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ORIGINAL ARTICLE

# An efficient computational technique for solving the Fokker–Planck equation with space and time fractional derivatives

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**Abstract** This paper presents numerical solutions of the linear and nonlinear Fokker–Planck partial differential equations [FPPDEs] with space and time fractional derivatives through analytical solutions. These are treated by two analytical methods, namely, fractional reduced differential transform method [FRDTM] and fractional variational iteration method [FVIM] followed by some examples. Numerical results obtained by both FRDTM and FVIM are compared with some existing methods in the literature. This comparison shows the supremacy of FRDTM over FVIM and existing methods in terms of accuracy, simplicity and reliability.


## 1. Introduction

The Fokker–Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles (Risken, 1989), that is, it expresses the change of probability of a random function in space and time, hence it is used to explain solute transport. Phenomena such as anomalous diffusion, continuous random walk, wave propagation, polymeric networks, charge carrier transport in amorphous semiconductors, DNA and RNA polymerases, the motion of ribosomes along mRNA and pattern formation are modeled by FPPDEs with space and time fractional derivatives (see Heinsalu et al., 2006; Yang et al., 2009; Zhuang et al., 2006/07 and references therein). These applications of FPPDE with space and time fractional derivatives have attracted us to make a study on it.

In the present investigation, we consider the numerical solution of FPPDE with space and time fractional derivatives of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A(x,t,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,t,u)\right] u(x,t), \quad t > 0, \quad 0 < \alpha, \beta \le 1, \tag{1}$$

subject to the initial condition,

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$$u(x,o) = g(x), \tag{2}$$

where $\alpha$ and $\beta$ are parameters describing the order of the fractional time and space derivatives, respectively (see Yan, 2013). The function $u(x,t)$ is assumed to be a causal function of time and space. It is interesting to note that for $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical FPPDE.

Qualitative properties like stability and convergence of (1) have been studied by the Yang et al. (2010). Various existing analytical and numerical methods have been used to solve (1) Garg and Manohar, 2011; Odibat and Momani, 2007; Ray and Gupta, 2014; Vanani and Aminataei, 2012; Yan, 2013; Yang et al., 2009; Yildrim, 2010. However, these analytical and numerical methods are not simple to apply and need tedious works and knowledge.

The differential transform method [DTM] is an analytical method which was first proposed by Zhou (1986) and its main applications therein are to solve both linear and nonlinear initial value problems in electric circuit analysis. Later, DTM has been used to solve partial differential equations (Soltanalizadeh, 2011; Soltanalizadeh and Yildirim, 2012). Another analytical version of DTM is the reduced differential transform method [RDTM]. Recently, the RDTM has been shown to be effective and reliable for handling linear and nonlinear partial differential equations and integral equations (one can refer Abazari and Kılıçman, 2013; Abazari and Soltanalizadeh, 2013; Saravanan and Magesh, 2013). This method has been developed by Keskin and Oturanc (2010) to solve the fractional partial differential equations with some modifications and it is named as fractional reduced differential transform method [FRDTM]. FRDTM has been successfully applied to solve many types of fractional partial differential equations (Gupta, 2011; Ray, 2013; Sohail and Mohyud-Din, 2012) and higher dimensional problems too (Srivastava et al., 2014). This literature survey shows that FRDTM has been used to solve the time fractional derivative problems but not on both space and time fractional derivatives. Here, we are the first to propose FRDTM for solving space and time fractional partial differential equations of the type (1).

The fractional variational iteration method [FVIM] was first proposed by Wu and Lee (2010). This technique is based on the modified Riemann–Liouville derivative. Recently, there are many interesting works that have been considered to solve various fractional differential equations (Elbeleze et al., 2013; Faraz et al., 2011; Song et al., 2013; Wu and Lee, 2010). A correction functional is constructed by the general Lagrange multiplier which can be found optimally through the variational theory. However, the general Lagrange multiplier cannot be identified directly by using integration by parts as in classical variational iteration method. To get this Lagrange multiplier, we need to use fractional integration by parts $\int_a^b u^{\alpha}(x)v(x)(dx)^{\alpha} = \alpha![u(x)v(x)]_a^b - \int_a^b u(x)v^{\alpha}(x)(dx)^{\alpha}$, which is mysterious to non mathematicians and it requires the complete knowledge of variational theory. Further, it leads to complicated computation and more time is consumed. But such a type of complicated computation will not occur in FRDTM. Inspiration behind the proposed FRDTM is to exhibit a solution scheme which is easy to understand. It is interesting to note that both FRDTM and FVIM provide the analytical solutions. Here, in this paper, the numerical solutions are obtained through the analytical solution. To show the supremacy of FRDTM over the other existing methods in the literature, the numerical results are compared for assessing the accuracy, simplicity and reliability.

