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Original article

Structure of analytical and numerical wave solutions for the Ito integro-differential equation arising in shallow water waves

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ABSTRACT

In this research, by implementing modified analytical and numerical methods, the construction of the analytical and numerical wave solutions for the Ito integro-differential dynamical equation are obtained. The central finite differences are employed to derive the numerical solutions of this equation. We applied the Taylor expansion to test the accuracy of the numerical solutions. We invoke the Von Neumann's stability to explore the stability. The comparison between the exact and numerical results is successfully obtained. We provide some graphical representations to illustrate this comparison and to show the behaviour of the travelling wave solutions. The error which arises from the performance of the used numerical method is investigated. The used methods can be utilized to deal with more nonlinear partial differential equations.

1. Introduction

Nonlinear evolution equations are commonly used to model and analyse most natural events. In particular, these equations play an essential role in describing and explaining some sophisticated phenomena arising in heat transfer, wave propagation, fluid dynamics, optics, elasticity, fibres, plasma physics, electrodynamics, biology, chemistry, ocean engineering, condensed matter physics and many other branches in nonlinear science(Helal and Seadawy, 2012; Khater et al., 2001; Khater et al., 2003; Khater et al., 2006; Calin Itu et al., 2019; Vlase et al., 2019; Praveen Agarwal, 2021; Agarwal et al., 2020; Rahmoune et al., 2020; Agarwal et al., 2020). The travelling wave solutions of such equations are utilized to clarify and interpret the physical mechanism in the real life. Consequently, there has been a great interest in the investigation of exact travelling wave solutions (Inc et al., 2020; Iqbal et al., 2018; Helal et al., 2014; Ozkan, 2020; Ahmad et al., 2020). Some scientists have devoted considerable effort to develop various approaches by which one can straightforwardly extract the exact travelling wave solutions for some partial differential equations. Among these approaches, we specify and state the following: the projective Riccati equation approach (Conte and Musette, 1992), the sine–cosine approach (Wazwaz, 2005; Wazwaz, 2004), the tanh-sech principal (Malflieta and Hereman, 1996; Wazwaz, 2004), the $exp(-f(\zeta))$-expansion process (Alharbi and Almatrafi, 2020), the Adomian decomposition technique (Adomain, 1994; Wazwaz, 2002), the complex hyperbolic function approach (Zayed et al., 2006; Chow, 1995), the rank analysis process (Feng, 2000), the extended tanh method (Fan, 2000; Wazwaz, 2007). For more information about other techniques, one can see Cesar and Gomez (2010), Alharbi et al. (2019), Alharbi and Almatrafi (2020), Alharbi and Almatrafi (2020), Abdelrahman et al. (2020), Alharbi et al. (2020), Alharbi et al. (2020), Alam and Tunç (2019), Shahida and Tunç (2019), Seadawy et al. (2019).

The $(1+1)$-dimensional Ito equation which is given by

$$V_{tt} + V_{xxxt} + 6V_xV_t + 3VV_{xt} + 3V_{xx}\int_{-\infty}^{x} V_t d\theta = 0, \tag{1}$$

was developed by Ito in 1980. According to Ito (1980), this equation is a generalization for the KdV equation. Several authors have discussed Eq. (1) in terms of its exact solutions. For instance, Zhao et al. (2010) used the extended homoclinic test approach to analyse soliton solutions and periodic type of soliton solutions of Eq. (1). The generalized direct algebraic method, the long simple equation

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approach and the modified F-expansion method have been successfully used in Seadawy et al. (2020) to obtain the exact solutions of Ito equation. Furthermore, the generalized Kudryashov technique has been applied in Gepreel et al. (2017) to construct the soliton solutions for Ito equation. The author in Akbari (2017) used the Kudryashov process to demonstrate the exact solutions of the generalized Ito integro-differential equation.

The construction of the exact solutions of Eq. (1) is remarkably limited to some approaches. Thus, we invoke the improved F-expansion process combined with Riccati equation to construct some exact travelling wave solutions for this equation. According to Islam et al. (2014), this method leads to more novel exact travelling wave solutions. The main strategy depends on reducing the integro-differential equation into an appropriate ordinary differential equation (ODE) solved by the considered method. Naturally, the achieved ODE does not provide the whole solutions of the considered PDE, but gives a class of solutions under some conditions. Since the traditional approaches cannot be sometimes applied to handle most nonlinear PDEs, we exploit the numerical methods to approximate the solutions. The finite difference formulae are executed to develop the numerical solutions of Ito equation. The exact and numerical solutions are found nearly the same in the behaviour with a very small error, as can be seen in the presented figures and the provided table.

