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Original article

A class of Langevin time-delay differential equations with general fractional orders and their applications to vibration theory

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ABSTRACT

Langevin differential equations with fractional orders play a significant role due to their applications in vibration theory, viscoelasticity and electrical circuits. In this paper, we mainly study the explicit analytical representation of solutions to a class of Langevin time-delay differential equations with general fractional orders, for both homogeneous and inhomogeneous cases. First, we propose a new representation of the solution via a recently defined delayed Mittag-Leffler type function with double infinite series to homogeneous Langevin differential equation with a constant delay using the Laplace transform technique. Second, we obtain exact formulas of the solutions of the inhomogeneous Langevin type delay differential equation via the fractional analogue of the variation constants formula and apply them to vibration theory. Moreover, we prove the existence and uniqueness problem of solutions of nonlinear fractional Langevin equations with constant delay using Banach's fixed point theorem in terms of a weighted norm with respect to exponential functions. Furthermore, the concept of stability analysis in the mean of solutions to Langevin time-delay differential equations based on the fixed point approach is proposed. Finally, an example is given to demonstrate the effectiveness of the proposed results.
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1. Introduction

In recent decades, fractional differential equations (FDEs) have attracted growing attention due to their extensive applications in mechanics (Mahmudov et al., 2020), time-delay systems (Huseynov and Mahmudov, 2020), electrical circuits (Ahmadova and Mahmudov, 2021), stability analysis (Cong et al., 2018), and stochastic analysis (Ahmadova and Mahmudov, 2020).

The classical Langevin equations (LEs) were proposed by a French physicist Paul Langevin in 1908 and he gave an exhaustive overview of Brownian motion. In the theory of Brownian motion, the classical Langevin equations are the important differential equations describing the progression of physical phenomena in fluctuating environments. However, for systems in complex media, the classical LEs cannot provide a sufficient and correct description of the dynamics. To deal with such problems, various generalizations of LEs have been introduced and studied in terms of various kinds of differential and integral operators of fractional-order. The fractional Langevin equations (FLEs), which are a generalization of the classical LEs, are of great interest not only from the point of view of the theory of stochastic processes (Mainardi and Pironi, 1996), but also by means of physical applications (Ahmadova and Mahmudov, 2021; Kobelev and Romanov, 2000).

FDEs containing not only one fractional derivative but also more than one fractional derivative are intensively studied in many complex systems. Recently, the physical processes have been represented by two main mathematical ways: multi-term equations (Camargo et al., 2009; Luchko and Gorenflo, 1999; Bazhlekova, 2013; Zhang and Hou, 2020) and multi-order systems (Wang and Ren, 2020; Huseynov et al., 2020; Ahmadova et al., 2021). Multi-term FDEs have been studied due to their applications in modelling and solved by various mathematical methods. In Luchko and Gorenflo (1999), Luchko and Gorenflo solved the multi-term FDEs with constant coefficients and with the Caputo fractional derivatives by using the method of operational calculus. Furthermore, in Bazhlekova (2013), Bazhlekova has considered the multi-term fractional relaxation equations with Caputo derivatives using the Laplace transform technique and studied the fundamental and impulse-response solutions of the initial value problem (IVP).

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In terms of numerical methods, Edwards et al. (2002) and Diethelm and Luchko (2004) considered the IVP for the general linear multi-term FDEs.

As one of the important special cases of multi-term DEs, FLEs with two different fractional derivatives have been discussed in Ahmadova and Mahmudov (2021), Cong et al. (2018), Ahmadova and Mahmudov (2020), Mainardi and Pironi (1996), Kobelev and Romanov (2000), Camargo et al. (2009), Luchko and Gorenflo (1999), Bazhlekova (2013), Zhang and Hou (2020), Wang and Ren (2020), Huseynov et al. (2020), Ahmadova et al. (2021), Edwards et al. (2002), Diethelm and Luchko (2004), Wang et al. (2020), Fazli et al. (2020), Baghani and Nieto (2019), Darzi et al. (2020), Baghani et al. (2021), Baghani (2017), Ahmad et al. (2012), Lim et al. (2008). Nowadays, the existence and uniqueness problem of solutions of initial and boundary value problems for nonlinear FLEs is extensively studied in Fazli et al. (2020), Baghani and Nieto (2019), Darzi et al. (2020), Baghani et al. (2021), Baghani (2017), Ahmad et al. (2012).

In Fazli et al. (2020), Fazli et al. have investigated the existence and uniqueness results for the following Cauchy problem of FLE involving two various fractional orders in sequential sense using the fixed point theorems of Banach and Weissinger:

$$\begin{cases} D^{\beta}_{0^+}(D^{\alpha}_{0^+}+\lambda)x(r)=f(r,x(r)), & 0<r\leqslant 1,\\ D^{i}_{0^+}x(0)=\mu_i, & 0\leqslant i<l,\\ D^{i}_{0^+}D^{\alpha}_{0^+}x(0)=\nu_i, & 0\leqslant i<n,\end{cases} \tag{1.1}$$

where $D^{\alpha}_{0^+}x(\cdot)$ and $D^{\beta}_{0^+}x(\cdot)$ are Caputo fractional derivatives of orders α and β in different intervals

$m-1<\alpha\leqslant m$ and $n-1<\beta\leqslant n$ with $l=\max\{m,n\}$ where $m,n\in\mathbb{N}, x(r)$ is the particle displacement, $\lambda\in\mathbb{R}$ is the friction coefficient, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a Lebesgue measurable function which represents a noise term.

In Baghani and Nieto, 2019, Baghani and Nieto have studied the existence and uniqueness results for the following boundary value problem of FLE involving two various fractional orders in sequential sense:

$$\begin{cases} D^{\beta}_{0^+}(D^{\alpha}_{0^+}+\lambda)x(r)=f(r,x(r)), & 0\leqslant r\leqslant 1,\\ x(0)=x(1), & \\ D^{2\alpha}_{0^+}x(1)+\lambda D^{\alpha}_{0^+}x(1)=0, & \end{cases} \tag{1.2}$$

where $D^{\alpha}_{0^+}x(\cdot)$and $D^{\beta}_{0^+}x(\cdot)$ are Caputo fractional derivatives of orders α and β in different intervals $0<\alpha\leqslant 1$ and $1<\beta\leqslant 2, \lambda\in\mathbb{R}$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Furthermore, Darzi et al. (2020) have considered the existence and uniqueness of initial value problem for nonlinear Langevin equation involving three fractional orders. In Baghani et al. (2021), Baghani et al. have analyzed existence, uniqueness and Hyers-Ulam stability results of solutions for the nonlinear fractional Langevin equation involving two fractional orders with three-point boundary conditions with the help of Krasnoselskii's fixed point theorem with respect to an appropriate weighted Banach space. In Baghani (2017), Baghani has mainly discussed the existence and uniqueness of a solution for the IVP of LEs. In Ahmad et al. (2012), Ahmad et al. discussed the nonlocal boundary value problem for nonlinear FLEs with two various fractional derivatives in different intervals. In accordance with an essential role of FLEs in applied sciences, these equations are extensively analyzed from analytical and numerical points of view. By means of analytical methods, in Lim et al. (2008), Lim et al. have studied the explicit analytical representation of solutions for a new type of FLEs with two Weyl fractional derivatives with the aid of Gauss's hypergeometric functions. In Ahmadova and Mahmudov (2021), Ahmadova and Mahmudov provide explicit formulas of solutions for linear FLEs with general fractional differential operators of Caputo type using Mittag-Leffler (M-L) type functions and studied their applications of FLEs to electrical circuit theory.

Fractional delay differential equations (FDDEs) are differential equations covering fractional-order differential operators and time-delays. Delay differential equations (DDEs) with fractional-order have gained considerable attention due to their applications in science, engineering and physics using appropriate numerical methods and graphical tools. Recently, the theory of FDDEs has also been well-established by means of analytical methods. Initially, Khusainov et al. (2005) has provided an analytical representation of the solution of a linear homogeneous matrix DE with permutable matrices in terms of infinite series. Note that the fractional analog of the same problem was considered by Li and Wang (2017), in particular in the case of $A=\Theta$, where Θ is a zero matrix. In another work, Li and Wang (2018) introduced a concept of delayed M-L type matrix function via a two-parameter M-L function and presented finite-time stability results for nonlinear FDDEs in the same special case. Mahmudov (2018) proposed a newly defined explicit formula to linear homogeneous and inhomogeneous fractional time-delayed systems via two-parameter M-L perturbation in the general case (i.e., A and B are arbitrary constant matrices). Huseynov and Mahmudov (2020) have provided a new representation of a solution through a delayed analog of three-parameter M-L functions under the assumptions in which A and B are commutative matrices. Moreover, Langevin type time-delay differential equations with two Riemann-Liouville fractional derivatives have considered by Mahmudov (2020). In Mahmudov (2020), Mahmudov has introduced an exact analytical formula for the solution of linear inhomogeneous FLE with a constant delay and studied the stability results of the solutions by means of a fixed point approach.

Stability analysis for fractional-order dynamical systems has been discussed over many years as one of the most essential topics in control engineering. During the last decades, a large number of papers related to stability theory have been published in the sense of Ulam-Hyers (Wang and Zhang, 2014; Mahmudov and Al-Khateeb, 2020; Ahmadova and Mahmudov, 2021; Peng and Wang, 2015). Wang and Zhang (2014) have studied Ulam-Hyers stability results of nonlinear FDEs with Hadamard fractional derivative in the weighted space of continuous functions. In Mahmudov and Al-Khateeb (2020), Mahmudov et al. have obtained several existence and Ulam-Hyers results for an IVP of time-delay Hadamard-type FDEs using a delayed perturbation of the M-L matrix functions with logarithms. In Ahmadova and Mahmudov (2021), Ahmadova and Mahmudov have studied stability results in Ulam-Hyers sense for the nonlinear fractional stochastic neutral differential equations system with the aid of weighted maximum norm and Itô's isometry in finite dimensional stochastic setting. Furthermore, Peng and Wang (2015) have studied stability results in terms of Ulam-Hyers for a multi-term FDEs using direct analysis methods.

Applications of FDEs include the study of vibration theory as a part of mechanical physics. In Liu and Duan (2015), Liu and Doan have discussed the asymptotic behavior of fundamental solutions of the fractional vibration equations (FVEs) where the damping term is characterized by means of Caputo type fractional derivative of order α satisfying $0<\alpha<1$ or $1<\alpha<2$. A detailed analysis for the analytical solutions is carried out via the Laplace integral transform and its complex inversion integral formula. In Gomez-Aguilar et al. (2012), Gomez-Aguilar et al. analyzed the analytical solutions of the mass-spring and spring-damper with regard to the classical M-L functions. In Wang and Hu (2010), ZaiHua and HaiYan have studied the asymptotic stability analysis of zero solution of a linear vibration system with fractional-order derivative of order $0<\alpha<2$. In Hong et al. (2006), Hong et al. have proposed an analytical scheme for a dynamic oscillatory system with a single degree of- freedom whose damping is described by fractional-order derivative of order β with $0<\beta<1$. Its analytical solutions are expressed by means of two-parameter M-L functions with the aid of fractional Green's function and Laplace transform technique. The solution in Hong et al. (2006) takes the form of a single power series and a generalized M-L function, but unlike Hong et al. we find an explicit analytic solution of the Cauchy problem for the following vibrational equations with two fractional orders $1<\alpha\leqslant 2$, $0<\beta\leqslant 1$ and a constant delay in terms of a new delayed analogue of M-L type function:

$$\begin{cases} M\left({}^{C}D^{\alpha}_{0^+}y\right)(r)+C\left({}^{C}D^{\beta}_{0^+}y\right)(r)+Ky(r-\tau)=\mathbb{F}_e(r), \quad r>0, \quad \tau>0, \\ y(r)=\varphi(r), -\tau\leqslant r\leqslant 0. \end{cases}$$

In the same vein as the above articles, our aim is to investigate an IVP for the following in-homogeneous Langevin time-delay DEs involving general fractional orders in Caputo sense as below:

$$\begin{cases} \left({}^{C}D^{\alpha}_{0^+}y\right)(r)-\mu\left({}^{C}D^{\beta}_{0^+}y\right)(r)-\lambda y(r-\tau)=g(r), \quad r\in(0,T], \quad \tau>0, \\ y(r)=\varphi(r), \quad -\tau\leqslant r\leqslant 0, \end{cases} \tag{1.3}$$

where $\left({}^{C}D^{\alpha}_{0^+}y\right)(\cdot)$and $\left({}^{C}D^{\beta}_{0^+}y\right)(\cdot)$ are Caputo fractional derivatives of orders α and β in different intervals $m-1<\alpha\leqslant m$, $m-2<\beta\leqslant m-1$, with $m\geqslant 2$, $\varphi:[-\tau,0]\to\mathbb{R}$ is an arbitrary $(m-1)$-times continuously differentiable real-valued function that determines initial conditions, $g:[0,T]\to\mathbb{R}$ is a continuous function, $\lambda,\mu$are real numbers and $T=n\tau$ for a fixed $n\in\mathbb{N}$.

Furthermore, unlike the above papers (Baghani and Nieto, 2019; Darzi et al., 2020), we present several important results on the existence & uniqueness, and Ulam-Hyers stability of solutions to fractional-order delayed Langevin equation with a constant delay by using the explicit analytical representation of solutions and properties of newly defined delayed Mittag-Leffler type scalar-valued functions whilst their technique is based on converting the fractional differential equation into the equivalent Volterra type integral equation:

$$\begin{cases} \left({}^{C}D^{\alpha}_{0^+}y\right)(r)-\mu\left({}^{C}D^{\beta}_{0^+}y\right)(r)-\lambda y(r-\tau)=g(r,y(r)), \quad r\in(0,T], \quad \tau>0, \\ y(r)=\varphi(r), \quad -\tau\leqslant r\leqslant 0, \end{cases} \tag{1.4}$$

where $m-1<\alpha\leqslant m, m-2<\beta\leqslant m-1, m\geqslant 2$ with $\alpha-\beta\geqslant 1$, and $g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a nonlinear perturbation.