## 2. Preliminaries

### 2.1. Fractional reduced differential transform method

In this section, we give some basic definitions and properties of the FRDTM which are used further in this paper.

**Definition 1.** Keskin and Oturanc (2010) Let $u(x,t)$ be an analytic function that is continuously differentiable with respect to time $t$ and space $x$ in the domain of interest then,

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=0}, \tag{3}$$

where $\alpha$ is a parameter describing the order of the time fractional derivative in the Caputo sense and the t-dimensional spectrum function $U_k(x)$ is the transformed function.

The differential inverse transform of $U_k(x)$ is defined as,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}. \tag{4}$$

The fundamental theorems of the FRDTM are derived from the fractional power series and the generalized Taylor series and they are given in the Table 1 (see Keskin and Oturanc, 2010 as follows:)

Further information about fractional derivatives and its properties can be found in Jumarie (2009) and Odibat and Momani (2007).

## 3. Description of the methods

In this section, we give the solution procedure of FRDTM and FVIM to solve (1).

**(a) Fractional reduced differential transform method**:

Applying the reduced differential transform on both sides of (1) and (2), we obtain,

$$RDT\left[\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right] = RDT\left[-\frac{\partial^{\beta}}{\partial x^{\beta}}A(x,t,u)u(x,t)\right] + RDT\left[\frac{\partial^{2\beta}}{\partial x^{2\beta}}B(x,t,u)u(x,t)\right], \tag{5}$$

$$RDT[u(x,0)] = RDT[g(x)]. \tag{6}$$

Using the fundamental Theorems 5, 6 and 7 in Table 1 on (5) and (6), we get the recurrence relations as,

**Table 1** Fundamental theorems of the FRDTM

| S. No | Original function | Transformed function |
|---|---|---|
| 1 | $w(x,t) = u(x,t) \pm v(x,t)$ | $W_k(x) = U_k(x) \pm V_k(x)$ |
| 2 | $w(x,t) = cu(x,t)$ | $W_k(x) = cU_k(x)$($c$ is constant) |
| 3 | $w(x,t) = x^m t^n$ | $W_k(x) = x^m \delta(k\alpha - n)$ |
| 4 | $w(x,t) = x^m t^n u(x,t)$ | $W_k(x) = x^m U_{k\alpha-n}(x)$ |
| 5 | $w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}(u(x,t))$ | $W_k(x) = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x)$ |
| 6 | $w(x,t) = u(x,t)v(x,t)$ | $W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$ |
| 7 | $w(x,t) = \frac{\partial}{\partial x}(u(x,t))$ | $W_k(x) = \frac{\partial}{\partial x}(U_k(x))$ |

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = -\frac{\partial^\beta}{\partial x^\beta}F_k(x) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}G_k(x), \quad (7)$$

where $U_k(x)$, $F_k(x)$ and $G_k(x)$ are the transformed functions of $u(x,t)$, $A(x,t,u)\ u(x,t)$ and $B(x,t,u)u(x,t)$, respectively.

$$U_0(x) = g(x). \quad (8)$$

From the Iterative calculations mentioned above (7), we get the inverse transform coefficients of $t^{k\alpha}$, $k = 0,1,2,\ldots.$ as,

$$U_1(x) = \eta_0(x), U_2(x) = \eta_1(x), U_3(x) = \eta_2(x), \ldots\ldots. \quad (9)$$

By substituting (8) and (9) in (4), we get the series solution. When we take $\alpha = 1 = \beta$, the series solution becomes exact solution. One can get the approximate solution by truncating the terms in the series solution.