2. Analysis of the improved F-expansion approach

In this section, we introduce and explain the analysis of the improved F-expansion technique combined with Riccati equation, as presented in Islam et al. (2014). Consider the Riccati equation on the form:

$$F\prime(\zeta) - F^2(\zeta) - r = 0, \tag{2}$$

where $\prime = \frac{d}{d\zeta}$, and r is a real parameter. The general solutions of Eq. (2) are given by

- If $r < 0$, the general solutions of Eq. (2) are formed as

$$F = -\sqrt{-r}\ tanh(\sqrt{-r}\zeta), \tag{3}$$

$$F = -\sqrt{-r}\ coth(\sqrt{-r}\zeta). \tag{4}$$

- If $r = 0$, the general solutions of Eq. (2) are expressed as

$$F = -\frac{1}{\zeta}. \tag{5}$$

- If $r > 0$, the general solutions of Eq. (2) become

$$F = \sqrt{r}\ tan(\sqrt{r}\zeta), \tag{6}$$

$$F = -\sqrt{r}\ cot(\sqrt{r}\zeta). \tag{7}$$

Now, we consider the following PDE:

$$G(v, v_x, v_t, v_{xx}, v_{xt}, \ldots) = 0, \tag{8}$$

where G is a polynomial in $v = v(x,t)$, and its partial derivatives. We now substitute the transform

$$v(x,t) = \Psi(\zeta), \qquad \zeta = kx - wt, \tag{9}$$

into Eq. (8) to reduce it into the following ODE:

$$P(\Psi, \Psi\prime, \Psi\prime\prime, \Psi\prime\prime\prime, \ldots) = 0, \tag{10}$$

where P is a polynomial in Ψ and its derivatives. Next, the travelling wave solutions are shown as follows:

$$\Psi(\zeta) = \sum_{i=0}^{M} \gamma_i (m + F(\zeta))^i + \sum_{j=1}^{M} \lambda_j (m + F(\zeta))^{-j}, \tag{11}$$

where the positive integer M is evaluated from the homogeneous balance, F satisfies Eq. (2) and the constants γ_i, λ_j, w and m are determined later. Substitute Eq. (11) into Eq. (10) to have a polynomial in F. Equate the coefficients of F to zero to end up with an algebraic system. The solutions of this system give the vales of the constants γ_i, λ_j, w and m. Inserting the values of these constants into Eq. (11) yields the travelling wave solutions.

3. Establishment of exact solutions

In this part, we derive new hyperbolic and trigonometric travelling wave solutions for the following equations:

$$V_{tt} + V_{xxxt} + 2\beta V_x V_t + \beta V V_{xt} + \beta V_{xx} \int_{-\infty}^{x} V_t d\theta = 0, \tag{12}$$

where β is an arbitrary constant. We first convert Eq. (12) into a new PDE by substituting $V = v_x$, into Eq. (12). Achieving this, we have

$$v_{xtt} + v_{xxxxt} + 2\beta\, v_{xx}\, v_{xt} + \beta\, v_x\, v_{xxt} + \beta\, v_{xxx}\, v_t = 0. \tag{13}$$

The transform

$$v(x,\,t) = \Psi(\zeta),\ \ \zeta = kx - wt, \tag{14}$$

where k and w are real numbers, is now substituted into Eq. (13) to give

$$w^2 k \Psi_{\zeta\zeta\zeta} - wk^4 \Psi_{\zeta\zeta\zeta\zeta\zeta} - 2\beta wk^3 \left(\Psi^2_{\zeta\zeta} + \Psi_{\zeta\zeta\zeta}\Psi_{\zeta}\right) = 0. \tag{15}$$

Integrating twice w.r.t. ζ and equating the integral constants to zero lead to

$$w\Psi_\zeta - k^3 \Psi_{\zeta\zeta\zeta} - \beta k^2 \Psi^2_\zeta = 0. \tag{16}$$

The homogeneous balance between $\Psi_{\zeta\zeta\zeta}$ and Ψ^2_ζ gives $M = 1$. Therefore, the travelling wave solution becomes

$$\Psi(\zeta) = \gamma_0 + \gamma_1 (m + F(\zeta)) + \frac{\lambda_1}{m + F(\zeta)}. \tag{17}$$

Next, we plug Eq. (17) into Eq. (16) and equate the coefficients of F^n, $n = 0, 1, 2, \ldots, 8$, to zero to obtain an algebraic system whose solutions are given by

$$\begin{aligned} &\text{Case 1}: \quad \gamma_1 = 0, \quad \lambda_1 = \tfrac{6(k^2 m^2 + k^2 r)}{\beta}, \quad w = -4rk^3, \\ &\text{Case 2}: \quad \gamma_1 = 0, \quad \lambda_1 = \tfrac{6k^2 r}{\beta}, \quad w = -4rk^3, m = 0, \\ &\text{Case 3}: \quad \gamma_1 = -\tfrac{6k^2}{\beta}, \quad \lambda_1 = 0, \quad w = -4rk^3, m = 0, \\ &\text{Case 4}: \quad \gamma_1 = -\tfrac{6k^2}{\beta}, \quad \lambda_1 = \tfrac{6k^2 r}{\beta}, \quad w = -16rk^3, m = 0. \end{aligned} \tag{18}$$