The structure of our research paper is outlined as follows. Section 2 is a preparatory section where we recall the main definitions, results and necessary lemmas from fractional calculus, special functions and FDEs. Section 3 is devoted to finding the analytical solutions of the homogeneous linear Langevin type time-delay DEs (1.3) with general fractional orders and we derive a special case when $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$ with the help of Laplace transform technique. In Section 4, we propose the structure of the analytical representation of solutions to the Cauchy problem for inhomogeneous time-delay FLEs using recently defined delayed M-L type functions. Moreover, we show that our analytical results in terms of M-L type functions coincides with the solutions in terms of Fox-Wright functions for delay-free systems. In Section 5, we prove the existence and uniqueness results of nonlinear fractional Langevin type DEs with a constant delay using the new appropriate norm concerning the exponential function in the weighted space of continuous functions. Section 6 is devoted to the presentation of the stability analysis of solutions of nonlinear FLEs with a constant delay in terms of Ulam-Hyers sense based on a fixed point approach. In Section 7, we present an application in the vibration theory and compare results for an *external force* $\mathbb{F}_e(r)$ in several interesting cases. To verify our main results obtained in Section 5, we present an example in Section 8, and Section 9 is devoted to the conclusion and future work.

To accomplish the introductory section, we present some notations which are used through the paper. Let $C([0,T],\mathbb{R})$ be endowed with the maximum norm, i.e., $\|y\|_C=\max\{|y(r)|, r\in[0,T]\}$for all $y(r)\in\mathbb{R}$ and $C^m([0,T],\mathbb{R})$ be the space of m times $(m\in\mathbb{N})$ continuously differentiable functions on a finite interval $[0,T]$ of the real line with the norm

$$\|y\|_{C^m}=\sum_{k=0}^{m}\max_{0\leqslant r\leqslant T}|y^{(k)}(r)|, \quad m=0,1,2,\ldots, \quad \|y\|_{C^0}\equiv\|y\|_C.$$

2. Preliminaries

We embark on this section by briefly introducing the essential structure of fractional calculus, special functions and FDEs (for the more salient details on the matter, see the textbooks (Kilbas et al., 2006; Podlubny, 1999). We begin by defining the basic gamma and beta functions which are playing a fundamental role for fractional calculus.

Definition 2.1 Whittaker and Watson, 1927. The Euler's gamma function is defined as:

$$\Gamma(\alpha)=\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau, \quad \alpha>0. \tag{2.1}$$

Definition 2.2. (Whittaker and Watson, 1927) The beta function is defined as below:

$$\mathcal{B}(c,d)=\int_0^1 \tau^{c-1}(1-\tau)^{d-1}d\tau, \quad c,d>0. \tag{2.2}$$

Furthermore, the beta function can be expressed with the aid of gamma functions (Whittaker and Watson, 1927) as below:

$$\mathcal{B}(c,d)=\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}, \quad c,d>0. \tag{2.3}$$

Definition 2.3 (*Kilbas et al., 2006; Podlubny, 1999*). The Riemann-Liouville (R-L) fractional integral of order $\alpha>0$ for a function $g\in C([0,\infty);\mathbb{R})$is defined by:

$$I^{\alpha}_{0^+}g)(r)=\frac{1}{\Gamma(\alpha)}\int_0^r (r-s)^{\alpha-1}g(s)\,ds, \quad r>0. \tag{2.4}$$

Definition 2.4 (*Kilbas et al., 2006; Podlubny, 1999*). The R-L fractional derivative of order $\alpha>0$ for a function $g\in C^m([0,\infty);\mathbb{R})$is defined by:

$$\begin{aligned}(D^{\alpha}_{0^+}g)(r)&=\frac{d^m}{dr^m}(I^{m-\alpha}_{0^+}g)(r):\\ &=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dr^m}\int_0^r (r-s)^{m-\alpha-1}g(s)ds,\\ &m-1<\alpha\leqslant m, \quad r>0.\end{aligned} \tag{2.5}$$

Definition 2.5 (*Kilbas et al., 2006; Podlubny, 1999*). The Caputo fractional derivative of order $\alpha>0$ for a function $g\in C^m([0,\infty);\mathbb{R})$is defined by

$$(D^{\alpha}_{0^+}g)(r) = I^{m-\alpha}_{0^+}\left(\frac{\mathrm{d}^m}{\mathrm{d}r^m}g\right)(r) := \frac{1}{\Gamma(m-\alpha)}\int_0^r (r-s)^{m-\alpha-1}\frac{\mathrm{d}^m}{\mathrm{d}s^m}g(s)\mathrm{d}s,$$
$$m-1<\alpha\leqslant m,\quad r>0. \tag{2.6}$$

The R-L fractional integral operator and the Caputo fractional derivative have the following properties for a function $g\in C^m([0,\infty);\mathbb{R})$ (Kilbas et al., 2006; Podlubny, 1999):

$$I^{\alpha}_{0^+}\left({}^CD^{\alpha}_{0^+}g\right)(r) = g(r) - \sum_{k=0}^{m-1}\tfrac{r^k}{\Gamma(k+1)}g^{(k)}(0),\quad m-1<\alpha\leqslant m,\quad r>0,$$
$${}^CD^{\alpha}_{0^+}\left(I^{\alpha}_{0^+}g\right)(r) = g(r),\quad m-1<\alpha\leqslant m,\quad r>0.$$

The relationship between the R-L and Caputo fractional derivatives of order $m-1<\alpha\leqslant m$ (Kilbas et al., 2006; Podlubny, 1999) is as follows:

$$\left({}^CD^{\alpha}_{0^+}g\right)(r) = \left(D^{\alpha}_{0^+}g\right)(r) - \sum_{k=0}^{m-1}\frac{r^{k-\alpha}}{\Gamma(k-\alpha+1)}g^{(k)}(0),\quad r>0. \tag{2.7}$$

The following results are useful in solving FDEs.

Definition 2.6 Whittaker and Watson, 1927. A function g on $[0,\infty)$ is said to be *exponentially bounded* if it satisfies an inequality of the form

$$\|g(r)\|\leqslant Le^{\sigma r},\quad r>T,$$

for some real constants $\sigma, L>0$ and $T>0$.

Definition 2.7 Whittaker and Watson, 1927. If $g:[0,\infty)\to\mathbb{R}$ is measurable and exponentially bounded on $[0,\infty)$, then the Laplace transform $\mathcal{L}\{g(r)\}(s)$ defined by

$$G(s) = \mathcal{L}\{g(r)\}(s) = \int_0^{\infty} e^{-sr}g(r)dr,\quad s\in\mathbb{C},$$

exists and is an analytic function of s for $Re(s)>0$.

The time shift property of the Laplace transform is defined by

$$\mathcal{L}\{g(r-a)\mathcal{H}(r-a)\}(s) = \exp(-as)G(s). \tag{2.8}$$

The inversion Laplace integral formula is defined by

$$\mathcal{L}^{-1}\{G(s)\}(r) := q\lim_{\eta\to\infty}\frac{1}{2\pi i}\int_{c-i\eta}^{c+i\eta} e^{sr}G(s)ds,\quad r\geqslant 0, \tag{2.9}$$

where $G(s)=\mathcal{L}\{g(r)\}(s), s\in\mathbb{C}$.

Definition 2.8. [Kilbas et al., 2006; Podlubny, 1999] The Laplace transform of Caputo fractional derivative of general order $m-1<\alpha\leqslant m$ is given by

$$\mathcal{L}\left\{\left({}^CD^{\alpha}_{0^+}y\right)(r)\right\}(s) = s^{\alpha}Y(s) - \sum_{k=0}^{m-1}s^{\alpha-k-1}y^{(k)}(0),$$

where $Y(s)$ represents the Laplace transform of the function $y(r)$ and $y^{(k)}(0), k=0,1,\ldots,m-1$ represent initial values of $y(r)$ at $r=0$.

Remark 2.1. In the special cases, the Laplace integral transform of the Caputo type fractional differential operator is defined by

- If $\alpha\in(0,1]$, then
$$\mathcal{L}\left\{\left({}^CD^{\alpha}_{0^+}y\right)(r)\right\}(s) = s^{\alpha}Y(s) - s^{\alpha-1}y_0,\quad \text{where}\quad y_0=y(0).$$
- If $\alpha\in(1,2]$, then
$$\mathcal{L}\left\{\left({}^CD^{\alpha}_{0^+}y\right)(r)\right\}(s) = s^{\alpha}Y(s) - s^{\alpha-1}y_0 - s^{\alpha-2}y'_0,\quad \text{where}\quad y_0 = y(0),\quad y'_0=y'(0).$$

Definition 2.9. [Whittaker and Watson, 1927] The convolution of two functions $g(r)$ and $h(r)$, given on $[0,\infty)$, is defined by the following integral

$$g*h := (g*h)(r) = \int_0^r g(s)h(r-s)\mathrm{d}s,\quad \text{for}\quad r\geqslant 0, \tag{2.10}$$

which has the commutativity property

$$g*h = h*g.$$

Theorem 2.1. [Whittaker and Watson, 1927] *The Laplace transform of convolution of two functions $g(r)$ and $h(r)$, given on $[0,\infty)$, is defined by*

$$\mathcal{L}\{(g*h)(r)\}(s) = \mathcal{L}\{g(r)\}(s)\mathcal{L}\{h(r)\}(s),\quad s\in\mathbb{C}. \tag{2.11}$$

Lemma 2.1 Whittaker and Watson, 1927. *Assume that Ω is a linear and bounded operator defined on a Banach space with $\|\Omega\|<1$. Then, $(I-\Omega)^{-1}$ is linear and bounded such that*

$$(I-\Omega)^{-1} = \sum_{j=0}^{\infty}\Omega^j.$$

The M-L function is a generalization of the exponential function, first introduced by Gösta Mittag-Leffler by using a single series (Mittag-Leffler, 1903). Extensions to two or three parameters are well known and thoroughly studied in textbooks such as Gorenflo et al. (2014), but these still involve single power series in one variable. Extensions to two or several variables have been studied more recently (Huseynov et al., 2020; Ahmadova et al., 2021; Saxena et al., 2011; Fernandez et al., 2020; Özarslan and Fernandez, 2021).

Definition 2.10 Mittag-Leffler, 1903. The classical M-L function is defined as

$$E_{\alpha}(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(k\alpha+1)},\quad \alpha>0,\quad u\in\mathbb{R}. \tag{2.12}$$

The two-parameter M-L function (Gorenflo et al., 2014) is defined by

$$E_{\alpha,\beta}(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(k\alpha+\beta)},\quad \alpha>0,\quad \beta\in\mathbb{R},\quad u\in\mathbb{R}. \tag{2.13}$$

The three-parameter Mittag-Leffler function (Prabhakar, 1971) is defined as

$$E^{\delta}_{\alpha,\beta}(u) = \sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(k\alpha+\beta)}\frac{u^k}{k!},\quad \alpha>0,\quad \beta,\delta\in\mathbb{R},\quad u\in\mathbb{R}, \tag{2.14}$$

where $(\delta)_k$ is the well-known Pochhammer symbol denoting $\frac{\Gamma(\delta+k)}{\Gamma(\delta)}$. These series are convergent, locally uniformly in u, provided the $\alpha\in\mathbb{R}$ with $\alpha>0$ condition is satisfied. Note that

$$E^1_{\alpha,\beta}(u) = E_{\alpha,\beta}(u),\quad E_{\alpha,1}(u) = E_{\alpha}(u),\quad E_1(u)=\exp(u),\quad u\in\mathbb{R}.$$

The next lemma includes Laplace transform of three-parameter M-L function which will be used throughout the proof of Lemma 3.2.

Lemma 2.2. For $\alpha>\beta>0, \mu\in\mathbb{R}, l\in\mathbb{N}_0=\{0,1,2,\ldots\}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^{\alpha}-\mu s^{\beta})^{l+1}}\right\}(r) = r^{(l+1)\alpha-1}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\mu^p r^{p(\alpha-\beta)}}{\Gamma(p(\alpha-\beta)+(l+1)\alpha)} := r^{(l+1)\alpha-1}E^{l+1}_{\alpha-\beta,(l+1)\alpha}(\mu r^{\alpha-\beta}),\quad Re(s)>0.$$

Proof. By using the Taylor series representation of $\frac{1}{(1-r)^{l+1}}, l \in \mathbb{N}_0$ of the form:

$$\frac{1}{(1-r)^{l+1}} = \sum_{p=0}^{\infty} \binom{l+p}{p} r^p, \quad |r| < 1,$$

we achieve that

$$\frac{1}{(s^\alpha - \mu s^\beta)^{l+1}} = \frac{1}{(s^\alpha)^{l+1}} \frac{1}{(1 - \frac{\mu}{s^{\alpha-\beta}})^{l+1}} = \frac{1}{s^{(l+1)\alpha}} \sum_{p=0}^{\infty} \binom{l+p}{p} \left(\frac{\mu}{s^{\alpha-\beta}}\right)^p = \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{\mu^p}{s^{p(\alpha-\beta)+(l+1)\alpha}}.$$

Taking inverse Laplace transform of the above function, we obtain that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s^\alpha - \mu s^\beta)^{l+1}}\right\}(r) &= \sum_{p=0}^{\infty} \mu^p \binom{l+p}{p} \mathcal{L}^{-1}\left\{\frac{1}{s^{p(\alpha-\beta)+(l+1)\alpha}}\right\}(r) \\ &= \sum_{p=0}^{\infty} \mu^p \binom{l+p}{p} \frac{r^{p(\alpha-\beta)+(l+1)\alpha-1}}{\Gamma(p(\alpha-\beta)+(l+1)\alpha)} \\ &= r^{(l+1)\alpha-1} E^{l+1}_{\alpha-\beta,(l+1)\alpha}(\mu r^{\alpha-\beta}). \end{aligned}$$

We have required an extra condition on s such that

$$s^{\alpha-\beta} > |\mu|,$$

for proper convergence of the series. But, this condition can be removed at the end of calculation since analytic continuation of both sides, to give the desired result for all $s \in \mathbb{C}$ which satisfying $Re(s) > 0$. □

Definition 2.11 Fernandez et al., 2020. We consider the bivariate Mittag-Leffler function defined by

$$E^\delta_{\alpha,\beta,\gamma}(u, v) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\delta)_{l+k}}{\Gamma(l\alpha + k\beta + \gamma)} \frac{u^l v^k}{l!k!}, \quad \alpha, \beta > 0, \\ \gamma, \delta \in \mathbb{R}, \quad u, v \in \mathbb{R}. \tag{2.15}$$

If we write $u = \lambda t^\alpha$ and $v = \mu t^\beta$ for a single variable t, and multiply by a power function $t^{\gamma-1}$, we derive the following univariate version

$$t^{\gamma-1} E^\delta_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\delta)_{l+k}}{\Gamma(l\alpha + k\beta + \gamma)} \frac{\lambda^l \mu^k}{l!k!} t^{l\alpha+k\beta+\gamma-1}. \tag{2.16}$$

Note that when $\delta = 1$,

$$\begin{aligned} E^1_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta) &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1)_{l+k}}{\Gamma(l\alpha+k\beta+\gamma)} \frac{\lambda^l \mu^k}{l!k!} t^{l\alpha+k\beta+\gamma-1} \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(l+k)!}{l!k!} \frac{\lambda^l \mu^k}{\Gamma(l\alpha+k\beta+\gamma)} t^{l\alpha+k\beta+\gamma-1} \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \binom{l+k}{k} \frac{\lambda^l \mu^k}{\Gamma(l\alpha+k\beta+\gamma)} t^{l\alpha+k\beta+\gamma-1}. \end{aligned}$$

For simplicity, we denote $E^1_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta) := E_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta)$ in our results for this paper.