**(b) Fractional variational iteration method**:

To solve (1) by means of FVIM, rewrite it in the form,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left[\frac{\partial^\beta}{\partial x^\beta}A(x,t,u) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}B(x,t,u)\right]u(x,t) = 0. \quad (10)$$

We can construct a correction functional for (10) as,

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \times \int_0^t \lambda(s)\left[\frac{\partial^\alpha}{\partial s^\alpha}u_n + \left(\frac{\partial^\beta}{\partial x^\beta}A - \frac{\partial^{2\beta}}{\partial x^{2\beta}}B\right)\tilde{u_n}(x,s)\right](ds)^\alpha, \quad (11)$$

where $\lambda$ is the general Lagrange multiplier, which can be identified optimally through variational theory. The function $\tilde{u_n}(x,t)$ is a restricted variation which means $\delta\tilde{u_n}(x,t) = 0$. By making the above functional stationary,

$$\delta u_{n+1}(x,t) = \delta u_n + \frac{\delta}{\Gamma(\alpha+1)}\int_0^t \lambda(s)\left[\frac{\partial^\alpha}{\partial s^\alpha}u_n(x,s)\right](ds)^\alpha. \quad (12)$$

By using fractional integration by parts, we may obtain,

$$\lambda = -1. \quad (13)$$

From (11) and (13), we get the following iteration formula,

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha+1)} \times \int_0^t \left[\frac{\partial^\alpha}{\partial s^\alpha}u_n(x,s) + \left(\frac{\partial^\beta}{\partial x^\beta}A - \frac{\partial^{2\beta}}{\partial x^{2\beta}}B\right)u_n(x,s)\right](ds)^\alpha. \quad (14)$$

When we start with the initial approximation $u_0(x,t) = g(x)$, then we can determine the approximations of $u_n(x,t), n \geqslant 1$. Finally, we approximate the solution as,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \quad (15)$$

## 4. Illustrative examples

Two different examples are considered in this section, to show the effectiveness of RDTM.

**Example 1.** Consider, the linear space-time fractional Fokker–Planck partial differential equation (Yan, 2013),

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial^\beta}{\partial x^\beta}\left(\frac{ux}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{ux^2}{12}\right), \quad t > 0, \quad x > 0, \quad 0 < \alpha, \beta \le 1, \quad (16)$$

with the initial condition,

$$u(x,0) = x^2. \quad (17)$$

**Case (i): FRDTM:** By taking the reduced differential transform on both sides of (16) and (17) and then applying appropriate results given in Table 1, the following recurrence relations are obtained,

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = \frac{-1}{6}\frac{\partial^\beta}{\partial x^\beta}(xU_k(x)) + \frac{1}{12}\frac{\partial^{2\beta}}{\partial x^{2\beta}}(x^2U_k(x)), \quad (18)$$

$$U_0(x) = x^2. \quad (19)$$

From the Iterative calculations mentioned above (18), we get the inverse transform coefficients of $t^{k\alpha}$ where $k = 0,1,2,\ldots.$ as,

$$U_1(x) = \frac{1}{\Gamma(1+\alpha)}\left(\frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)}\right),$$

$$U_2(x) = \frac{1}{\Gamma(1+2\alpha)}\left[\frac{-\Gamma(6-2\beta)x^{5-3\beta}}{3\Gamma(5-2\beta)\Gamma(6-3\beta)} + \frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)} - \frac{\Gamma(6-\beta)x^{5-3\beta}}{12\Gamma(4-\beta)\Gamma(6-3\beta)}\right], \ldots\ldots. \quad (20)$$

Substituting (19) and (20) in (4), we get the series solution as,

$$u(x,t) = x^2 + \frac{1}{\Gamma(1+\alpha)}\left(\frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)}\right)t^\alpha + \frac{1}{\Gamma(1+2\alpha)}\left[\frac{-\Gamma(6-2\beta)x^{5-3\beta}}{3\Gamma(5-2\beta)\Gamma(6-3\beta)} + \frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)} - \frac{\Gamma(6-\beta)x^{5-3\beta}}{12\Gamma(4-\beta)\Gamma(6-3\beta)}\right]t^{2\alpha} + \ldots\ldots. \quad (21)$$

**Case (ii): FVIM:**

According to the formula (14), the iteration formula of (16) is given by,

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \left[\frac{\partial^\alpha}{\partial s^\alpha}u_n(x,s) + \frac{1}{6}\left(\frac{\partial^\beta}{\partial x^\beta}(xu_n(x,s)) - \frac{1}{12}\frac{\partial^{2\beta}}{\partial x^{2\beta}}(x^2u_n(x,s))\right)\right](ds)^\alpha. \quad (22)$$