Here, $v_{i,j}$ and $V_{i,j}$ are the exact solutions of Eqs. (13) and (12), respectively. The index i indicates the solution number while j illustrates the case number. Hence, the exact solutions of Eqs. (13) and (12) can be simply formed as follows:

1. If $r < 0$ and $k = -1$. We have
Case 1:

$$v_{1,1}(x,t) = \gamma_0 + \frac{6\left(k^2 m^2 + k^2 r\right)}{\beta} \left(m + \sqrt{-r}\,tanh(\sqrt{-r}(kx + 4rk^3 t))\right)^{-1}. \tag{19}$$

Hence, the exact solution of Eq. (12), which is determined by taking the first derivative of $v_{1,1}$ with respect to x, is

$$V_{1,1}(x,t) = \frac{6kr\left(k^2 m^2 + k^2 r\right) sech^2\left(\sqrt{-r}\left(4k^3 rt + kx\right)\right)}{\beta\left(\sqrt{-r}\tanh\left(\sqrt{-r}\left(4k^3 rt + kx\right)\right) + m\right)^2}. \tag{20}$$

Case 2:

$$v_{1,2}(x,t) = \gamma_0 - \frac{6k^2 r}{\beta}\left(m + \sqrt{-r}\,tanh(\sqrt{-r}(kx + 4rk^3 t))\right)^{-1}. \tag{21}$$

Therefore, the exact solution of Eq. (12) obtaining by taking the first derivative of $v_{1,2}$ with respect to x, is

$$V_{1,2}(x,t) = \frac{6k^3 r^2 sech^2\left(\sqrt{-r}\left(4k^3 rt + kx\right)\right)}{\beta\left(\sqrt{-r}\tanh\left(\sqrt{-r}\left(4k^3 rt + kx\right)\right) + m\right)^2}. \tag{22}$$

Case 3:

$$v_{1,3}(x,t) = \gamma_0 - \frac{6k^2}{\beta}\left(m + \sqrt{-r}\,tanh(\sqrt{-r}(kx + 4k^3 rt))\right). \tag{23}$$

Thus, the exact solution of Eq. (12) is shown as follows:

$$V_{1,3}(x,t) = \frac{6k^3 r sech^2\left(\sqrt{-r}\left(4k^3 rt + kx\right)\right)}{\beta}. \tag{24}$$

Case 4:

$$v_{1,4}(x,t) = \gamma_0 - \frac{6k^2}{\beta}\left(\sqrt{-r}\,tanh(\sqrt{-r}(kx + 16k^3 rt))\right) + \frac{6k^2 r}{\beta\left(\sqrt{-r}\,tanh(\sqrt{-r}(kx + 16k^3 rt))\right)}. \tag{25}$$

Hence, the exact solution of Eq. (12) is expressed as

$$V_{1,4}(x,t) = \frac{6r\,\mathrm{csch}^2\left(\sqrt{-r}(-16rt - x)\right)}{\beta} - \frac{6r\,\mathrm{sech}^2\left(\sqrt{-r}(-16rt - x)\right)}{\beta}. \tag{26}$$

2. If $r > 0$, $k = +1$ then

Case 1:

$$v_{2,1}(x,t) = \gamma_0 + \frac{6\left(k^2 m^2 + k^2 r\right)}{\beta}\left(m + \sqrt{r}\,tan(\sqrt{r}(kx + 4rk^3 t))\right)^{-1}. \tag{27}$$

Thus, the exact solution of Eq. (12) is given by

$$V_{2,1}(x,t) = -\frac{6r(m^2 + r)\sec^2\left(\sqrt{r}(4rt + x)\right)}{\beta\left(m + \sqrt{r}\tan\left(\sqrt{r}(4rt + x)\right)\right)^2}. \tag{28}$$

Case 2:

$$v_{2,2}(x,t) = \gamma_0 + \frac{6k^2 r}{\beta}\left(\sqrt{r}\,tan(\sqrt{r}(kx + 4rk^3 t))\right)^{-1}. \tag{29}$$

Therefore, the exact solution of Eq. (12) is shown as follows:

$$V_{2,2}(x,t) = -\frac{6r\csc^2\left(\sqrt{r}(4rt + x)\right)}{\beta}. \tag{30}$$