Now, we consider another special function which will be introduced later in Section 4.

Definition 2.12. Let $\lambda_i, \mu_j \in \mathbb{R}, \alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Generalized Wright function or more appropriately Fox-Wright function ${}_p\Psi_q(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined by

$${}_p\Psi_q(r) = {}_p\Psi_q\left[\begin{matrix}(\lambda_i, \alpha_i)_{1,p} \\ (\mu_j, \beta_j)_{1,q}\end{matrix} \middle| r\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\lambda_i + \alpha_i k)}{\prod_{j=1}^{q}\Gamma(\mu_j + \beta_j k)} \frac{r^k}{k!}. \tag{2.17}$$

This Fox-Wright function was established by Fox (1928) and Wright (1935). If the following condition is satisfied:

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1,$$

then the series in (2.17) is convergent for arbitrary $r \in \mathbb{R}$.

In terms of Laplace integral transform method, Kilbas et al. (2006) have considered the Cauchy problem for (1.3) without delay by using generalized Wright functions, in both homogeneous and inhomogeneous cases. It is necessary to note that our results by means of a recently defined M-L type functions with double infinite series are identical with the results in terms of Fox-Wright functions in Kilbas et al. (2006).

3. Exact analytical solution of linear homogeneous FLE with a constant delay: delayed M-L type function approach

Definition 3.1. [Mahmudov, 2020] Delayed analogue of M-L type function generated by $\lambda, \mu \in \mathbb{R}$ of three parameters $\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, \mu; \cdot) : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, \mu; r) = \sum_{l=0}^{\infty}\sum_{p=0}^{\infty} \binom{l+p}{p} \frac{\lambda^l \mu^p}{\Gamma(l\alpha + p\beta + \gamma)} (r - l\tau)^{l\alpha+p\beta+\gamma-1} \mathcal{H}(r - l\tau), \\ \alpha > 0, \quad \beta, \gamma \in \mathbb{R}, \tag{3.1}$$

where $\mathcal{H}(\cdot) : \mathbb{R} \to \mathbb{R}$ is the Heaviside function defined as follows

$$\mathcal{H}(r) = \begin{cases} 1, & r \geqslant 0, \\ 0, & r < 0. \end{cases}$$

Lemma 3.1. *Let $\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, \mu; r)$ be defined by (3.1). Then the following statements hold true:*

(i) If $\mu = 0$, then

$$\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, 0; r) := \mathcal{E}^\tau_{\alpha,\gamma}(\lambda; r), \tag{3.2}$$

where $\mathcal{E}^\tau_{\alpha,\gamma}(\lambda; \cdot) : \mathbb{R} \to \mathbb{R}$ is the delayed analogue of classical M-L type function of two parameters;

(ii) If $\lambda = 0$, then

$$\mathbb{E}^\tau_{\alpha,\beta,\gamma}(0, \mu; r) := r^{\gamma-1} E_{\beta,\gamma}(\mu r^\beta), \quad r > 0. \tag{3.3}$$

Proof. (i) If $\mu = 0$, then $\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, 0; r)$ coincides with $\mathcal{E}^\tau_{\alpha,\gamma}(\lambda; r)$:

$$\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, 0; r) = \sum_{l=0}^{\infty} \lambda^l \frac{(r - l\tau)^{l\alpha+\gamma-1}}{\Gamma(l\alpha + \gamma)} \mathcal{H}(r - l\tau) := \mathcal{E}^\tau_{\alpha,\gamma}(\lambda; r).$$

Furthermore, this case was considered in Huseynov and Mahmudov (2020), Li and Wang (2017), Li and Wang (2018).

(ii) If $\lambda = 0$, then $\mathbb{E}^\tau_{\alpha,\beta,\gamma}(0, \mu; r)$ coincides with classical Mittag-Leffler type function of two parameters $r^{\gamma-1} E_{\beta,\gamma}(\mu r^\beta)$ for $r > 0$. Trivially, from the definition of $\mathbb{E}^\tau_{\alpha,\beta,\gamma}(\lambda, \mu; r)$, we have

$$\mathbb{E}^\tau_{\alpha,\beta,\gamma}(0, \mu; r) = \sum_{p=0}^{\infty} \mu^p \frac{r^{p\beta+\gamma-1}}{\Gamma(p\beta + \gamma)} \mathcal{H}(r) := r^{\gamma-1} E_{\beta,\gamma}(\mu r^\beta), \quad r > 0.$$

In addition, this case was investigated in Kilbas et al. (2006). □

Lemma 3.2. *For* $\alpha>0,\alpha>\beta,\alpha>\gamma,\lambda,\mu\in\mathbb{R}$, *the following result holds true:*

$$\mathcal{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right\}(r)=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-l\tau)^{l\alpha+p(\alpha-\beta)+\alpha-\gamma-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha-\gamma)}\mathcal{H}(r-l\tau)=\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\gamma}(\lambda,\mu;r).,\quad Re(s)>0.$$

Proof. In accordance with the well-known Neumann series (2.1), $\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}$ can be written via a series expansion as follows:

$$\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}=\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}}\frac{1}{1-\frac{\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}}}=\sum_{l=0}^{\infty}\frac{\lambda^{l}e^{-ls\tau}s^{\gamma}}{(s^{\alpha}-\mu s^{\beta})^{l+1}}.$$

Then applying Lemma 2.2 to the last expression, we acquire that

$$\begin{aligned}\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}&=\sum_{l=0}^{\infty}\frac{\lambda^{l}e^{-ls\tau}s^{\gamma}}{s^{(l+1)\alpha}}\frac{1}{(1-\frac{\mu}{s^{\alpha-\beta}})^{l+1}}\\&=\sum_{l=0}^{\infty}\frac{\lambda^{l}e^{-ls\tau}s^{\gamma}}{s^{(l+1)\alpha}}\sum_{p=0}^{\infty}\binom{l+p}{p}\left(\frac{\mu}{s^{\alpha-\beta}}\right)^{p}\\&=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}e^{-ls\tau}}{s^{(l+1)\alpha+p(\alpha-\beta)-\gamma}}.\end{aligned}$$

Since the time delay property of the Laplace integral transform (2.8), we have

$$\mathcal{L}\{g(r-\tau)\mathcal{H}(r-\tau)\}(s)=e^{-s\tau}G(s).$$

Then, by taking inverse Laplace transform of the aforementioned function, we attain

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right\}(r)&=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\lambda^{l}\mu^{p}\mathcal{L}^{-1}\left\{\frac{e^{-ls\tau}}{s^{l\alpha+p(\alpha-\beta)+\alpha-\gamma}}\right\}(r)\\&=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-l\tau)^{l\alpha+p(\alpha-\beta)+\alpha-\gamma-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha-\gamma)}\mathcal{H}(r-l\tau)\\&=\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\gamma}(\lambda,\mu;r).\end{aligned}$$

We have required extra conditions on s, namely:

$$s^{\alpha-\beta}>|\mu|\quad\text{and}\quad|s^{\alpha}-\mu s^{\beta}|>|\lambda|e^{-s\tau},$$

for convergence of the series. However, these conditions can be removed at the end of evaluation by analytic continuation, to get the desired result for all $s\in\mathbb{C}$ with $Re(s)>0$. The proof is complete. □

The following lemma will be of significance for the results of next theorem.

Lemma 3.3. *For any* $r\in\mathbb{R}$ *and any parameters* $\alpha,\beta,\gamma,\nu,\tau,\lambda,\mu\in\mathbb{R}$ *satisfying* $\alpha,\beta,\nu,\tau>0$ *and* $\gamma-1>\lfloor\nu\rfloor$, *we have:*

$${}^{C}D^{\nu}_{0+}\left(\mathbb{E}^{\tau}_{\alpha,\beta,\gamma}(\lambda,\mu;s)\right)(r)=\mathbb{E}^{\tau}_{\alpha,\beta,\gamma-\nu}(\lambda,\mu;r).$$

Proof. We will make use of the well-known formula (Kilbas et al., 2006)

$${}^{C}D^{\mu}_{0+}\left(\frac{s^{\xi}}{\Gamma(\xi+1)}\right)(r)=\begin{cases}\frac{r^{\xi-\mu}}{\Gamma(\xi-\mu+1)}, & \xi>\lfloor\mu\rfloor,\\ 0, & \xi=0,1,2,\ldots,[\mu],\\ \textit{undefined}, & \text{otherwise}.\end{cases}\tag{3.4}$$

Therefore, given the condition $\gamma-1>\lfloor\nu\rfloor$, we can obtain that

$$\begin{aligned}&{}^{C}D^{\nu}_{0+}\left(\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^{l}\mu^{p}\frac{(l+p)!}{l!p!}\frac{(s-l\tau)^{l\alpha+p\beta+\gamma-1}\mathcal{H}(s-l\tau)}{\Gamma(l\alpha+p\beta+\gamma)}\right)(r)\\&=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^{l}\mu^{p}\frac{(l+p)!}{l!p!}{}^{C}D^{\nu}_{0+}\left(\frac{(s-l\tau)^{l\alpha+p\beta+\gamma-1}\mathcal{H}(s-l\tau)}{\Gamma(l\alpha+p\beta+\gamma)}\right)(r)\\&=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!}{l!p!}\frac{\lambda^{l}\mu^{p}(r-l\tau)^{l\alpha+p\beta+\gamma-\nu-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p\beta+\gamma-\nu)}\\&=\mathbb{E}^{\tau}_{\alpha,\beta,\gamma-\nu}(\lambda,\mu;r),\quad r\in\mathbb{R}.\end{aligned}$$

This completes the proof. □

Now, we discuss an IVP for the following linear homogeneous FLE with a constant delay:

$$\begin{cases}({}^{C}D^{\alpha}_{0^{+}}y)(r)-\mu\left({}^{C}D^{\beta}_{0^{+}}y\right)(r)-\lambda y(r-\tau)=0, & r\in(0,T],\quad\tau>0,\\ y(r)=\varphi(r), & -\tau\leqslant r\leqslant0,\end{cases}\tag{3.5}$$

where involving general fractional orders as $m-1<\alpha\leqslant m$ and $m-2<\beta\leqslant m-1, m\geqslant2$.

Theorem 3.1. *A unique analytical solution* $y\in C^{m}([-\tau,T],\mathbb{R})$ *of the Cauchy problem (3.5) has the following form:*

$$\begin{aligned}y(r)&=(1+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau))\varphi_{0}+(r+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau))\varphi'_{0}\\&+\cdots+\left(\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right)\varphi^{(m-2)}_{0}\\&+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi^{(m-1)}_{0}+\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds\\&=\sum_{j=0}^{m-2}\left[\frac{r^{j}}{\Gamma(j+1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi^{(j)}_{0}+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi^{(m-1)}_{0}\\&+\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds,\quad r>0.\end{aligned}\tag{3.6}$$

Proof. Let $T=\infty$. We suppose that (1.3) has a unique m times continuously differentiable solution y,g is continuous and exponentially bounded, and ${}^{C}D^{\beta}_{0^{+}}y$ (or ${}^{C}D^{\alpha}_{0^{+}}y$) is exponentially bounded on $[0,\infty)$, then y and ${}^{C}D^{\beta}_{0^{+}}y$ (or ${}^{C}D^{\alpha}_{0^{+}}y$) are exponentially bounded on $[0,\infty)$, thus their Laplace transforms exist. Then we can acquire an integral representation of solution to the linear homogeneous Langevin type time-delay equation with general fractional orders in Caputo's sense.

Firstly, we are applying Laplace integral transform to the both sides of (3.5) with the aid of following relations:

$$\mathcal{L}\left\{({}^{C}D^{\alpha}_{0^{+}}y)(r)\right\}(s)=s^{\alpha}Y(s)-\sum_{k=0}^{m-1}s^{\alpha-k-1}\varphi^{(k)}_{0},\quad m-1<\alpha\leqslant m,$$

$$\mathcal{L}\left\{({}^{C}D^{\beta}_{0^{+}}y)(r)\right\}(s)=s^{\beta}Y(s)-\sum_{k=0}^{m-2}s^{\beta-k-1}\varphi^{(k)}_{0},\quad m-2<\beta\leqslant m-1,$$

and first, by using substitution $r-\tau=\theta$, we obtain that

$$\begin{aligned}\mathcal{L}\{y(r-\tau)\}(s)&=\int_{0}^{\infty}e^{-sr}y(r-\tau)dr=e^{-s\tau}\int_{-\tau}^{\infty}e^{-s\theta}y(\theta)d\theta\\&=e^{-s\tau}\left(\int_{-\tau}^{0}e^{-s\theta}y(\theta)d\theta+\int_{0}^{\infty}e^{-s\theta}y(\theta)d\theta\right)\\&=e^{-s\tau}Y(s)+\int_{-\tau}^{0}e^{-s(\tau+\theta)}\varphi(\theta)d\theta.\end{aligned}$$

Since $\tau+\theta=r$, we acquire that

$$\begin{aligned}\mathcal{L}\{y(r-\tau)\}(s)&=e^{-s\tau}Y(s)+\int_{0}^{\tau}e^{-st}\varphi(r-\tau)dr\\&=e^{-s\tau}Y(s)+\int_{0}^{\infty}e^{-sr}\tilde{\varphi}(r-\tau)dr\\&=e^{-s\tau}Y(s)+\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s),\end{aligned}$$

where $\tilde{\varphi}(\cdot):\mathbb{R}\to\mathbb{R}$ is the unit-step function defined as follows:

$$\tilde{\varphi}(r)=\{\ \varphi(r),\quad -\tau\leqslant r\leqslant 0, 0,\qquad r>0.$$