Taking initial approximation as $u_0(x,t) = x^2$, we get the following approximations,

$$u_1(x,t) = x^2 - \frac{1}{\Gamma(1+\alpha)}\left(\frac{x^{3-\beta}}{\Gamma(4-\beta)} - \frac{2x^{4-2\beta}}{\Gamma(5-2\beta)}\right)t^\alpha,$$

$$u_2(x,t) = x^2 - \frac{1}{\Gamma(1+\alpha)}\left(\frac{x^{3-\beta}}{\Gamma(4-\beta)} - \frac{2x^{4-2\beta}}{\Gamma(5-2\beta)}\right)t^\alpha - \frac{1}{\Gamma(1+2\alpha)}\left[\frac{\Gamma(6-2\beta)x^{5-3\beta}}{3\Gamma(5-2\beta)\Gamma(6-3\beta)} - \frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} + \frac{\Gamma(6-\beta)x^{5-3\beta}}{12\Gamma(4-\beta)\Gamma(6-3\beta)} - \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]t^{2\alpha}, \quad (23)$$

and so on, in the same manner the rest of the components of the iteration formula (22) can be obtained.

When $\alpha = \beta = 1$, Eqs. (21) and (23) are reduced to

$$u(x,t) = x^2\left(1 + \frac{t}{2} + \frac{\left(\frac{t}{2}\right)^2}{2!} + \ldots\ldots\right) = x^2 e^{\frac{t}{2}},$$

which is the exact solution of (16).

Table 2 shows the comparison of exact solutions with the approximate solutions of different methods ADM, VIM, HWM, ILTM, FVIM and FRDTM for various values of $x$ and $t$ when $\alpha = \beta = 1$. Given numerical results of ADM, VIM and HWM in Table 2 have been taken from Ray and Gupta (2014) and the numerical results of ILTM, FVIM and FRDTM have been constructed for first three terms of the analytical solution by using the Matlab version 1.0.0.1. It is found that the solutions obtained by the present method are better than the ADM, VIM, HWM, ILTM and coincide with FVIM as well as exact solution. Accuracy wise, FRDTM shows better performance over the other techniques.

ADM uses complicated Adomian polynomial and noise term phenomena, VIM involves estimation of the Lagrange multiplier and noise term phenomena, HWM involves reduction of fractional PDE to system of equations added complexity to the respective techniques while such complexities are not occured in the solution procedure of FRDTM. It shows the simplicity of FRDTM.

In Vanani and Aminataei (2012), the authors took seven iterations to show the ability of their method but here we consider only first three iterations. It shows the rate of convergence, constancy and reliability of the FRDTM. In addition, Tables 3 and 4 are given for further reference.

**Table 2** Comparison of numerical results obtained by ADM, VIM, HWM, ILTM, FVIM, FRDTM with the exact solution of the linear FPPDE at various points of $x$ and $t$ when $\alpha = \beta = 1$.

| $t$ | $x$ | ADM | VIM | HWM | ILTM | FVIM | FRDTM | EXACT |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 0.25 | 0.069062 | 0.069062 | 0.0689468 | 0.0693 | 0.0691 | 0.0691 | 0.0691 |
| | 0.50 | 0.276259 | 0.27625 | 0.274611 | 0.2771 | 0.2763 | 0.2763 | 0.2763 |
| | 0.75 | 0.621563 | 0.621563 | 0.619337 | 0.6234 | 0.6216 | 0.6216 | 0.6217 |
| 0.4 | 0.25 | 7.63E−02 | 7.63E−02 | 0.0753937 | 0.0771 | 0.0762 | 0.0762 | 0.0763 |
| | 0.5 | 0.305 | 0.305 | 0.299222 | 0.3083 | 0.305 | 0.305 | 0.3054 |
| | 0.75 | 0.68625 | 0.68625 | 0.676175 | 0.6938 | 0.6863 | 0.6863 | 0.687 |
| 0.6 | 0.25 | 0.084062 | 0.084063 | 0.0818405 | 0.0859 | 0.0841 | 0.0841 | 0.0844 |
| | 0.50 | 3.36E−01 | 3.36E−01 | 0.323833 | 0.3438 | 0.3362 | 0.3362 | 0.3375 |
| | 0.75 | 0.756562 | 0.756562 | 0.733012 | 0.7734 | 0.7566 | 0.7566 | 0.7593 |