Case 3:

$$v_{2,3}(x,t) = \gamma_0 - \frac{6k^2}{\beta}\left(m + \sqrt{r}\,tan(\sqrt{r}(kx + 4rk^3 t))\right). \tag{31}$$

Hence, the exact solution of Eq. (12) is given by

$$V_{2,3}(x,t) = -\frac{6r\sec^2\left(\sqrt{r}(4rt + x)\right)}{\beta}. \tag{32}$$

Case 4:

$$v_{2,4}(x,t) = \gamma_0 - \frac{6k^2}{\beta}\left(\sqrt{r}\,tan(\sqrt{r}(kx + 16rk^3 t))\right) + \frac{6k^2 r}{\beta\left(\sqrt{r}\,tan(\sqrt{r}(kx + 16rk^3 t))\right)}. \tag{33}$$

Thus, the exact solution of Eq. (12) is illustrated as

$$V_{2,4}(x,t) = -\frac{6r\csc^2\left(\sqrt{r}(16rt + x)\right)}{\beta} - \frac{6r\sec^2\left(\sqrt{r}(16rt + x)\right)}{\beta}. \tag{34}$$

4. Establishment of numerical solutions

This section is devoted to utterly demonstrate the numerical results of Eqs. (12) and (13) by executing the central finite differences. The numerical solutions are investigated on the interval $[a, b]$, as illustrated in Fig. 1. Here, a and b denote the endpoints of the physical domain. We begin with generating the numerical solutions of Eq. (13) by reforming Eq. (13) into the following system:

$$\begin{aligned} &v_t = g_x, \\ &g_{xxt} + Q_{xx} = 0, \end{aligned} \tag{35}$$

where

$$Q = g_{xxx} + \beta(vg_x)_x - \beta v g_{xx}. \tag{36}$$

The variables x and t indicate the spatial and time variables, respectively. Modifying Eq. (13) into a system leads a scheme much faster and more stable. Let $U = g_{xx}$. Then, system (35) becomes

$$\begin{aligned} &v_t = g_x, \\ &U_t + Q_{xx} = 0, \end{aligned} \tag{37}$$

where

$$Q = g_{xxx} + \beta(vg_x)_x - \beta v g_{xx}.$$

The initial conditions of system (37) is generated by introducing the function g which reads

$$g(x,t) = \int_{x_L}^{x} \frac{\partial v(\xi, t)}{\partial t} d\xi. \tag{38}$$

Eq. (19) is selected to build the initial condition of v. Hence, Eq. (38) becomes

$$g(x,t) = \frac{24r^2 k^5 (m^2 + r)}{\beta k(rm - \sqrt{-r^3}\tanh(-\sqrt{-r}(-4rk^3 t - kx)))}. \tag{39}$$

Thus, the initial conditions of system (37) are shown as

$$\begin{aligned} &v_{1,1}(x,0) = \gamma_0 + \frac{6\left(k^2 m^2 + k^2 r\right)}{\beta}\left(m + \sqrt{-r}\,tanh(\sqrt{-r}(kx))\right)^{-1}, \\ &g(x,0) = \frac{24r^2 k^5\left(m^2 + r\right)}{\beta k(rm - \sqrt{-r^3}\tanh(-\sqrt{-r}(-kx)))}. \end{aligned} \tag{40}$$

It is worth noting that Fig. 1 is perfectly utilized to deduce the relevant boundary conditions which are

$$\begin{aligned} &v_x = g_x = 0, \quad \text{at} \quad x = a, \\ &v_x = g_x = 0, \quad \text{at} \quad x = b, \end{aligned} \tag{41}$$

where a and b are the endpoints of the physical domain. Eqs. (41) are also used to determine the fictitious points which we invoke to find the spatial derivatives at the boundaries of the domain. The interval $[a, b]$ is divided into sub-intervals such that

$$x_n = a + (n-1)\Delta x, \qquad \forall n = 1, 2, \ldots, N+1. \tag{42}$$

Here, Δx indicates the space increment. It is important to mention that the numerical results at the mesh node are characterized by v_n and g_n. In addition, we semi-discretize the spatial derivatives