Then, we achieve the following result

$$\begin{aligned} s^{\alpha}Y(s)\quad & -\sum_{k=0}^{m-1}s^{\alpha-k-1}\varphi_0^{(k)}-\mu\left(s^{\beta}Y(s)-\sum_{k=0}^{m-2}s^{\beta-k-1}\varphi_0^{(k)}\right)\\ & -\lambda(e^{-s\tau}Y(s)+\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s))=0. \end{aligned}$$

Next, we write above expression in the following explicit form

$$\begin{aligned} [s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}]Y(s) &=\left(s^{\alpha-1}-\mu s^{\beta-1}\right)\varphi_0+\left(s^{\alpha-2}-\mu s^{\beta-2}\right)\varphi_0'\\ &+\dots+\left(s^{\alpha-m+1}-\mu s^{\beta-m+1}\right)\varphi_0^{(m-2)}+s^{\alpha-m}\varphi_0^{(m-1)}\\ &+\lambda\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s). \end{aligned}\tag{3.7}$$

Then, we solve (3.7) with respect to $Y(s)$,

$$\begin{aligned} Y(s) &=\frac{s^{\alpha-1}-\mu s^{\beta-1}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0+\frac{s^{\alpha-2}-\mu s^{\beta-2}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0'\\ &+\dots+\frac{s^{\alpha-m+1}-\mu s^{\beta-m+1}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0^{(m-2)}+\frac{s^{\alpha-m}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0^{(m-1)}+\lambda\frac{\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s)}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\\ &=s^{-1}\left[\frac{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}+\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0+s^{-2}\left[\frac{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}+\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0'\\ &+\dots+s^{-(m-1)}\left[\frac{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}+\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0^{(m-2)}+\frac{s^{\alpha-m}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0^{(m-1)}+\lambda\frac{\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s)}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\\ &=s^{-1}\left[1+\frac{\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0+s^{-2}\left[1+\frac{\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0'\\ &+\dots+s^{-(m-1)}\left[1+\frac{\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right]\varphi_0^{(m-2)}+\frac{s^{\alpha-m}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0^{(m-1)}+\lambda\frac{\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s)}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\\ &=\left(s^{-1}\varphi_0+s^{-2}\varphi_0'+\dots+s^{-(m-1)}\varphi_0^{(m-2)}\right)\left(1+\frac{\lambda e^{-s\tau}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\right)\\ &+\frac{s^{\alpha-m}}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}\varphi_0^{(m-1)}+\lambda\frac{\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s)}{s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}}. \end{aligned}\tag{3.8}$$

In accordance with Lemma 2.1, we have

$$\begin{aligned} &\left(s^{\alpha}-\mu s^{\beta}-\lambda e^{-s\tau}\right)^{-1}=\left(s^{\alpha}-\mu s^{\beta}\right)^{-1}\left(1-\lambda\left(s^{\alpha}-\mu s^{\beta}\right)^{-1}e^{-s\tau}\right)^{-1}\\ &=\left(s^{\alpha}-\mu s^{\beta}\right)^{-1}\left(1+\lambda\left(s^{\alpha}-\mu s^{\beta}\right)^{-1}e^{-s\tau}+\lambda^{2}\left(s^{\alpha}-\mu s^{\beta}\right)^{-2}e^{-2s\tau}+\dots+\lambda^{l}\left(s^{\alpha}-\mu s^{\beta}\right)^{-l}e^{-ls\tau}+\dots\right)\\ &=\left(s^{\alpha}-\mu s^{\beta}\right)^{-1}\sum_{l=0}^{\infty}\lambda^{l}\left(s^{\alpha}-\mu s^{\beta}\right)^{-l}e^{-ls\tau}=\sum_{l=0}^{\infty}\lambda^{l}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-ls\tau}. \end{aligned}\tag{3.9}$$

By inserting (3.9) into (3.8), we achieve

$$\begin{aligned} Y(s) &=\left[s^{-1}\varphi_0+s^{-2}\varphi_0'+\dots+s^{-(m-1)}\varphi_0^{(m-2)}\right]\\ &+\left[s^{-1}\varphi_0+s^{-2}\varphi_0'+\dots+s^{-(m-1)}\varphi_0^{(m-2)}\right]\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-(l+1)s\tau}\\ &+s^{\alpha-m}\sum_{l=0}^{\infty}\lambda^{l}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-ls\tau}\varphi_0^{(m-1)}\\ &+\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-ls\tau}\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s)\\ &=\left[s^{-1}\varphi_0+s^{-2}\varphi_0'+\dots+s^{-(m-1)}\varphi_0^{(m-2)}\right]\\ &+s^{-1}\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-(l+1)s\tau}\varphi_0\\ &+s^{-2}\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-(l+1)s\tau}\varphi_0'\\ &+\dots+s^{-(m-1)}\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-(l+1)s\tau}\varphi_0^{(m-2)}\\ &+s^{\alpha-m}\sum_{l=0}^{\infty}\lambda^{l}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-ls\tau}\varphi_0^{(m-1)}\\ &+\sum_{l=0}^{\infty}\lambda^{l+1}\left(s^{\alpha}-\mu s^{\beta}\right)^{-(l+1)}e^{-ls\tau}\mathcal{L}\{\tilde{\varphi}(r-\tau)\}(s). \end{aligned}\tag{3.10}$$

Taking inverse Laplace transform of (3.10) and applying Lemma 2.2, Lemma 3.2, and formulas (2.8) and (2.11), we find an explicit representation of solution for a Cauchy problem (3.5):

$$\begin{aligned} y(r) &=\left[\varphi_0+r\varphi_0'+\dots+\frac{r^{m-2}}{\Gamma(m-1)}\varphi_0^{(m-2)}\right]\\ &+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\varphi_0\\ &+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)}\varphi_0'\\ &+\dots+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+m-1)}\varphi_0^{(m-2)}\\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m)}\varphi_0^{(m-1)}\\ &+\lambda\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-l\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathcal{H}(r-l\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\tilde{\varphi}(s-\tau)ds\\ &=\left(1+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\right)\varphi_0\\ &+\left(r+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)}\right)\varphi_0'\\ &+\dots+\left(\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+m-1)}\right)\varphi_0^{(m-2)}\\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m)}\varphi_0^{(m-1)}\\ &+\lambda\int_{-\tau}^{r-\tau}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^{l}\mu^{p}(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\tilde{\varphi}(s)ds\\ &=\left(1+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right)\varphi_0+\left(r+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau)\right)\varphi_0'\\ &+\dots+\left(\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right)\varphi_0^{(m-2)}+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)}\\ &+\lambda\int_{-\tau}^{r-\tau}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\tilde{\varphi}(s)ds\\ &=\sum_{j=0}^{m-2}\left[\frac{r^{j}}{\Gamma(j+1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)}+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)}\\ &+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds. \end{aligned}\tag{3.11}$$

where we have used the following results:

$$\begin{aligned} r\geqslant\tau\quad\Rightarrow\quad&\int_{-\tau}^{r-\tau}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\tilde{\varphi}(s)\\ ds=&\int_{-\tau}^{0}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds, \end{aligned}\tag{3.12}$$

and

$$\begin{aligned} r<\tau\quad\Rightarrow\quad&\int_{-\tau}^{r-\tau}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\tilde{\varphi}(s)ds\\ =&\int_{-\tau}^{r-\tau}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds. \end{aligned}\tag{3.13}$$

If we combine the above cases (3.12) and (3.13), we can derive the following desired result:

$$\begin{aligned} &\int_{-\tau}^{r-\tau}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\tilde{\varphi}(s)ds\\ &=\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds. \end{aligned}\tag{3.14}$$

Verification by substitution. Having found explicit form for $y(r)$, it remains to verify that $y(r)$ is an exact analytical solution of (3.5), indeed. First of all, it should be noted that the Caputo fractional derivative of constant function is equal to zero. Now, we will apply Lemma 3.3 to show that $y(r)$ is a solution of (3.5) by direct substitution method. To do so, we compute the fractional differentiation of $y(r)$ by applying Lemma 3.3 and then starting from the series (3.1) to the formula (3.4) and make the use of the following eminent Pascal's identity

$$\binom{l+p}{p}=\binom{l+p-1}{p}+\binom{l+p-1}{p-1},\quad \text{for}\quad l,p\geqslant 1,$$

as below

$$
\begin{aligned}
\left({}^{C}D_{0^+}^{\alpha}y\right)(r) &= {}^{C}D_{0^+}^{\alpha}\Big((1+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau))\varphi_0+(r+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau))\varphi_0' \\
&+\cdots+\left(\tfrac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right)\varphi_0^{(m-2)} \\
&+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)}+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s\Big) \\
&= {}^{C}D_{0^+}^{\alpha}\Bigg[\left(1+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\right)\varphi_0 \\
&+\left(r+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)}\right)\varphi_0' \\
&+\ldots+\left(\tfrac{r^{m-2}}{\Gamma(m-1)}+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+m-1)}\right)\varphi_0^{(m-2)} \\
&+\left(\tfrac{r^{m-1}}{\Gamma(m)}\varphi_0^{(m-1)}+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^l\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m)}\varphi_0^{(m-1)}\right. \\
&\left.+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m)}\varphi_0^{(m-1)}\right) \\
&+\Bigg(\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\tfrac{(r-\tau-s)^{\alpha-1}\mathcal{H}(r-\tau-s)}{\Gamma(\alpha)}\varphi(s)\mathrm{d}s \\
&+\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\varphi(s)\mathrm{d}s \\
&+\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\varphi(s)\mathrm{d}s\Bigg)\Bigg] \\
&=\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\varphi_0 \\
&+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\varphi_0' \\
&+\ldots+\lambda\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-1)}\varphi_0^{(m-2)} \\
&+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^l\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-\alpha)}\varphi_0^{(m-1)} \\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-\alpha)}\varphi_0^{(m-1)} \\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta))}\varphi(s)\mathrm{d}s \\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l\mu^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta))}\varphi(s)\mathrm{d}s.
\end{aligned}
$$

From now on, using again Pascal's rule for binomial coefficients, we find that

$$
\begin{aligned}
\left({}^{C}D_{0^+}^{\alpha}y\right)(r) &= \Bigg[\lambda\mathcal{H}(r-\tau)+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+1)} \\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\Bigg]\varphi_0 \\
&+\Bigg[\lambda(r-\tau)\mathcal{H}(r-\tau)+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+2)} \\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\Bigg]\varphi_0' \\
&+\cdots+\Bigg[\lambda\tfrac{(r-\tau)^{m-2}}{\Gamma(m-1)}\mathcal{H}(r-\tau)+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-1)} \\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^{l+1}\mu^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-1)}\Bigg]\varphi_0^{(m-2)} \\
&+\Bigg[\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^{l+1}\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-\alpha)} \\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^{l+1}\mu^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+p(\alpha-\beta)+m-\alpha)}\Bigg]\varphi_0^{(m-1)} \\
&+\Bigg[\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^l\mu^p(r-(l+1)\tau-\theta)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta))}\varphi(s)\mathrm{d}s \\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l\mu^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+p(\alpha-\beta))}\varphi(s)\mathrm{d}s\Bigg] \\
&=\Bigg[\lambda\mathcal{H}(r-\tau)+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+2}\mu^p(r-(l+2)\tau)^{(l+1)\alpha+p(\alpha-\beta)}\mathcal{H}(r-(l+2)\tau)}{\Gamma((l+1)\alpha+p(\alpha-\beta)+1)} \\
&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+1}\mu^{p+1}(r-(l+1)\tau)^{l\alpha+(p+1)(\alpha-\beta)}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)}\Bigg]\varphi_0 \\
&+\Bigg[\lambda(r-\tau)\mathcal{H}(r-\tau)+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+2}\mu^p(r-(l+2)\tau)^{(l+1)\alpha+p(\alpha-\beta)+1}\mathcal{H}(r-(l+2)\tau)}{\Gamma((l+1)\alpha+p(\alpha-\beta)+2)} \\
&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+1}\mu^{p+1}(r-(l+1)\tau)^{l\alpha+(p+1)(\alpha-\beta)+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+2)}\Bigg]\varphi_0' \\
&+\cdots+\Bigg[\lambda\tfrac{(r-\tau)^{m-2}}{\Gamma(m-1)}\mathcal{H}(r-\tau)+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+2}\mu^p(r-(l+2)\tau)^{(l+1)\alpha+p(\alpha-\beta)+m-2}\mathcal{H}(r-(l+2)\tau)}{\Gamma((l+1)\alpha+p(\alpha-\beta)+m-1)} \\
&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+1}\mu^{p+1}(r-(l+1)\tau)^{l\alpha+(p+1)(\alpha-\beta)+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+m-1)}\Bigg]\varphi_0^{(m-2)} \\
&+\Bigg[\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+2}\mu^p(r-(l+2)\tau)^{(l+1)\alpha+p(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-(l+2)\tau)}{\Gamma((l+1)\alpha+p(\alpha-\beta)+m-\alpha)} \\
&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+1}\mu^{p+1}(r-(l+1)\tau)^{l\alpha+(p+1)(\alpha-\beta)+m-\alpha-1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+m-\alpha)}\Bigg]\varphi_0^{(m-1)} \\
&+\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+2}\mu^p(r-(l+2)\tau-s)^{(l+1)\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+2)\tau-s)}{\Gamma((l+1)\alpha+p(\alpha-\beta))}\varphi(s)\mathrm{d}s \\
&+\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^{l+1}\mu^{p+1}(r-(l+1)\tau-s)^{l\alpha+(p+1)(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+(p+1)(\alpha-\beta))}\varphi(s)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&=\left[\lambda\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-2\tau)+\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+1}(\lambda,\mu;r-\tau)\right]\varphi_0 \\
&+\left[\lambda(r-\tau)\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-2\tau)+\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+2}(\lambda,\mu;r-\tau)\right]\varphi_0' \\
&+\ldots+\left[\lambda\tfrac{(r-\tau)^{m-2}}{\Gamma(m-1)}\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-2\tau)+\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+m-1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(m-2)} \\
&+\left[\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r-\tau)+\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m-\beta}(\lambda,\mu;r-\tau)\right]\varphi_0^{(m-1)} \\
&+\left[\lambda^2\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s+\lambda\mu\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s\right].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mu\left({}^{C}D_{0^+}^{\beta}y\right)(r) &= {}^{C}D_{0^+}^{\beta}\Big((1+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau))\varphi_0+(r+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau))\varphi_0' \\
&+\cdots+\left(\tfrac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right)\varphi_0^{(m-2)} \\
&+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)}+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s\Big) \\
&=\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+1}(\lambda,\mu;r-\tau)\varphi_0+\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+2}(\lambda,\mu;r-\tau)\varphi_0' \\
&+\ldots+\lambda\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta+m-1}(\lambda,\mu;r-\tau)\varphi_0^{(m-2)}+\mu\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m-\beta}(\lambda,\mu;r-\tau)\varphi_0^{(m-1)} \\
&+\lambda\mu\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s,
\end{aligned}
$$

and

$$\begin{aligned}\lambda y(r-\tau) &= \left[\lambda\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-2\tau)\right]\varphi_0 \\ &+\left[\lambda(r-\tau)\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-2\tau)\right]\varphi_0' \\ &+\ldots+\left[\lambda\tfrac{(r-\tau)^{m-2}}{\Gamma(m-1)}\mathcal{H}(r-\tau)+\lambda^2\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-2\tau)\right]\varphi_0^{(m-2)} \\ &+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r-\tau)\varphi_0^{(m-1)}+\lambda^2\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds.\end{aligned}$$

Taking a linear combination, we find that the following desired result:

$$\left({}^CD^{\alpha}_{0^+}y\right)(r)-\mu\left({}^CD^{\beta}_{0^+}y\right)(r)-\lambda y(r-\tau)=0,\quad r>0.$$

This completes the proof. □

4. Integral representation of solution to linear inhomogeneous Langevin type DDEs with general fractional orders

In this section, we find the explicit formula of solutions to linear inhomogeneous fractional Langevin type DDEs with constant coefficients by applying the classical ideas to find solution of (1.3).