**Table 3** Absolute errors between the solution obtained by ADM, VIM, HWM, ILTM, FVIM, FRDTM and the exact solution of the linear FPPDE.

| $t$ | $x$ | $E_1$[a] | $E_2$[b] | $E_3$[c] | $E_4$[d] | $E_5$[e] | $E_6$[f] |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.25 | 3.8E−05 | 3.8E−05 | 0.0001532 | 0.0002 | 0 | 0 |
| | 0.50 | 4.1E−05 | 5E−05 | 0.001689 | 0.0008 | 0 | 0 |
| | 0.75 | 0.000137 | 0.000137 | 0.002363 | 0.0017 | 1E−04 | 1E−04 |
| 0.4 | 0.25 | 5E−05 | 5E−05 | 0.0009063 | 0.0008 | 0.0001 | 0.0001 |
| | 0.5 | 0.0004 | 0.0004 | 0.006178 | 0.0029 | 0.0004 | 0.0004 |
| | 0.75 | 0.00075 | 0.00075 | 0.010825 | 0.0068 | 0.0007 | 0.0007 |
| 0.6 | 0.25 | 0.000338 | 0.000337 | 0.0025595 | 0.0015 | 0.0003 | 0.0003 |
| | 0.50 | 0.00125 | 0.00125 | 0.013667 | 0.0063 | 0.0013 | 0.0013 |
| | 0.75 | 0.002738 | 0.002738 | 0.026288 | 0.0141 | 0.0027 | 0.0027 |

[a] |Exact − ADM|.
[b] |Exact − VIM|.
[c] |Exact − HWM|.
[d] |Exact − ILTM|.
[e] |Exact − FVIM|.
[f] |Exact − FRDTM|.

**Table 4** Comparison of numerical results obtained by ADM, VIM, OTM, HWM, ILTM, FVIM with FRDTM solution of the linear FPPDE at various points of $x$ and $t$.

| $(\alpha, \beta)$ | $t$ | $x$ | ADM | VIM | OTM | HWM | ILTM | FVIM | FRDTM |
|---|---|---|---|---|---|---|---|---|---|
| (0.5,0.5) | 0.2 | 0.25 | 0.06044 | 0.06111 | 0.061929 | 0.0601168 | 0.0605 | 0.0604 | 0.0604 |
| | | 0.50 | 0.244329 | 0.24618 | 0.248365 | 0.244247 | 0.2446 | 0.2443 | 0.2443 |
| | | 0.75 | 0.559866 | 0.56056 | 0.562348 | 0.559936 | 0.5609 | 0.5599 | 0.5599 |
| | 0.4 | 0.25 | 5.96E−02 | 6.00E−02 | 6.14E−02 | 0.0591215 | 0.0597 | 0.0596 | 0.0596 |
| | | 0.5 | 0.242066 | 0.24303 | 0.246833 | 0.241821 | 0.2426 | 0.2421 | 0.2421 |
| | | 0.75 | 0.558992 | 0.55902 | 0.562276 | 0.558771 | 0.5611 | 0.559 | 0.559 |
| | 0.6 | 0.25 | 0.059004 | 0.05898 | 0.060883 | 0.0583544 | 0.0591 | 0.059 | 0.059 |
| | | 0.50 | 2.40E−01 | 2.40E−01 | 2.45E−01 | 0.239941 | 0.2411 | 0.2404 | 0.2404 |
| | | 0.75 | 0.558407 | 0.55777 | 0.562273 | 0.557834 | 0.5615 | 0.5584 | 0.5584 |
| (0.75,0.75) | 0.2 | 0.25 | 0.063002 | 0.062922 | 0.06292 | 0.0633685 | 0.0631 | 0.063 | 0.063 |
| | | 0.50 | 0.258161 | 0.256856 | 0.256782 | 0.256326 | 0.2587 | 0.2582 | 0.2582 |
| | | 0.75 | 0.592855 | 0.58779 | 0.588104 | 0.595415 | 0.5946 | 0.5929 | 0.5929 |
| | 0.4 | 0.25 | 6.33E−02 | 6.33E−02 | 6.33E−02 | 0.063968 | 0.0636 | 0.0634 | 0.0634 |
| | | 0.5 | 0.264157 | 0.262868 | 0.262916 | 0.260722 | 0.2658 | 0.2642 | 0.2642 |
| | | 0.75 | 0.615589 | 0.610213 | 0.611786 | 0.618446 | 0.6205 | 0.6156 | 0.6156 |
| | 0.6 | 0.25 | 0.063713 | 0.063642 | 0.063669 | 0.0644986 | 0.0641 | 0.0637 | 0.0637 |
| | | 0.50 | 2.70E−01 | 2.69E−01 | 2.69E−01 | 0.264632 | 0.2726 | 0.2697 | 0.2697 |
| | | 0.75 | 0.636878 | 0.631709 | 0.634637 | 0.639038 | 0.6458 | 0.6369 | 0.6369 |