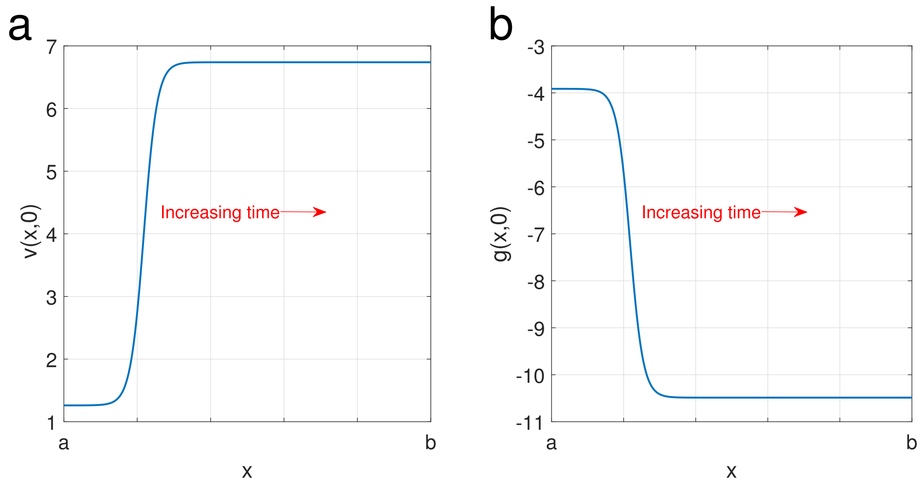

Fig. 1. (a) and (b) represent the initial conditions for v and g, respectively, using Eqs. (40). The parameter values are taken by $\gamma_0=-2$, $r=-0.3$, $\beta=1.2$, $m=1.2$.

whereas the temporal derivatives are kept continuous. Consequently, the new system is given by

$$\begin{aligned}
&v_t|_n=\tfrac{1}{2\Delta x}\left(g_{n+1}-g_{n-1}\right)=0,\\
&U_t|_n=-\tfrac{Q_{n+1}-2Q_n+Q_{n-1}}{\Delta x^2},\\
&Q_n=g_{xxx}|_n+\beta\left(vg_x\right)_x\big|_n-\beta\, v_n\,\tfrac{g_{n+1}-2g_n+g_{n-1}}{\Delta x^2}.
\end{aligned}\tag{43}$$

The discretization of the term $g_{xxx}|_n$ and the term $\left(vg_x\right)_x\big|_n$ can be expressed as

$$\begin{aligned}
&g_{xxx}|_n=\tfrac{g_{n+2}-3g_{n+1}+3g_n-g_{n-1}}{2\Delta x^3},\\
&\left(vg_x\right)_x\big|_n=\tfrac{\left(v_{n+1}+v_n\right)\left(g_{n+1}-g_n\right)-\left(v_n+v_{n-1}\right)\left(g_n-g_{n-1}\right)}{2\Delta x^2}.
\end{aligned}$$

The conditions (41) are upgraded with their ODE form

$$\begin{aligned}
&v_{t,1}=v_{t,N+1}=0,\\
&g_{t,1}=g_{t,N+1}=0,
\end{aligned}\tag{44}$$

Here, $\Delta x=(b-a)/N$. It is to noted that the method of lines is applied here. This approach depends on DASPK solver (Brown et al., 1994) which employs backward differentiation formulas so as to roughly specify time derivatives. Furthermore, the Jacobian matrix of the linearised system is approximated using LU factorization. In order to achieve less bandwidth for the matrix, a unique system numbering for the variables $v_1, g_1, v_2, g_2, \ldots, v_{N-1}, g_{N-1}, v_N, g_N, v_{N+1}, g_{N+1}$, is used.

4.1. Accuracy of the scheme

Taylor expansion is applied on the numerical scheme to manifest its accuracy. The accuracy is studied by substituting Taylor expansion into the scheme and simplifying the result. This gives us that the accuracy is from $O(\Delta t^2, \Delta x^2)$. As a result, the accuracy of the schemes are from the second order in time and space.

4.2. Stability of the scheme

Von Neumann's stability is carried out in this subsection to examine the stability of the numerical scheme. Assume that $\alpha=v(x_n,t_m)$. Then, system (37) is converted into

$$\begin{aligned}
&v_t=g_x,\\
&g_{txx}+Q_{xx}=0,\\
&Q_{xx}=\left(g_{xxx}+\beta\alpha g_{xx}-\beta\alpha g_{xx}\right).
\end{aligned}\tag{45}$$

System (45) is clearly discretized as

$$\begin{aligned}
&v_n^{m+1}-v_n^m=\tfrac{\Delta t}{2\Delta x}\left(g_{n+1}-g_{n-1}\right),\\
&g_{xx}|_n^{m+1}-g_{xx}|_n^m+\tfrac{\Delta t}{\Delta x^2}\left(Q_{n+1}^{m+1}-2Q_n^{m+1}+Q_{n-1}^{m+1}\right)=0,\\
&Q_n^{m+1}=\tfrac{1}{\Delta x}\left(g_{xx}|_{n+1}^{m+1}-g_{xx}|_n^{m+1}\right).
\end{aligned}\tag{46}$$