Consider the following two Caputo type FDDEs with constant coefficients:

$$\begin{cases}({}^CD^{\alpha}_{0^+}y)(r)-\mu({}^CD^{\beta}_{0^+}y)(r)-\lambda y(r-\tau)=g(r), & r>0,\quad \tau>0,\\ y(r)\equiv 0, & -\tau\leqslant r\leqslant 0,\end{cases}\tag{4.1}$$

and

$$\begin{cases}({}^CD^{\alpha}_{0^+}y)(r)-\mu({}^CD^{\beta}_{0^+}y)(r)-\lambda y(r-\tau)=0, & r>0,\quad \tau>0,\\ y(r)=\varphi(r), & -\tau\leqslant r\leqslant 0.\end{cases}\tag{4.2}$$

The following lemma plays a significant role in the proof of next theorem which can be derived from classical ideas to find solution of linear FDDEs.

Lemma 4.1. *If $y_1(r)$ and $y_2(r)$ are the solutions systems (4.1) and (4.2), respectively, then $y(r)=y_1(r)+y_2(r)$ is the general solution of system (1.3).*

Note that the solution $y_2(r)$ of (4.2) have studied in Section 3. In other words, to achieve our target we need to find $y_1(r)$ which is a particular solution of (1.3).

Lemma 4.2. *Let $m-1<\alpha<m, m-2<\beta\leqslant m-1$ for $m\geqslant 2$. Then, we have the following relation:*

$$\begin{aligned}&\int_{\eta+l\tau}^{r}(r-s)^{m-1-\alpha}(s-l\tau-\eta)^{l\alpha+p(\alpha-\beta)+\alpha-1}ds \\ &=(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{B}(m-\alpha,(l+1)\alpha+p(\alpha-\beta)),\end{aligned}\tag{4.3}$$

where $\mathcal{B}$ is a beta function.

Proof. By using the substitution $u=\frac{r-s}{r-l\tau-\eta}$ and formula (2.2), we achieve

$$\begin{aligned}&\int_{\eta+l\tau}^{r}(r-s)^{m-1-\alpha}(s-l\tau-\eta)^{l\alpha+p(\alpha-\beta)+\alpha-1}ds \\ &=(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\int_0^1 u^{m-1-\alpha}(1-u)^{l\alpha+p(\alpha-\beta)+\alpha-1}du \\ &=(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{B}(m-\alpha,(l+1)\alpha+p(\alpha-\beta)).\end{aligned}$$

The proof is complete.

Theorem 4.1. *A solution $\hat{y}\in C^m([0,T],\mathbb{R})$ of (1.3) satisfying zero initial conditions $\hat{y}(r)\equiv 0,\ r\in[-\tau,0),\quad \hat{y}^{(k)}(0)\equiv 0,\ 0\leqslant k\leqslant m-1$ has the following form:*

$$\hat{y}(r)=\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s)ds,\quad r>0.\tag{4.4}$$

Proof. By using the method of variation of constants, any solution $\hat{y}$ of inhomogeneous system should be satisfied the following form:

$$\hat{y}(r)=\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)h(s)ds,\quad r>0,\tag{4.5}$$

where $h(s),\ 0\leqslant s\leqslant r$ is a sought after vector function and $\hat{y}(0)=0$.

Because of these homogeneous initial values, it follows that in this case, for any given order either in the interval of $(m-1,m]$ for $m\geqslant 2$, the Caputo and R-L fractional derivatives are equal in accordance with the relation (2.7). Therefore, in the work below we will apply R-L fractional derivative instead of Caputo fractional derivative one to verify the solution of differential equation with fractional-orders. Having Caputo fractional differentiation on both sides of (4.5) and in accordance with Lemma 4.2, we attain the following result:

$$\begin{aligned}&({}^CD^{\alpha}_{0^+}\hat{y})(r)=(D^{\alpha}_{0^+}\hat{y})(r)=\tfrac{1}{\Gamma(m-\alpha)}\tfrac{d^m}{dr^m}\int_0^r(r-s)^{m-1-\alpha}\int_0^s\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;s-\eta)h(\eta)d\eta ds \\ &=\tfrac{1}{\Gamma(m-\alpha)}\tfrac{d^m}{dr^m}\int_0^r\int_0^s(r-s)^{m-1-\alpha}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;s-\eta)h(\eta)d\eta ds \\ &=\tfrac{1}{\Gamma(m-\alpha)}\tfrac{d^m}{dr^m}\int_0^r\int_{\eta+l\tau}^r(r-s)^{m-1-\alpha}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;s-\eta)h(\eta)d\eta ds \\ &=\tfrac{1}{\Gamma(m-\alpha)}\tfrac{d^m}{dr^m}\int_0^r h(\eta)\left(\int_{\eta+l\tau}^r(r-s)^{m-1-\alpha}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;s-\eta)ds\right)d\eta \\ &=\tfrac{1}{\Gamma(m-\alpha)}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^p\binom{l+p}{p}\tfrac{d^m}{dr^m}\int_0^r h(\eta)\left(\int_{\eta+l\tau}^r(r-s)^{m-1-\alpha}\tfrac{(s-l\tau-\eta)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}ds\right)d\eta \\ &=\tfrac{1}{\Gamma(m-\alpha)}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^p\binom{l+p}{p}\tfrac{d^m}{dr^m}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma((l+1)\alpha+p(\alpha-\beta))}h(\eta)d\eta \\ &\times\mathcal{B}(m-\alpha,(l+1)\alpha+p(\alpha-\beta)) \\ &=\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^p\binom{l+p}{p}\tfrac{d^m}{dr^m}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+m+p(\alpha-\beta))}h(\eta)d\eta.\end{aligned}$$

With the aid of following binomial identity:

$$\binom{l+p}{p}=\binom{l+p-1}{p}+\binom{l+p-1}{p-1},\quad l,p\geqslant 1,$$

and applying Leibniz rule for higher-order derivatives (Huseynov et al., 2021) (see Theorem 3.2), we attain

$$\begin{aligned}({}^CD^{\alpha}_{0^+}\hat{y})(r)&=(D^{\alpha}_{0^+}\hat{y})(r)=\tfrac{d^m}{dr^m}\int_0^r\tfrac{(r-\eta)^{m-1}\mathcal{H}(r-\eta)}{\Gamma(m)}h(\eta)d\eta \\ &+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^p\binom{l+p-1}{p}\tfrac{d^m}{dr^m}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+m+p(\alpha-\beta))}h(\eta)d\eta \\ &+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\lambda^l\mu^p\binom{l+p-1}{p-1}\tfrac{d^m}{dr^m}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+m+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+m+p(\alpha-\beta))}h(\eta)d\eta \\ &=h(r)+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^p\binom{l+p-1}{p}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+p(\alpha-\beta))}h(\eta)d\eta \\ &+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\lambda^l\mu^p\binom{l+p-1}{p-1}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+p(\alpha-\beta))}h(\eta)d\eta \\ &=h(r)+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^{l+1}\mu^p\binom{l+p}{p}\int_0^r\tfrac{(r-(l+1)\tau-\eta)^{(l+1)\alpha+p(\alpha-\beta)-1}\mathcal{H}(r-(l+1)\tau-\eta)}{\Gamma((l+1)\alpha+p(\alpha-\beta))}h(\eta)d\eta \\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\lambda^l\mu^{p+1}\binom{l+p}{p}\int_0^r\tfrac{(r-l\tau-\eta)^{l\alpha+(p+1)(\alpha-\beta)-1}\mathcal{H}(r-l\tau-\eta)}{\Gamma(l\alpha+(p+1)(\alpha-\beta))}h(\eta)d\eta \\ &=h(r)+\lambda\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-\eta)h(\eta)d\eta+\mu\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha-\beta}(\lambda,\mu;r-\eta)h(\eta)d\eta \\ &=\mu({}^CD^{\beta}_{0^+}\hat{y})(r)+\lambda y(r-\tau)+h(r)=\mu({}^CD^{\beta}_{0^+}\hat{y})(r)+\lambda y(r-\tau)+g(r).\end{aligned}$$

Therefore, we acquire that $h(r)=g(r)$ for $r\in[0,T]$. The proof is complete. □

Finally, using Lemma 4.1 to combine the result of Theorem 3.5 with that of Theorem 4.1, we obtain the following general solution of IVP for FLE with a constant delay.

Theorem 4.2. *A unique analytical solution $y \in C^m([-\tau, T], \mathbb{R})$ of the Cauchy problem (1.3) has the following form:*

$$\begin{aligned} y(r) &= \left(1 + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right)\varphi_0 + \left(r + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau)\right)\varphi_0' \\ &+ \cdots + \left(\tfrac{r^{m-2}}{\Gamma(m-1)} + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right)\varphi_0^{(m-2)} + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} \\ &+ \lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s)ds \\ &:= \sum_{j=0}^{m-2}\left[\tfrac{r^j}{\Gamma(j+1)} + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)} + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} \\ &+ \lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s)ds, \quad r > 0. \end{aligned} \tag{4.6}$$

Proof. The proof of theorem is straightway. Hence, we pass over it. □

Corollary 4.1. (The case of $\tau = 0$). *The unique analytical solution $y \in C^m([0, T], \mathbb{R})$ of the following Cauchy problem for Langevin type linear inhomogeneous FDE:*

$$\begin{cases} \left({}^C D^{\alpha}_{0+} y\right)(r) - \mu\left({}^C D^{\beta}_{0+} y\right)(r) - \lambda y(r) = g(r), \quad r \in (0, T], \\ y^{(k)}(0) = y_0^{(k)}, \quad 0 \leqslant k \leqslant m-1, \end{cases} \tag{4.7}$$

can be represented by as follows

$$\begin{aligned} y(r) &= \sum_{j=0}^{m-2}\left[\tfrac{r^j}{\Gamma(j+1)} + \lambda r^{\alpha+j} E_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda r^{\alpha}, \mu r^{\alpha-\beta})\right] y_0^{(j)} + r^{m-1} E_{\alpha,\alpha-\beta,m}(\lambda r^{\alpha}, \mu r^{\alpha-\beta}) y_0^{(m-1)} \\ &+ \textstyle\int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}\left(\lambda (r-s)^{\alpha}, \mu (r-s)^{\alpha-\beta}\right) g(s)ds, \quad r > 0. \end{aligned} \tag{4.8}$$

Proof. Using the formula (4.6), we derive the following explicit representation of the solution to the Cauchy problem of the Langevin equation with two incommensurate fractional-orders (4.7): □

$$\begin{aligned} y(r) &= \left(1 + \lambda \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\right) y_0 \\ &+ \left(r + \lambda \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)}\right) y_0' \\ &+ \cdots + \left(\tfrac{r^{m-2}}{\Gamma(m-1)} + \lambda \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+\alpha+m-2}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+m-1)}\right) y_0^{(m-2)} \\ &+ \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+m-1}}{\Gamma(l\alpha+p(\alpha-\beta)+m)} y_0^{(m-1)} \\ &+ \int_0^r \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p (r-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} g(s)ds \\ &= (1 + \lambda r^{\alpha} E_{\alpha,\alpha-\beta,\alpha+1}(\lambda r^{\alpha}, \mu r^{\alpha-\beta})) y_0 + (r + \lambda r^{\alpha+1} E_{\alpha,\alpha-\beta,\alpha+2}(\lambda r^{\alpha}, \mu r^{\alpha-\beta})) y_0' \\ &+ \cdots + \left(\tfrac{r^{m-2}}{\Gamma(m-1)} + \lambda r^{\alpha+m-2} E_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda r^{\alpha}, \mu r^{\alpha-\beta})\right) y_0^{(m-2)} \\ &+ r^{m-1} E_{\alpha,\alpha-\beta,m}(\lambda r^{\alpha}, \mu r^{\alpha-\beta}) y_0^{(m-1)} \\ &+ \textstyle\int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}\left(\lambda (r-s)^{\alpha}, \mu (r-s)^{\alpha-\beta}\right) g(s)ds \\ &:= \sum_{j=0}^{m-2}\left[\tfrac{r^j}{\Gamma(j+1)} + \lambda r^{\alpha+j} E_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda r^{\alpha}, \mu r^{\alpha-\beta})\right] y_0^{(j)} + r^{m-1} E_{\alpha,\alpha-\beta,m}(\lambda r^{\alpha}, \mu r^{\alpha-\beta}) y_0^{(m-1)} \\ &+ \textstyle\int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}\left(\lambda (r-s)^{\alpha}, \mu (r-s)^{\alpha-\beta}\right) g(s)ds, \quad r > 0. \end{aligned} \tag{4.9}$$

It is important to note that these results (4.8) coincide with the analytical solutions of Langevin FDE with general fractional orders which considered by Ahmadova and Mahmudov in Ahmadova and Mahmudov (2021) with the help of bivariate Mittag-Leffler type functions. Furthermore, the authors in Ahmadova and Mahmudov (2021) have investigated application of fractional-order Langevin equations to the electrical circuit theory.

