1^4} + \frac{\partial^4 U_0}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U_0}{\partial x_2^2 \partial t^2} \\ &\quad + \frac{\partial^4 U_0}{\partial x_2^2 \partial x_1^2} + \frac{\partial^4 U_0}{\partial x_2^4} - \frac{\partial U_0}{\partial t} + 2U_0 = 0, \\ &\frac{\partial U_1}{\partial t}(0, x_1, x_2) = 2\sinh(x_1 + x_2), \\ &p^2 : \frac{\partial^3 U_2}{\partial t^3} - \frac{\partial^3 U_1}{\partial t \partial x_1^2} - \frac{\partial^3 U_1}{\partial t \partial x_2^2} - \frac{\partial^4 U_1}{\partial x_1^2 \partial t^2} + \frac{\partial^4 U_1}{\partial x_1^4} \\ &\quad + \frac{\partial^4 U_1}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U_1}{\partial x_2^2 \partial t^2} + \frac{\partial^4 U_1}{\partial x_2^2 \partial x_1^2} \\ &\quad + \frac{\partial^4 U_1}{\partial x_2^4} - \frac{\partial U_1}{\partial t} + 2U_1 = 0, \\ &\frac{\partial^2 U_2}{\partial t^2}(0, x_1, x_2) = 4\sinh(x_1 + x_2), \\ &\vdots \\ &p^j : \frac{\partial^3 U_j}{\partial t^3} - \frac{\partial^3 U_{j-1}}{\partial t \partial x_1^2} - \frac{\partial^3 U_{j-1}}{\partial t \partial x_2^2} - \frac{\partial^4 U_{j-1}}{\partial x_1^2 \partial t^2} + \frac{\partial^4 U_{j-1}}{\partial x_1^4} \\ &\quad + \frac{\partial^4 U_{j-1}}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U_{j-1}}{\partial x_2^2 \partial t^2} + \frac{\partial^4 U_{j-1}}{\partial x_2^2 \partial x_1^2} \\ &\quad + \frac{\partial^4 U_{j-1}}{\partial x_2^4} - \frac{\partial U_{j-1}}{\partial t} + 2U_{j-1} = 0, \\ &\vdots \end{aligned}$$

We take

$$U_0 = u_0 = \sinh(x_1 + x_2). \tag{28}$$

So we have

$$U_j = -\int_0^t \int_0^t \times \int_0^t \left( \begin{array}{c} -\frac{\partial^3 U_{j-1}}{\partial \xi_1 \partial x_1^2} - \frac{\partial^3 U_{j-1}}{\partial \xi_1 \partial x_2^2} - \frac{\partial^4 U_{j-1}}{\partial x_1^2 \partial \xi_1^2} + \frac{\partial^4 U_{j-1}}{\partial x_1^4} + \frac{\partial^4 U_{j-1}}{\partial x_1^2 \partial x_2^2} - \frac{\partial^4 U_{j-1}}{\partial x_2^2 \partial \xi_1^2} \\ + \frac{\partial^4 U_{j-1}}{\partial x_2^2 \partial x_1^2} + \frac{\partial^4 U_{j-1}}{\partial x_2^4} - \sum_{k=0}^{j-1} \frac{\partial U_k}{\partial \xi_1} \frac{\partial U_{j-1-k}}{\partial \xi_1} + 4\sum_{k=0}^{j-1} U_k U_{j-1-k} \end{array} \right) d\xi_1 d\xi_2 dt. \tag{29}$$

With the aid of the initial approximation given by Eq. (28) and the iteration formula (29) we get the other of component as follows

$$U_1 = \frac{4}{3}t^3 \sinh(x_1 + x_2),$$

$$U_2 = \frac{2}{3}t^4 \sinh(x_1 + x_2),$$

$$U_3 = \frac{4}{15}t^4 \sinh(x_1 + x_2),$$

$$\vdots$$

Therefore, the approximate solution can be readily obtained by

$$u = U_0 + U_1 + U_2 + \cdots = e^{2t}\sinh(x_1 + x_2),$$

hence, $u = e^{2t}\sinh(x_1 + x_2)$, which is an exact solution of Example 4.

## 4. Conclusion

In this work, we used homotopy perturbation method for solving nonlinear partial differential parabolic–hyperbolic equations. The results have been approved the efficiency of this method for solving these problems. The solution obtained by homotopy perturbation method is valid for not only weakly nonlinear equations but also for strong ones. Furthermore, accurate solutions were derived from first-order approximations in the examples presented in this paper.

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