Next, Von Neumann's concepts of v_n^m and g_n^m are given by

$$v_n^m=\gamma^m e^{i\Delta x\,\xi\, n},\qquad g_n^m=\mu^m e^{i\Delta x\,\xi\, n}.\tag{47}$$

Here, γ, $\mu$and ξ are constants. Insert Eqs. (47) into system (46) and simplify to have

$$\begin{aligned}
&v_n^{m+1}-s_1 g_n^{m+1}=v_n^m,\\
&0\,v_n^{m+1}+(1-s_2)g_n^{m+1}=g_n^m.
\end{aligned}\tag{48}$$

System (48) can be simply written as

$$\begin{bmatrix}1 & s_1\\ 0 & 1-s_2\end{bmatrix}\begin{bmatrix}v\\ g\end{bmatrix}^{m+1}=\begin{bmatrix}v\\ g\end{bmatrix}^{m}.\tag{49}$$

Therefore,

$$\begin{bmatrix}v\\ g\end{bmatrix}^{m+1}=A\begin{bmatrix}v\\ g\end{bmatrix}^{m},\tag{50}$$

where

$$A=\begin{bmatrix}1 & \frac{s_1}{1-s_2}\\ 0 & \frac{1}{1-s_2}\end{bmatrix}.$$

s_1 and s_2 are given by

$$\begin{aligned}
&s_1=\tfrac{\Delta t}{\Delta x}\, i\sin(\xi\Delta x),\\
&s_2=\tfrac{4\Delta t}{\Delta x^3}\sin^2\left(\tfrac{\xi\Delta x}{2}\right)\left(e^{i\xi\Delta x}-1\right),
\end{aligned}$$

where Δt is the time step. The eigenvalue of the matrix A is given by

$$\lambda_{1,2}=\left\{-\frac{1}{s_2-1},1\right\}.\tag{51}$$

The stability occurs if the maximum eigenvalues of A is less than or equal one. However, this is obviously satisfied in the obtained eigenvalues in which the maximum eigenvalue is one. As a consequence, we conclude that the numerical scheme is unconditionally stable.

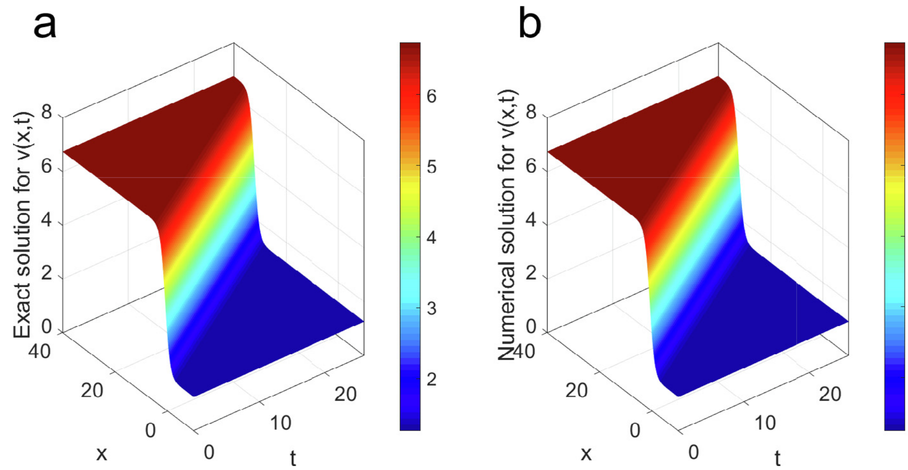

Fig. 2. Graphical comparison between the analytical solution of $v(x,t)$ (left) and the numerical solution of $v(x,t)$ (right). The values of the used parameters are $\gamma_0 = -2, r = -0.3, \beta = 1.2, m = 1.2$.

Table 1
L_2 error and CPU time taken to reach $t = 10$ for the numerical approach.

Δx	L_2 error	CPU
$1.000e-01$	$3.206e-01$	7.26×10^{-3} s
$5.000e-02$	$1.575e-01$	1.49×10^{-2} s
$1.000e-02$	$6.800e-03$	7.14×10^{-2} s
$5.000e-03$	$1.730e-03$	1.40×10^{-1} s
$2.000e-03$	$2.796e-04$	3.20×10^{-1} s
$1.000e-03$	$7.640e-05$	6.80×10^{-1} s

5. Result and discussion

In this paper, we have precisely deduced novel exact travelling wave solutions on the form of hyperbolic and trigonometric functions for Eqs. (12) and (13) using the improved F-expansion process combined with Riccati equation. The presented solutions are more general than the solutions investigated by Gepreel et al. (2017) and by Akbari (2017). Akbari (2017) determined one rational solution for the generalized Ito integro-differential equation via the Kudryashov method whereas the authors in Gepreel et al. (2017) applied the generalized Kudryashov technique to extract three rational solutions. The finite difference formulae are perfectly employed in this paper. The achieved numerical solutions are super acceptable. Therefore, the applied methods are more reliable and powerful.