Remark 4.1. The Cauchy problem (4.7) has also a solution in terms of Fox-Wright functions below:

$$\begin{aligned} y(r) &= \sum_{j=0}^{m-2}\Bigg\{\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+j}}{l!}\,{}_1\Psi_1\left[\begin{matrix}(l+1,1)\\(l\alpha+j+1,\alpha-\beta)\end{matrix}\Big|\mu r^{\alpha-\beta}\right] \\ &-\mu\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+j+\alpha-\beta}}{l!}\,{}_1\Psi_1\left[\begin{matrix}(l+1,1)\\(l\alpha+j+\alpha-\beta+1,\alpha-\beta)\end{matrix}\Big|\mu r^{\alpha-\beta}\right]\Bigg\} y_0^{(j)} \\ &+\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+m-1}}{l!}\,{}_1\Psi_1\left[\begin{matrix}(l+1,1)\\(l\alpha+m,\alpha-\beta)\end{matrix}\Big|\mu r^{\alpha-\beta}\right] y_0^{(m-1)} \\ &+\textstyle\int_0^r (r-s)^{\alpha-1} G_{\alpha,\beta;\lambda,\mu}(r-s) g(s)ds, \quad r > 0, \end{aligned}$$

where

$$G_{\alpha,\beta;\lambda,\mu}(r) := \sum_{l=0}^{\infty}\frac{\lambda^l r^{l\alpha}}{l!}\Psi_1^1\left[\begin{matrix}(l+1,1)\\(l\alpha+\alpha,\alpha-\beta)\end{matrix}\Big|\mu r^{\alpha-\beta}\right].$$

Proof. Using the definition of Fox-Wright function (2.17) and Pascal's identity for binomial coefficients, we arrive at

$$\begin{aligned} y(r) &= \sum_{j=0}^{m-2}\Bigg\{\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+j}}{l!}\sum_{p=0}^{\infty}\tfrac{\Gamma(l+1+p)}{\Gamma(l\alpha+1+p(\alpha-\beta))}\tfrac{\mu^p r^{p(\alpha-\beta)}}{p!} \\ &-\mu\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+j+\alpha-\beta}}{l!}\sum_{p=0}^{\infty}\tfrac{\Gamma(l+1+p)}{\Gamma(l\alpha+\alpha-\beta+j+1+p(\alpha-\beta))}\tfrac{\mu^p r^{p(\alpha-\beta)}}{p!}\Bigg\} y_0^{(j)} \\ &+\sum_{l=0}^{\infty}\tfrac{\lambda^l r^{l\alpha+m-1}}{l!}\sum_{p=0}^{\infty}\tfrac{\Gamma(l+1+p)}{\Gamma(l\alpha+p(\alpha-\beta)+m)}\tfrac{\lambda^p r^{p(\alpha-\beta)}}{p!} \\ &+\int_0^r\sum_{l=0}^{\infty}\tfrac{\lambda^l (r-s)^{l\alpha+\alpha-1}}{l!}\sum_{p=0}^{\infty}\tfrac{\Gamma(l+1+p)}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\tfrac{\lambda^p (r-s)^{p(\alpha-\beta)}}{p!} g(s)ds \\ &= \sum_{j=0}^{m-2}\Bigg\{\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+j}}{\Gamma(l\alpha+p(\alpha-\beta)+1+j+1)} \\ &-\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^{p+1} r^{l\alpha+(p+1)(\alpha-\beta)+j}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+j+1)}\Bigg\} y_0^{(j)} \\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+m-1}}{\Gamma(l\alpha+p(\alpha-\beta)+m)} y_0^{(m-1)} \\ &+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p (r-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} g(s)ds \\ &= \sum_{j=0}^{m-2}\Bigg\{\tfrac{r^j}{\Gamma(j+1)} + \sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+j}}{\Gamma(l\alpha+p(\alpha-\beta)+j+1)} \\ &+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+j}}{\Gamma(l\alpha+p(\alpha-\beta)+j+1)} \\ &-\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+j}}{\Gamma(l\alpha+p(\alpha-\beta)+j+1)}\Bigg\} y_0^{(j)} \\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p r^{l\alpha+p(\alpha-\beta)+m-1}}{\Gamma(l\alpha+p(\alpha-\beta)+m)} y_0^{(m-1)} \\ &+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\tfrac{\lambda^l \mu^p (r-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} g(s)ds \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-2}\left\{\frac{r^j}{\Gamma(j+1)} + \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^l\mu^p r^{l\alpha+p(\alpha-\beta)+\alpha+j}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+j+1)}\right\}y_0^{(j)}\\
&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^l\mu^p r^{l\alpha+p(\alpha-\beta)+m-1}}{\Gamma(l\alpha+p(\alpha-\beta)+m)}y_0^{(m-1)}\\
&+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^l\mu^p(r-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)ds\\
&= \sum_{j=0}^{m-2}\left\{\frac{r^j}{\Gamma(j+1)} + \lambda r^{\alpha+j}E_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda r^\alpha,\mu r^{\alpha-\beta})\right\}y_0^{(j)}\\
&+r^{m-1}E_{\alpha,\alpha-\beta,m}(\lambda r^\alpha,\mu r^{\alpha-\beta})y_0^{(m-1)} + \int_0^r(r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(\lambda r^\alpha,\mu r^{\alpha-\beta})g(s)ds, \quad r>0.
\end{aligned}
$$

Therefore, our solution in terms of recently defined M-L type functions coincide with the solution by means of Fox-Wright functions shown in Kilbas et al. (2006) (see pp. 314 and 323). □

Corollary 4.2. (The case of $\lambda = 0$). *The unique analytical solution* $y \in C^m([0,T],\mathbb{R})$ *of the IVP for following linear inhomogeneous FDE with two incommensurate fractional orders*

$$
\begin{cases}
({}^CD_{0^+}^\alpha y)(r) - \mu\left({}^CD_{0^+}^\beta y\right)(r) = g(r), & r\in(0,T],\\
y^{(k)}(r) = y_0^{(k)}, & 0\leqslant k\leqslant m-1,
\end{cases}
\tag{4.10}
$$

can be represented by as below

$$
\begin{aligned}
y(r) &= \sum_{j=0}^{m-2}\frac{r^j}{\Gamma(j+1)}y_0^{(j)} + r^{m-1}E_{\alpha-\beta,m}(\mu r^{\alpha-\beta})y_0^{(m-1)}\\
&+\int_0^r(r-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(\mu(r-s)^{\alpha-\beta})g(s)ds, \quad r>0.
\end{aligned}
\tag{4.11}
$$

Proof. With the help of the formula (4.6), we derive the following explicit representation of solution for a fractional-order dynamical system (4.10):

$$
\begin{aligned}
y(r) &= \left[y_0 + ry_0' + \ldots + \frac{r^{m-2}}{\Gamma(m-1)}y_0^{(m-2)}\right]\\
&+\sum_{p=0}^{\infty}\frac{\mu^p r^{p(\alpha-\beta)+m-1}}{\Gamma(p(\alpha-\beta)+m)}y_0^{(m-1)} + \int_0^r\sum_{p=0}^{\infty}\frac{\mu^p(r-s)^{p(\alpha-\beta)+\alpha-1}}{\Gamma(p(\alpha-\beta)+\alpha)}g(s)ds\\
&= \sum_{j=0}^{m-2}\frac{r^j}{\Gamma(j+1)}y_0^{(j)} + r^{m-1}E_{\alpha-\beta,m}(\mu r^{\alpha-\beta})y_0^{(m-1)}\\
&+\int_0^r(r-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(\mu(r-s)^{\alpha-\beta})g(s)ds, \quad r>0.
\end{aligned}
\tag{4.12}
$$

It is interesting to note that these solutions (4.11) are identical with the solutions of FDE in Caputo's sense that considered by Kilbas et al. (2006) by means of two-parameter M-L or Wiman's functions. □

Corollary 4.3. (The case of $\mu = 0$). *The unique analytical solution* $y \in C^m([0,T],\mathbb{R})$ *of the Cauchy problem for following linear inhomogeneous FDE with a constant delay*

$$
\begin{cases}
({}^CD_{0^+}^\alpha y)(r) - \lambda y(r-\tau) = g(r), & r\in(0,T], \quad \tau>0,\\
y(r) = \varphi(r), & -\tau\leqslant r\leqslant 0,
\end{cases}
\tag{4.13}
$$

can be expressed by as follows

$$
\begin{aligned}
y(r) &= \sum_{j=0}^{m-2}\left[\frac{r^j}{\Gamma(j+1)} + \lambda\mathcal{E}_{\alpha,\alpha+j+1}^\tau(\lambda;r-\tau)\right]\varphi_0^{(j)} + \mathcal{E}_{\alpha,\alpha+m}^\tau(\lambda;r)\varphi_0^{(m-1)}\\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-\tau-s)\varphi(s)ds + \int_0^r\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-s)g(s)ds, \quad r>0.
\end{aligned}
\tag{4.14}
$$

Proof. With the aid of the formula (4.6), we obtain the following explicit representation of solution for fractional-order time-delay system (4.13):

$$
\begin{aligned}
y(r) &= \left[\varphi_0 + r\varphi_0' + \ldots + \frac{r^{m-2}}{\Gamma(m-1)}\varphi_0^{(m-2)}\right]\\
&+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+1)}\varphi_0\\
&+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+2)}\varphi_0'\\
&+\cdots+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+m-1)}\varphi_0^{(m-2)}\\
&+\sum_{l=0}^{\infty}\frac{\lambda^l(r-l\tau)^{l\alpha+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+m)}\varphi_0^{(m-1)}\\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau-s)^{l\alpha+\alpha-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+\alpha)}\varphi(s)ds\\
&+\int_0^r\sum_{l=0}^{\infty}\frac{\lambda^l(r-l\tau-s)^{l\alpha+\alpha-1}\mathcal{H}(r-l\tau-s)}{\Gamma(l\alpha+\alpha)}g(s)ds\\
&= \left(1+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+1)}\right)\varphi_0\\
&+\left(r+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha+1}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+2)}\right)\varphi_0'\\
&+\ldots+\left(\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau)^{l\alpha+\alpha+m-2}\mathcal{H}(r-(l+1)\tau)}{\Gamma(l\alpha+\alpha+m-1)}\right)\varphi_0^{(m-2)}\\
&+\sum_{l=0}^{\infty}\frac{\lambda^l(r-l\tau)^{l\alpha+m-1}\mathcal{H}(r-l\tau)}{\Gamma(l\alpha+m)}\varphi_0^{(m-1)}\\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\sum_{l=0}^{\infty}\frac{\lambda^l(r-(l+1)\tau-s)^{l\alpha+\alpha-1}\mathcal{H}(r-(l+1)\tau-s)}{\Gamma(l\alpha+\alpha)}\varphi(s)ds\\
&+\int_0^r\sum_{l=0}^{\infty}\frac{\lambda^l(r-l\tau-s)^{l\alpha+\alpha-1}\mathcal{H}(r-l\tau-s)}{\Gamma(l\alpha+\alpha)}g(s)ds\\
&= \left(1+\lambda\mathcal{E}_{\alpha,\alpha+1}^\tau(\lambda;r-\tau)\right)\varphi_0 + \left(r+\lambda\mathcal{E}_{\alpha,\alpha+2}^\tau(\lambda;r-\tau)\right)\varphi_0'\\
&+\cdots+\left(\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathcal{E}_{\alpha,\alpha+m-1}^\tau(\lambda;r-\tau)\right)\varphi_0^{(m-2)} + \mathcal{E}_{\alpha,m}^\tau(\lambda,\mu;r)\varphi_0^{(m-1)}\\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-\tau-s)\varphi(s)ds + \int_0^r\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-s)g(s)ds\\
&:= \sum_{j=0}^{m-2}\left[\frac{r^j}{\Gamma(j+1)}+\lambda\mathcal{E}_{\alpha,\alpha+j+1}^\tau(\lambda;r-\tau)\right]\varphi_0^{(j)} + \mathcal{E}_{\alpha,m}^\tau(\lambda;r)\varphi_0^{(m-1)}\\
&+\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-\tau-s)\varphi(s)ds + \int_0^r\mathcal{E}_{\alpha,\alpha}^\tau(\lambda;r-s)g(s)ds, \quad r>0.
\end{aligned}
\tag{4.15}
$$

Furthermore, it should be noted that above results (4.14) are the general case of the solutions of fractional-order time-delay differential equation with constant coefficients which considered by Li and Wang (2017) and Li and Wang (2018) in case of $\alpha\in(0,1]$. □

4.1. Special case: the solutions of delayed FLEs

In this subsection, we provide the special case of (1.3) where $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$.

Firstly, we consider the homogeneous linear delayed FLE in the following form

$$
\begin{cases}
({}^CD_{0^+}^\alpha y)(r) - \mu\left({}^CD_{0^+}^\beta y\right)(r) - \lambda y(r-\tau) = 0, & r\in(0,T], \quad \tau>0,\\
y(r) = \varphi(r), & -\tau\leqslant r\leqslant 0,
\end{cases}
\tag{4.16}
$$

as well as the corresponding in-homogeneous delayed FLE in the form

$$\begin{cases} ({}^{C}D^{\alpha}_{0^+}y)(r) - \mu\left({}^{C}D^{\beta}_{0^+}y\right)(r) - \lambda y(r-\tau) = g(r), & r \in (0,T], \quad \tau > 0, \\ y(r) = \varphi(r), & -\tau \leqslant r \leqslant 0, \end{cases} \tag{4.17}$$

where $\varphi \in C^1([-\tau, 0], \mathbb{R})$ is an arbitrary continuously differentiable real-valued function that determines initial conditions.