**Example 2.** Consider, the nonlinear space-time fractional Fokker–Planck partial differential equation (Yan, 2013; Yang et al., 2009),

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial^\beta}{\partial x^\beta}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}(u^2), \quad t > 0,$$
$$x > 0, \quad 0 < \alpha, \beta \le 1, \tag{24}$$

with the initial condition,

$$u(x,0) = x^2. \tag{25}$$

**Case (i): FRDTM:**

By taking the reduced differential transform on both sides of (24) and (25) and then applying appropriate results given in Table 1, the following recurrence relations are obtained.

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) = -4\frac{\partial^\beta}{\partial x^\beta}\left(\frac{1}{x}\sum_{k_1=0}^{k} U_{k_1}(x)U_{k-k_1}(x)\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\sum_{k_1=0}^{k} U_{k_1}(x)U_{k-k_1}(x)\right) + \frac{1}{3}\frac{\partial^\beta}{\partial x^\beta}(xU_k(x)), \tag{26}$$

$$U_0(x) = x^2. \tag{27}$$

From the Iterative calculations mentioned above (26), we get the inverse transform coefficients of $t^{k\alpha}$ where $k = 0, 1, 2, \ldots$ as,

$$U_1(x) = \frac{1}{\Gamma(1+\alpha)}\left[x^{3-\beta}\left(\frac{24x^{1-\beta}}{\Gamma(5-2\beta)} - \frac{22}{\Gamma(4-\beta)}\right)\right],$$

$$U_2(x) = \frac{1}{\Gamma(1+2\alpha)}\left[\frac{-184}{\Gamma(5-2\beta)}\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(5-2\beta)} + \frac{506}{3}\frac{\Gamma(6-2\beta)x^{5-3\beta}}{\Gamma(4-\beta)\Gamma(6-3\beta)} + \frac{48\Gamma(6-\beta)x^{5-3\beta}}{\Gamma(5-2\beta)\Gamma(6-3\beta)} - \frac{44\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(4-\beta)\Gamma(7-4\beta)}\right], \ldots\ldots \tag{28}$$

Substituting (27) and (28) in (4), we get the series solution as,

$$u(x,t) = x^2 + \frac{1}{\Gamma(1+\alpha)}\left[x^{3-\beta}\left(\frac{24x^{1-\beta}}{\Gamma(5-2\beta)} - \frac{22}{\Gamma(4-\beta)}\right)\right]t^\alpha + \frac{1}{\Gamma(1+2\alpha)}\left[\frac{-184}{\Gamma(5-2\beta)}\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(5-2\beta)} + \frac{506}{3}\frac{\Gamma(6-2\beta)x^{5-3\beta}}{\Gamma(4-\beta)\Gamma(6-3\beta)} + \frac{48\Gamma(6-\beta)x^{5-3\beta}}{\Gamma(5-2\beta)\Gamma(6-3\beta)} - \frac{44\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(4-\beta)\Gamma(7-4\beta)}\right]t^{2\alpha}\ldots\ldots \tag{29}$$

**Case (ii): FVIM:**

According to the formula (14), the iteration formula of (24) is given by,

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha+1)} \times \int_0^t \left[\frac{\partial^\alpha}{\partial s^\alpha}u_n + \frac{\partial^\beta}{\partial x^\beta}\left(\frac{4u_n^2}{x} - \frac{xu_n}{3}\right) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}(u_n^2)\right](ds)^\alpha. \tag{30}$$