The central finite differences have accurately presented approximated solutions. For instance, Fig. 2 shows that the exact travelling wave solution and the numerical solution of $v(x,t)$ are almost coincident and concomitant. Furthermore, the 2D diagram in Fig. 7 illustrates that the performance of the numerical method for different values of Δx where the smallest value is the best. In particular, when we take $\Delta x = 0.1$, the resulting L_2 error of the numerical solution (dashed yellow line) is large while this error is strongly reduced when we use $\Delta x = 0.001$, (dashed blue line). Note that all figures are plotted under the values $\gamma_0 = -2, r = -0.3, \beta = 1.2, m = 1.2$. L_2 error and CPU time can be easily observed in Table 1. We begin with $\Delta x = 1.000e-01$ where the L_2 error reaches $3.206e-01$ in 7.26×10^{-3} second. This error arrives at $1.730e-03$ for smaller $\Delta x = 5.000e-03$ during 1.40×10^{-1} seconds which increases. However, the method gives suitable performance and shrinks the error for small $\Delta x = 1.000e-03$ with a slight increment in the CPU time (6.80×10^{-1} s) (see Figs. 3–6).

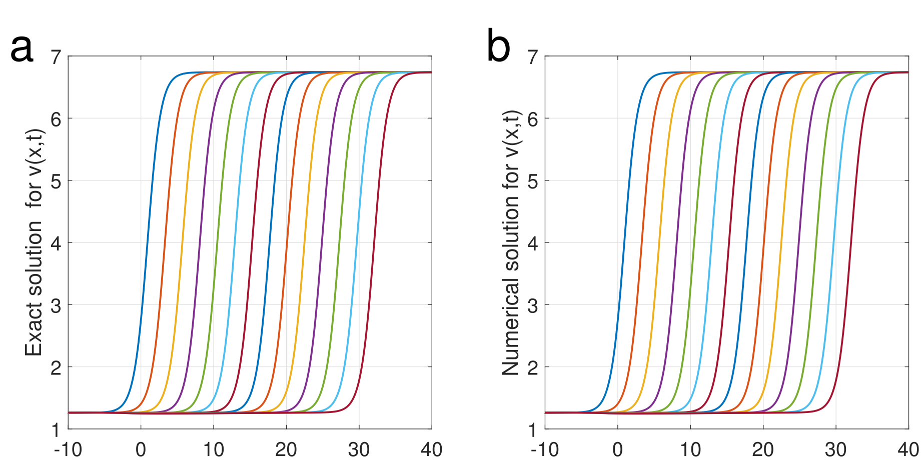

Fig. 3. Graphs (a) and (b) represent the time evolution of a single travelling wave for the exact and numerical results of $v(x,t)$.

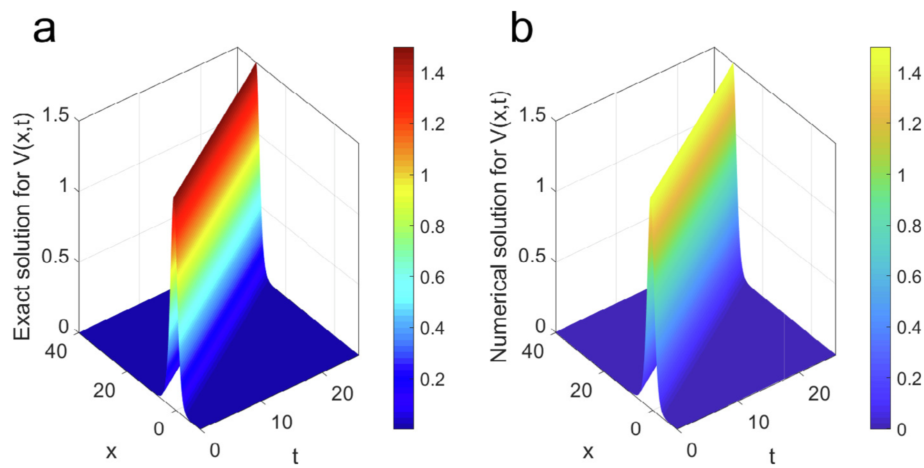

Fig. 4. 3D surfaces sketching a single travelling wave for exact and numerical solutions of $V(x,t)$. The plots also compare the characteristics between the two solutions.