Theorem 4.3. *The unique analytical solution $y \in C^2([0,T],\mathbb{R})$ of the Cauchy problem (4.16) for linear homogeneous FLE with a constant delay can be represented by*

$$\begin{aligned} y(r) &= \left(1 + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right)\varphi_0 + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,2}(\lambda,\mu;r)\varphi_0' \\ &+\lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds, \quad r > 0. \end{aligned} \tag{4.18}$$

Theorem 4.4. *A unique analytical solution $y \in C^2([0,T],\mathbb{R})$ of the IVP (4.17) for linear inhomogeneous time-delay FLE has the form:*

$$\begin{aligned} y(r) &= \left(1 + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right)\varphi_0 + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,2}(\lambda,\mu;r)\varphi_0' \\ &+\lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s)ds, \quad r > 0. \end{aligned} \tag{4.19}$$

Proof. The proofs of Theorem 4.3 and 4.4 are straightforward approaches by following the general case above, so we omit it here. □

5. Existence and uniqueness problem for nonlinear time-delay FLE

In this section, firstly, we consider the Cauchy problem for nonlinear FLE with a constant delay:

$$\begin{cases} ({}^{C}D^{\alpha}_{0^+}y)(r) - \mu({}^{C}D^{\alpha}_{0^+}y)(r) - \lambda y(r-\tau) = g(r, y(r)), & r \in (0,T], \quad \tau > 0, \\ y(r) = \varphi(r), & -\tau \leqslant r \leqslant 0, \end{cases} \tag{6.1}$$

where $m-1 < \alpha \leqslant m, m-2 < \beta \leqslant m-1, m \geqslant 2$ with $\alpha - \beta \geqslant 1, y(\cdot) \in \mathbb{R}, g(\cdot, y(\cdot)) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a nonlinear perturbation and also a continuous function. Moreover, for definiteness, we will also assume that

$$(r \mapsto g(r, 0)) \in C([0,\infty),\mathbb{R}). \tag{6.2}$$

Then, in accordance with Theorem 4.2, we acquire the solution of nonlinear fractional Langevin type DDE (6.1) as follows:

$$\begin{aligned} y(r) &= \sum_{j=0}^{m-2}\left[\frac{r^j}{\Gamma(j+1)} + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)} + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} \\ &+\lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s,y(s))ds, \quad r > 0. \end{aligned}$$

Lemma 5.1. *The following estimation holds true:*

$|\mathbb{E}_{\alpha,\alpha-\beta,\alpha+k}(\lambda,\mu;r)| \leqslant r^{\alpha+k-1}\exp(|\lambda| r^{\alpha} + |\mu| r^{\alpha-\beta})$ for $k = 0, 1, \ldots, m-1$.

Proof. Firstly, we need to estimate $\mathbb{E}_{\alpha,\alpha-\beta,\alpha+k}(\lambda,\mu;r)$ as below:

$$\begin{aligned} |\mathbb{E}_{\alpha,\alpha-\beta,\alpha+k}(\lambda,\mu;r)| &\leqslant \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{|\lambda|^l|\mu|^p r^{l\alpha+p(\alpha-\beta)+\alpha+k-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+k)} \\ &= r^{\alpha+k-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!}{l!p!}|\lambda|^l|\mu|^p\frac{r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+k)}. \end{aligned}$$

Since $m-1 < \alpha \leqslant m, m-2 < \beta \leqslant m-1$ and $\alpha - \beta \geqslant 1$ for $m \geqslant 2$, we have

$$\Gamma(l\alpha + p(\alpha-\beta) + \alpha + k) > \Gamma(l+p+1) = (l+p)!$$

It follows that

$$\begin{aligned} |\mathbb{E}_{\alpha,\alpha-\beta,\alpha+k}(\lambda,\mu;r)| &\leqslant r^{\alpha+k-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!}{l!p!}|\lambda|^l|\mu|^p\frac{r^{l\alpha+p(\alpha-\beta)}}{(l+p)!} \\ &= r^{\alpha+k-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{|\lambda|^l|\mu|^p}{l!p!}r^{l\alpha+p(\alpha-\beta)} \\ &= r^{\alpha+k-1}\sum_{l=0}^{\infty}\frac{|\lambda|^l r^{l\alpha}}{l!}\sum_{p=0}^{\infty}\frac{|\mu|^p r^{p(\alpha-\beta)}}{p!} \\ &= r^{\alpha+k-1}\exp(|\lambda| r^{\alpha})\exp(|\mu| r^{\alpha-\beta}) \\ &= r^{\alpha+k-1}\exp(|\lambda| r^{\alpha} + |\mu| r^{\alpha-\beta}). \end{aligned}$$

The proof is complete. □

Corollary 5.1. *For $m \geqslant 2$, the following result holds:*

$$|\mathbb{E}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)| \leqslant r^{m-1}\exp(|\lambda| r^{\alpha} + |\mu| r^{\alpha-\beta}).$$

Theorem 5.1. *Suppose that the following assumptions hold true:*

(H_1) $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function;
(H_2) there exists $L_g > 0$ such that g satisfying the Lipschitz condition:

$$|g(r,y) - g(r,z)| \leqslant L_g|y - z|, \forall (r,y),(r,z) \in [0,T] \times \mathbb{R};$$

Then, the problem (6.1) has a unique global continuous solution on $[0,T]$.

Proof. Let a ball be defined as $\mathcal{B}_R := q\{y \in C([0,T],\mathbb{R}) : \|y\|_{\omega} \leqslant R, \omega > 0\}$where R is positive constant with

$$R \geqslant \frac{K|\varphi_0^{(j)}|\omega^{\alpha} + T^{m-1}N\omega^{\alpha}|\varphi_0^{(m-1)}| + N\Gamma(\alpha)(|\lambda|\|\varphi\|_{\omega} + M)}{\omega^{\alpha} - N\Gamma(\alpha)L_g}, \tag{6.3}$$

where

$$\begin{aligned} K &= \sum_{j=0}^{m-2}\left(\frac{T^j}{\Gamma(j+1)} + |\lambda| T^{\alpha+j}N\right), \quad M \\ &= \max\left\{\frac{|g(r,0)|}{\exp(\omega r)}, 0 \leqslant r \leqslant T\right\}, \quad N = \exp(|\lambda| T^{\alpha} + |\mu| T^{\alpha-\beta}). \end{aligned}$$

Now, we define an integral operator $\mathcal{P}$ on $\mathcal{B}_R$ as follows:

$$\mathcal{P} : C([0,T],\mathbb{R}) \supset \mathcal{B}_R \ni y \mapsto \mathcal{P}(y) := (r \mapsto (\mathcal{P}y)(r)) \in C([0,T],\mathbb{R}),$$

via the following formula

$$\begin{aligned} (\mathcal{P}y)(r) &= \sum_{j=0}^{m-2}\left[\frac{r^j}{\Gamma(j+1)} + \lambda \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)} + \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} \\ &+\lambda \int_{-\tau}^{\min\{r-\tau,0\}} \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s,y(s))ds, \quad 0 \leqslant r \leqslant T. \end{aligned} \tag{6.4}$$

It is evident that $\mathcal{P}$ is well-defined due to (H_1). Thus, the existence of solution of the Cauchy problem (6.1) is equivalent to that the integral operator $\mathcal{P}$ has a fixed point on B_R. We will apply contraction mapping principle or Banach's fixed point theorem to prove that $\mathcal{P}$ has a unique fixed point.

However, we will not use maximum norm on $C([0,T],\mathbb{R})$. Because, the selection of maximum norm only derives us to a local solution defined on the subinterval of $[0,T]$. Let $C([0,T],\mathbb{R})$ be endowed with the weighted maximum norm $\|\cdot\|_\omega$ where $\omega>0$ with regard to exponential function, defined as:

$$\|y\|_\omega := q\max_{0\leqslant r\leqslant T}\left\{\frac{|y(r)|}{\exp(\omega r)}\right\}, \quad \forall y\in C([0,T],\mathbb{R}).$$

Since two norms $\|\cdot\|_\infty$ and $\|\cdot\|_\omega$ are equivalent, $(C([0,T],\mathbb{R}),\|\cdot\|_\omega)$ is also a Banach space. The proof is divided into two steps:

Step 1: We prove that $\mathcal{P}(B_R)\subset B_R$.

Now take $\forall r\in[0,T]$ and $\forall y\in B_R$. By using (H_2) via Lemma 5.1 and Corollary 5.1, we acquire:

$$\begin{aligned}
\frac{|(\mathcal{P}y)(r)|}{\exp(\omega r)} &\leqslant \frac{1}{\exp(\omega r)}\left|1+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right||\varphi_0| \\
&+\frac{1}{\exp(\omega r)}\left|r+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau)\right||\varphi_0'| \\
&+\cdots+\frac{1}{\exp(\omega r)}\left|\frac{r^{m-2}}{\Gamma(m-1)}+\lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right||\varphi_0^{(m-2)}| \\
&+\frac{1}{\exp(\omega r)}\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\right|\left|\varphi_0^{(m-1)}\right| \\
&+\frac{1}{\exp(\omega r)}\left|\lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)ds\right| \\
&+\frac{1}{\exp(\omega r)}\left|\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s,y(s))ds\right| \\
&\leqslant|\varphi_0|+|\lambda|\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(\lambda,\mu;r-\tau)\right||\varphi_0| \\
&+r|\varphi_0'|+|\lambda|\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+2}(\lambda,\mu;r-\tau)\right||\varphi_0'| \\
&+\cdots+\frac{r^{m-2}}{\Gamma(m-1)}|\varphi_0^{(m-2)}|+|\lambda|\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+m-1}(\lambda,\mu;r-\tau)\right||\varphi_0^{(m-2)}| \\
&+\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\right||\varphi_0^{(m-1)}|+\frac{|\lambda|}{\exp(\omega r)}\int_{-\tau}^0\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\right||\varphi(s)|ds \\
&+\frac{1}{\exp(\omega r)}\int_0^r\left|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)\right||((g(s,y(s))-g(s,0))+g(s,0))|ds \\
&\leqslant|\varphi_0|+|\lambda|r^\alpha\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})|\varphi_0| \\
&+r|\varphi_0'|+|\lambda|r^{\alpha+1}\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})|\varphi_0'| \\
&+\cdots+\frac{r^{m-2}}{\Gamma(m-1)}|\varphi_0^{(m-2)}|+|\lambda|r^{m+\alpha-2}\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})|\varphi_0^{(m-2)}| \\
&+r^{m-1}\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})|\varphi_0^{(m-1)}| \\
&+\frac{|\lambda|}{\exp(\omega r)}\int_{-\tau}^0(r-\tau-s)^{\alpha-1}\exp(|\lambda|(r-\tau-s)^\alpha+|\mu|(r-\tau-s)^{\alpha-\beta})\frac{\exp(\omega s)}{\exp(\omega s)}|\varphi(s)|ds \\
&+\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(|\lambda|(r-s)^\alpha+|\mu|(r-s)^{\alpha-\beta})\frac{\exp(\omega s)}{\exp(\omega s)}|g(s,y(s))-g(s,0)|ds \\
&+\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(|\lambda|(r-s)^\alpha+|\mu|(r-s)^{\alpha-\beta})\frac{\exp(\omega s)}{\exp(\omega s)}|g(s,0)|ds
\end{aligned}$$

By making use of the substitution $r-s=u$ and Lipschitz condition (H_2), we obtain:

$$\begin{aligned}
\frac{|(\mathcal{P}y)(r)|}{\exp(\omega r)} &\leqslant \sum_{j=0}^{m-2}\frac{r^j}{\Gamma(j+1)}|\varphi_0^{(j)}|+\sum_{j=0}^{m-2}|\lambda|r^{\alpha+j}\exp(|\lambda|r^\alpha|+\mu|r^{\alpha-\beta})|\varphi_0^{(j)}| \\
&+r^{m-1}\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})|\varphi_0^{(m-1)}| \\
&+\frac{|\lambda|}{\exp(\omega r)}\int_{-\tau}^0(r-\tau-s)^{\alpha-1}\frac{\exp(\omega s)}{\exp(\omega s)}|\varphi(s)|ds\times\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta}) \\
&+\frac{L_g}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\frac{\exp(\omega s)}{\exp(\omega s)}|y(s)|ds\times\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta}) \\
&+\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\frac{\exp(\omega s)}{\exp(\omega s)}|g(s,0)|ds\times\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta}) \\
&\leqslant\sum_{j=0}^{m-2}\frac{T^j}{\Gamma(j+1)}|\varphi_0^{(j)}|+\sum_{j=0}^{m-2}|\lambda|T^{\alpha+j}\exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta})|\varphi_0^{(j)}| \\
&+T^{m-1}\exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta})|\varphi_0^{(m-1)}| \\
&+\frac{|\lambda|}{\exp(\omega r)}\int_0^\tau(r-s)^{\alpha-1}\exp(\omega(s-\tau))ds\max_{0\leqslant r\leqslant T}\left\{\frac{|\varphi(r)|}{\exp(\omega r)}\right\}exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta}) \\
&+\frac{L_g}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\max_{0\leqslant r\leqslant T}\left\{\frac{|y(r)|}{\exp(\omega r)}\right\}exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta}) \\
&+\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\max_{0\leqslant r\leqslant T}\left\{\frac{|g(s,0)|}{\exp(\omega r)}\right\}exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta}) \\
&=\sum_{j=0}^{m-2}\frac{T^j}{\Gamma(j+1)}|\varphi_0^{(j)}|+\sum_{j=0}^{m-2}|\lambda|T^{\alpha+j}N|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}| \\
&+\frac{N|\lambda|}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\|\varphi\|_\omega \\
&+\frac{L_gN}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\|y\|_\omega+\frac{MN}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds
\end{aligned}$$

$$\begin{aligned}
&=\sum_{j=0}^{m-2}\left(\frac{T^j}{\Gamma(j+1)}+|\lambda|T^{\alpha+j}N\right)|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}| \\
&+\frac{N|\lambda|}{\exp(\omega r)}\int_0^r u^{\alpha-1}\exp(\omega r)\exp(-\omega u)du\|\varphi\|_\omega \\
&+\frac{L_gN}{\exp(\omega r)}\int_0^r u^{\alpha-1}\exp(\omega r)\exp(-\omega u)du\|y\|_\omega \\
&+\frac{MN}{\exp(\omega r)}\int_0^r u^{\alpha-1}\exp(\omega r)\exp(-\omega u)du \\
&=K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+N|\lambda|\int_0^r u^{\alpha-1}\exp(-\omega u)du\|\varphi\|_\omega \\
&+L_gN\int_0^r u^{\alpha-1}\exp(-\omega u)du\|y\|_\omega+MN\int_0^r u^{\alpha-1}\exp(-\omega u)du \\
&=K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+\frac{N|\lambda|}{\omega^\alpha}\int_0^{\omega r}v^{\alpha-1}\exp(-v)dv\|\varphi\|_\omega \\
&+\frac{L_gN}{\omega^\alpha}\int_0^{\omega r}v^{\alpha-1}\exp(-v)dv\|y\|_\omega+\frac{MN}{\omega^\alpha}\int_0^{\omega r}v^{\alpha-1}\exp(-v)dv \\
&=K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+\frac{N}{\omega^\alpha}\int_0^{\omega r}v^{\alpha-1}\exp(-v)dv(|\lambda|\|\varphi\|_\omega+L_g\|y\|_\omega+M) \\
&\leqslant K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+\frac{N}{\omega^\alpha}\int_0^{\infty}v^{\alpha-1}\exp(-v)dv(|\lambda|\|\varphi\|_\omega+L_g\|y\|_\omega+M) \\
&=K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+\frac{N\Gamma(\alpha)}{\omega^\alpha}(|\lambda|\|\varphi\|_\omega+L_g\|y\|_\omega+M) \\
&\leqslant K|\varphi_0^{(j)}|+T^{m-1}N|\varphi_0^{(m-1)}|+\frac{N\Gamma(\alpha)}{\omega^\alpha}(|\lambda|\|\varphi\|_\omega+L_gR+M).
\end{aligned}$$

Taking maximum over $[0,T]$ and using the inequality (6.3), we acquire the following relation:

$$\|\mathcal{P}y\|_\omega\leqslant R.$$

Therefore, $\mathcal{P}:\mathcal{B}_R\to\mathcal{B}_R$. In other words, $\mathcal{P}$ is well-defined on $\mathcal{B}_R$.