Taking initial approximation as $u_0(x,t) = x^2$, we get the following approximations,

$$u_1(x,t) = x^2 - \frac{1}{\Gamma(1+\alpha)}\left(\frac{22x^{3-\beta}}{\Gamma(4-\beta)} - \frac{24x^{4-2\beta}}{\Gamma(5-2\beta)}\right)t^\alpha,$$

$$u_2(x,t) = x^2 - \frac{1}{\Gamma(1+\alpha)}\left(\frac{22x^{3-\beta}}{\Gamma(4-\beta)} - \frac{24x^{4-2\beta}}{\Gamma(5-2\beta)}\right)t^\alpha - \frac{1}{\Gamma(1+2\alpha)}\left[\frac{-506\Gamma(5-\beta)x^{4-2\beta}}{3\Gamma(5-2\beta)\Gamma(4-\beta)} + \frac{184\Gamma(6-2\beta)x^{5-3\beta}}{\Gamma(6-3\beta)\Gamma(5-2\beta)} + \frac{44\Gamma(6-\beta)x^{5-3\beta}}{\Gamma(4-\beta)\Gamma(6-3\beta)} - \frac{48\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]t^{2\alpha} - \frac{4\Gamma(2\alpha+1)}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} \times \left[\frac{484\Gamma(6-2\beta)x^{5-3\beta}}{[\Gamma(4-\beta)]^2\Gamma(6-3\beta)} + \frac{576\Gamma(8-4\beta)x^{7-5\beta}}{[\Gamma(5-2\beta)]^2\Gamma(8-5\beta)}\right.$$

$$-\frac{1056\Gamma(7-3\beta)x^{6-4\beta}}{\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(7-4\beta)}-\frac{121\Gamma(7-2\beta)x^{6-4\beta}}{[\Gamma(4-\beta)]^2\Gamma(7-4\beta)} -\frac{144\Gamma(9-4\beta)x^{8-6\beta}}{[\Gamma(5-2\beta)]^2\Gamma(9-6\beta)}+\frac{264\Gamma(8-3\beta)x^{7-5\beta}}{\Gamma(4-\beta)\Gamma(5-2\beta)\Gamma(8-5\beta)}\Bigg]t^{3\alpha}, \quad (31)$$

and so on, in the same manner the rest of the components of the iteration formula (22) can be obtained.

It is noticed that the coefficient of $t^{3\alpha}$ in (31) is not exact one. This noise term occurred while calculating $u_2(x,t)$. So, we have to take this term in to the next approximation to get the exact one, which leads to repeated calculations. Due to this, it consumes more time to converge. This kind of disadvantages did not appear while using FRDTM. The values of $\alpha=\beta=1$ is the only case for which we know the exact solution $u(x,t)=x^2e^t$.

Tables 5 and 6 show the comparison of exact solutions with the approximate solutions of different methods HPTM, ADM, VIM, ILTM, FVIM and FRDTM for various values of $x$ and $t$ when $\alpha=\beta=1$. Given numerical results of HPTM, ADM and VIM in Tables 5 and 6 have been taken from Yang et al. (2009) and the numerical results of ILTM, FVIM and FRDTM have been constructed for first three terms of the analytical solution by using the Matlab version 1.0.0.1. Also from Table 6, it is found that the solutions obtained by the present method are better than the HPTM, ADM, VIM, ILTM, FVIM and identical with the exact solution. It shows the better accuracy of the FRDTM.

Now for non-linear FPPDE, HPTM which is a hybrid method involving Laplace transform and homotopy perturbation methods, ADM uses complicated Adomian polynomial and noise term phenomena and VIM involves estimation of

**Table 5** Comparison of numerical results obtained by HPTM, ADM, VIM, ILTM, FVIM, FRDTM with the exact solution of the non linear FPPDE at various points of $x$ and $t$ when $\alpha=\beta=1$.