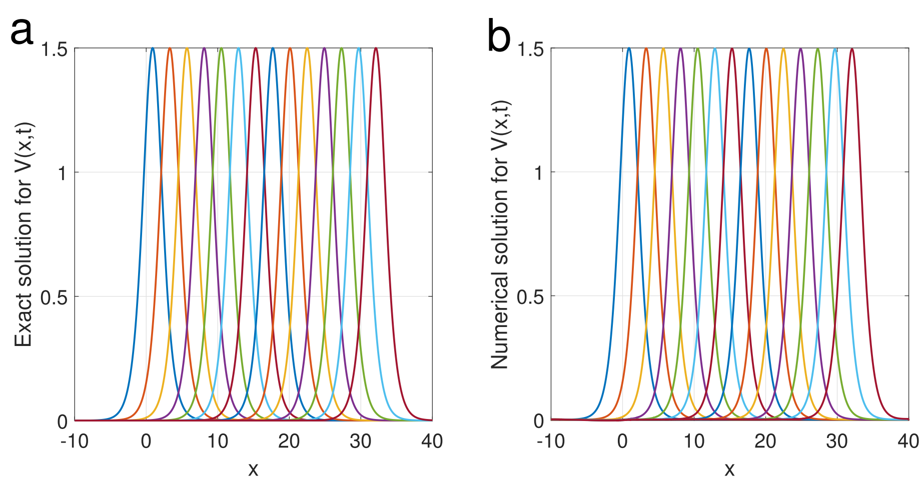

Fig. 5. The time development for a solitary wave solution of the exact solution of $V(x,t)$ (left) and the numerical solution of $V(x,t)$ (right).

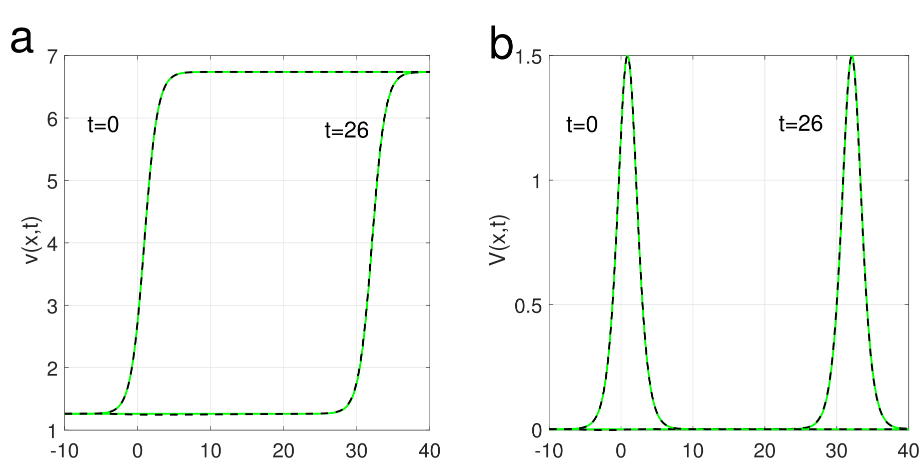

Fig. 6. Diagram (a) shows that the exact solution of $v(x,t)$ is almost the same as its numerical solution while Figure (b) illustrates the behaviour of the exact and numerical solutions of $V(x,t)$.

6. Conclusion

The improved F-expansion approach combined with Riccati equation and the central finite differences have been utilized in this research to establish the exact travelling wave solutions and the numerical solutions, respectively, for Eqs. (12) and (13). The correctness of the accomplished solutions is verified by substituting the solutions into the leading equations. The numerical solu-

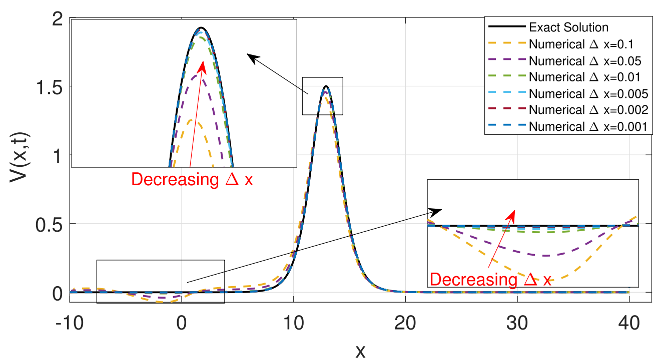

Fig. 7. Comparison between the exact and numerical solutions of $V(x,t)$ for various values of Δx. The numerical solution approaches the exact solution if Δx is very small.

tions approximately approach to the exact solutions for small Δx, as can be seen in Fig. 7. More precisely, L_2 error rapidly declines for smaller Δx, as presented in Table 1. The numerical scheme is found unconditionally stable via Von Neumann stability. Its accuracy is from second order in time and space. The applied processes lead to practical and powerful results. Eventually, we conclude that the employed techniques can be certainly used to deal with more complicated nonlinear evolution equations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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