| $t$ | $x$ | HPTM | ADM | VIM | ILTM | FVIM | FRDTM | EXACT |
|---|---|---|---|---|---|---|---|---|
| 0.06 | 0.25 | 0.066367 | 0.066367 | 0.066363 | 0.0663 | 0.0662 | 0.0664 | 0.0664 |
| | 0.50 | 0.265468 | 0.265468 | 0.26545 | 0.2654 | 0.2653 | 0.2655 | 0.2655 |
| | 0.75 | 0.597303 | 0.597303 | 0.597262 | 0.5972 | 0.5971 | 0.5973 | 0.5973 |
| | 1 | 1.06197 | 1.06197 | 1.0618 | 1.0617 | 1.0617 | 1.0618 | 1.0618 |
| 0.2 | 0.25 | 0.076417 | 7.64E−02 | 7.63E−02 | 0.0761 | 0.0761 | 0.0762 | 0.0763 |
| | 0.5 | 0.305667 | 0.305667 | 0.305 | 0.3048 | 0.305 | 0.305 | 0.3054 |
| | 0.75 | 0.68775 | 0.68775 | 0.68625 | 0.6862 | 0.686 | 0.6863 | 0.687 |
| | 1 | 1.22267 | 1.22267 | 1.22 | 1.219 | 1.219 | 1.22 | 1.2214 |
| 0.4 | 0.25 | 0.093833 | 0.093833 | 0.0925 | 0.0923 | 0.0924 | 0.0925 | 0.093239 |
| | 0.50 | 0.37533 | 3.75E−01 | 3.70E−01 | 0.368 | 0.367 | 0.37 | 0.373 |
| | 0.75 | 0.8445 | 8.45E−01 | 8.33E−01 | 0.8321 | 0.8319 | 0.8325 | 0.8392 |
| | 1 | 1.50133 | 1.50133 | 1.48 | 1.478 | 1.477 | 1.48 | 1.4918 |

**Table 6** Absolute errors between the solution obtained by HPTM, ADM, VIM, ILTM, FVIM, FRDTM and the exact solution of the non-linear FPPDE.

| $t$ | $x$ | $E_1$[a] | $E_2$[b] | $E_3$[c] | $E_4$[d] | $E_5$[e] | $E_6$[f] |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.25 | 3.3E−05 | 3.3E−05 | 3.7E−05 | 0 | 0 | 0 |
| | 0.50 | 3.2E−05 | 3.2E−05 | 5E−05 | 0 | 0 | 0 |
| | 0.75 | 3E−06 | 3E−06 | 3.8E−05 | 0 | 0 | 0 |
| | 1 | 0.00017 | 0.00017 | 0 | 0 | 0 | 0 |
| 0.4 | 0.25 | 0.000117 | 0.000116 | 5E−05 | 0.0001 | 0.0001 | 0.0001 |
| | 0.5 | 0.000267 | 0.000267 | 0.0004 | 0.0004 | 0.0004 | 0.0004 |
| | 0.75 | 0.00075 | 0.00075 | 0.00075 | 0.0007 | 0.0007 | 0.0007 |
| | 1 | 0.00127 | 0.00127 | 0.0014 | 0.0014 | 0.0014 | 0.0014 |
| 0.6 | 0.25 | 0.000594 | 0.000594 | 0.000739 | 0.000739 | 0.000739 | 0.000739 |
| | 0.50 | 0.00233 | 0.00233 | 0.003 | 0.003 | 0.003 | 0.003 |
| | 0.75 | 0.0053 | 0.0053 | 0.0067 | 0.0067 | 0.0067 | 0.0067 |
| | 1 | 0.00953 | 0.00953 | 0.0118 | 0.0118 | 0.0118 | 0.0118 |

[a] |Exact – HPTM|.
[b] |Exact – ADM|.
[c] |Exact – VIM|.
[d] |Exact – ILTM|.
[e] |Exact – FVIM|.
[f] |Exact – FRDTM|.

the Lagrange multiplier and noise term phenomena which added complexity to the respective techniques while such complexities are not occured in the solution procedure of FRDTM. It shows the simplicity of the FRDTM.

In Yang et al. (2009), the author has mentioned that the VIM took fifteen iterations but here we consider only first three iterations. Also, from Table 6 we can see the rate of convergence, constancy and reliability of the FRDTM.

## 5. Conclusions

We have applied FRDTM and FVIM to solve the linear and nonlinear FPPDEs with space and time fractional derivatives. Numerical solution has been obtained through analytical solution. Comparison has been made with FVIM and the existing methods in the literature and it reveals that FRDTM overcomes the complexity like noise term phenomena, calculation of complicated Lagrange multiplier and redundant calculations. It has been shown that FRDTM is far better than FVIM and the existing methods by means of accuracy, simplicity and reliability.

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