Step 2: We show that $\mathcal{P}$ is a contractive mapping.

We need to show that $\mathcal{P}$ is a contraction on $\mathcal{B}_R$. To see this, let $\forall y,z\in\mathcal{B}_R$. Note that

$$\begin{aligned}(\mathcal{P}y)(r)-(\mathcal{P}z)(r)=\int_0^r&\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)(g(s,y(s))\\
&-g(s,z(s)))ds,\quad r>0.\end{aligned} \tag{6.6}$$

Thus, for any $r\in[0,T]$, from Lemma 5.1 and (H_2)-Lipschitz condition, it follows that

$$\begin{aligned}
\frac{|(\mathcal{P}y)(r)-(\mathcal{P}z)(r)|}{\exp(\omega r)} &=\frac{1}{\exp(\omega r)}\left|\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)(g(s,y(s))-g(s,z(s)))ds\right| \\
&\leqslant\frac{1}{\exp(\omega r)}\int_0^r|\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)|\frac{\exp(\omega s)}{\exp(\omega s)}|g(s,y(s))-g(s,z(s))|ds \\
&\leqslant L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\frac{\exp(\omega s)}{\exp(\omega s)}|y(s)-z(s)|ds \\
&\leqslant L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\max_{0\leqslant r\leqslant T}\left\{\frac{|y(r)-z(r)|}{\exp(\omega r)}\right\} \\
&=L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\exp(\omega r)}\int_0^r(r-s)^{\alpha-1}\exp(\omega s)ds\times\|y-z\|_\omega \\
&=L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\exp(\omega r)}\int_0^r u^{\alpha-1}\exp(\omega r)\exp(-\omega u)du\times\|y-z\|_\omega \\
&=L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\int_0^r u^{\alpha-1}\exp(-\omega u)du\times\|y-z\|_\omega \\
&=L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\omega^\alpha}\int_0^{\omega r}v^{\alpha-1}\exp(-v)dv\times\|y-z\|_\omega \\
&\leqslant L_g\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{1}{\omega^\alpha}\int_0^{\infty}v^{\alpha-1}\exp(-v)dv\times\|y-z\|_\omega \\
&=\exp(|\lambda|r^\alpha+|\mu|r^{\alpha-\beta})\frac{L_g\Gamma(\alpha)}{\omega^\alpha}\times\|y-z\|_\omega \\
&\leqslant\exp(|\lambda|T^\alpha+|\mu|T^{\alpha-\beta})\frac{L_g\Gamma(\alpha)}{\omega^\alpha}\times\|y-z\|_\omega \\
&:=\frac{NL_g\Gamma(\alpha)}{\omega^\alpha}\|y-z\|_\omega
\end{aligned}$$

Taking maximum over $[0,T]$, we achieve the following result:

$$\|\mathcal{P}(y)-\mathcal{P}(z)\|_{\omega} \leqslant \frac{NL_g\Gamma(\alpha)}{\omega^{\alpha}}\|y-z\|_{\omega}. \tag{6.7}$$

If we choose $\omega > (NL_g\Gamma(\alpha))^{1/\alpha}$, then $\mathcal{P}$ is a contraction. So, by Banach's fixed point theorem, there exists a unique fixed point of $\mathcal{P}$ which is just the unique global continuous solution of the IVP (6.1) as a desired result. The proof is complete. □

Remark 5.1. In the proof of Theorem 5.1, the existence interval $[0,T]$ does not depend on the parameters of (6.1). Therefore, by repeating the arguments, it can be easily seen that if the assumptions (H_1) and (H_2) satisfy for all $t \in [0,\infty)$, then the assertion of this theorem hold on the half real line $\mathbb{R}_+$, that is, for any $(m-1)$-times continuously differentiable initial data $\varphi : [-\tau, 0] \to \mathbb{R}$, the nonlinear fractional-order LE with a constant delay (6.1) has a unique global continuous solution on $[0,\infty)$.

6. Ulam-Hyers stability analysis on FLE with a constant delay

In this section, we are going to discuss stability of the Langevin type DDE (6.1) with two different Caputo fractional derivatives in Ulam-Hyers sense on the time interval $[0,T]$.

Let $\varepsilon > 0$. Consider the Cauchy problem for Caputo type fractional Langevin DDE (6.1) and following inequality:

$$\left|({}^C D^{\alpha}_{0^+} z)(r) - \mu({}^C D^{\beta}_{0+} z)(r) - \lambda z(r-\tau) - g(r, z(r))\right| \leqslant \varepsilon, \quad \text{for} \quad r \in [0,T]. \tag{7.1}$$

Definition 6.1. The Eq. (6.1) is Ulam-Hyers stable if there exists $\theta > 0$such that for each $\varepsilon > 0$ and for each solution $z \in C([0,T],\mathbb{R})$of the inequality (7.1) there exists a solution $y \in C([0,T],\mathbb{R})$ of the Eq. (6.1) satisfying the inequality with respect to a weighted norm:

$$\|y-z\|_{\omega} \leqslant \theta\varepsilon, \quad r \in [0,T]. \tag{7.2}$$

Remark 6.1. A function $z \in C([0,T],\mathbb{R})$ is a solution of the inequality (7.1) if and only if there exists a function $f \in C([0,T],\mathbb{R})$ which satisfying the following conditions:

(i) $|f(r)| \leqslant \varepsilon$;
(ii) $(D^{\alpha}_{0^+} z)(r) - \mu(D^{\beta}_{0+} z)(r) - \lambda z(r-\tau) - g(r, z(r)) := f(r), \quad r \in [0,T]$.

According to the Remark 6.1, the solution of following equation:

$$({}^C D^{\alpha}_{0^+} z)(r) - \mu({}^C D^{\beta}_{0+} z)(r) - \lambda z(r-\tau) = g(r, z(r)) + f(r), \quad r \in [0,T], \tag{7.3}$$

can be represented by

$$\begin{aligned} z(r) &= \sum_{j=0}^{m-2}\left[\tfrac{r^j}{\Gamma(j+1)} + \lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)} \\ &+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} + \lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s \\ &+\int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s,z(s))\mathrm{d}s + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)f(s)\mathrm{d}s \\ &:= (\mathcal{P}z)(r) + \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)f(s)\mathrm{d}s, \quad 0 \leqslant r \leqslant T. \end{aligned} \tag{7.4}$$

By using Lemma 5.1, the difference $z(r) - (\mathcal{P}z)(r)$ can be evaluated as follows:

$$\begin{aligned} |z(r) - (\mathcal{P}z)(r)| &\leqslant \left|\int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)f(s)\mathrm{d}s\right| \\ &\leqslant \int_0^r |\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)||f(s)|\mathrm{d}s \\ &\leqslant \varepsilon r^{\alpha-1}\exp(|\lambda|r^{\alpha} + |\mu|r^{\alpha-\beta})\int_0^r \mathrm{d}s \\ &\leqslant \varepsilon T^{\alpha}\exp(|\lambda|T^{\alpha} + |\mu|T^{\alpha-\beta}) := \varepsilon T^{\alpha}N. \end{aligned} \tag{7.5}$$

Eventually, we are ready to state and prove the Ulam-Hyers stability result for FLE with a constant delay:

Theorem 6.1. *Assume that $(H_1)-(H_2)$ are satisfied. Then the Eq. (6.1) is Ulam-Hyers stable on $[0,T]$.*

Proof. Suppose that $z \in C([0,T],\mathbb{R})$ is a solution of the inequality (7.1). Let y be a unique solution of the Cauchy problem for Langevin type fractional-order DDE (6.1), that is

$$\begin{aligned} y(r) &= \sum_{j=0}^{m-2}\left[\tfrac{r^j}{\Gamma(j+1)} + \lambda\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+j+1}(\lambda,\mu;r-\tau)\right]\varphi_0^{(j)} \\ &+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,m}(\lambda,\mu;r)\varphi_0^{(m-1)} + \lambda\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-\tau-s)\varphi(s)\mathrm{d}s \\ &+\int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)g(s,y(s))\mathrm{d}s := (\mathcal{P}y)(r), \quad 0 \leqslant r \leqslant T. \end{aligned} \tag{7.6}$$

By using estimation (6.7) and (7.5), we have:

$$\begin{aligned} \frac{|y(r)-z(r)|}{\exp(\omega r)} &= \tfrac{1}{\exp(\omega r)}\left|(\mathcal{P}y)(r) - (\mathcal{P}z)(r) - \int_0^r \mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)f(s)\mathrm{d}s\right| \\ &\leqslant \tfrac{1}{\exp(\omega r)}|(\mathcal{P}y)(r) - (\mathcal{P}z)(r)| + \int_0^r |\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(\lambda,\mu;r-s)||f(s)|\mathrm{d}s \\ &\leqslant \tfrac{L_g\Gamma(\alpha)}{\omega^{\alpha}}\exp(|\lambda|T^{\alpha} + |\mu|T^{\alpha-\beta})\|y-z\|_{\omega} + \varepsilon T^{\alpha}\exp(|\lambda|T^{\alpha} + |\mu|T^{\alpha-\beta}) \\ &:= \tfrac{L_g N\Gamma(\alpha)}{\omega^{\alpha}}\|y-z\|_{\omega} + \varepsilon T^{\alpha}N. \end{aligned}$$

Taking maximum over $[0,T]$, we acquire

$$\|y-z\|_{\omega} \leqslant \frac{L_g N\Gamma(\alpha)}{\omega^{\alpha}}\|y-z\|_{\omega} + \varepsilon T^{\alpha}N,$$

that yields that

$$\|y-z\|_{\omega} \leqslant \varepsilon\frac{T^{\alpha}N}{1-\frac{\Gamma(\alpha)L_g N}{\omega^{\alpha}}}.$$

By choosing $\omega > (\Gamma(\alpha)L_g N)^{\frac{1}{\alpha}}$ which implies that

$$\|y-z\|_{\omega} \leqslant \theta\varepsilon \tag{7.7}$$

where

$$\theta := \frac{T^{\alpha}N}{1-\frac{\Gamma(\alpha)L_g N}{\omega^{\alpha}}} > 0.$$

The proof is complete. □

Remark 6.2. In the proof of Theorem (6.1), the Ulam-Hyers stability interval $[0,T]$ does not depend on the parameters of (6.1) and (7.1). Therefore, by repeating the arguments, it can be easily seen that if the assumptions (H_1) and (H_2) satisfy for all $t \in [0,\infty)$, then the assertion of this theorem hold on the half real line $\mathbb{R}_+$, that is, for any $(m-1)$-times continuously differentiable initial data $\varphi(\cdot) : [-\tau, 0] \to \mathbb{R}$, the fractional nonlinear delayed LE with two incommensurate fractional-orders (6.1) is stable in Ulam-Hyers's sense on $[0,\infty)$.

7. Application to the vibration theory: spring-mass-damper systems

In this section, we discuss fractional-order vibration systems which is widely used in physics and mechanical engineering sciences.

For the sake of a physical law, it is known that the *spring force* is defined by

$$\mathbb{F}_s = -Ky(r), \quad r > 0,$$

and the *damping force* is given by

$$\mathbb{F}_d = -C\left({}^{C}D^{\beta}_{0^+}y\right)(r), \quad r > 0, \quad 0 < \beta \leqslant 1,$$

where K – is *a constant of spring stiffness*, C – is *a constant of the viscous damping*.

If there exist three kinds of forces namely: $\mathbb{F}_s$, $\mathbb{F}_d$ and the external force $\mathbb{F}_e$, then in accordance with the Newton's second law, the motion of the *mass M* along a vertical straight line is illustrated by the following fractional-order vibration DDE with a single degree of freedom:

$$My''(r) + C\left({}^{C}D^{\beta}_{0^+}y\right)(r) + Ky(r-\tau) = \mathbb{F}_e(r), \quad r > 0, \quad \tau > 0, \tag{8.1}$$

with initial condition

$$y(r) = \varphi(r), \quad -\tau \leqslant r \leqslant 0, \tag{8.2}$$

where ${}^{C}D^{\beta}_{0^+}y$is the fractional derivative of order $\beta \in (0, 1]$ in Caputo's sense of *the displacement y*. In the particular case, ${}^{C}D^{\beta}_{0^+}y$ can be reduced to the ordinary differential operator whenever $\beta = 1$. In such case, we acquire the following classical vibration DE with a constant delay:

$$My''(r) + Cy'(r) + Ky(r-\tau) = \mathbb{F}_e(r), \quad r > 0, \quad \tau > 0, \tag{8.3}$$

with initial condition

$$y(r) = \varphi(r), \quad -\tau \leqslant r \leqslant 0. \tag{8.4}$$

The simplest model of fractional vibration system with linear viscous damping (8.1) can be demonstrated as shown in Fig. 1. Furthermore, in Fig. 1 we have considered forced vibrations with fractional-order linear viscous damping under *disturbing external force* $\mathbb{F}_e$ on the vibrating body. In the special case, if *an external force* equals to zero i.e., $\mathbb{F}_e(r) = 0$ this system so-called *free* vibrations with viscous damping.

Here the coefficients of *mass* (M), *spring stiffness* (K) and *viscous damping* (C) are connected as positive real constants.

7.1. A new representation of the solution of fractional vibration differential equation (FVE) with a constant delay

The second order in-homogeneous ODEs with fractional-order are arising in the field of vibration theory based on the well-known principle of Newton's second law (William, 1983). Our target is model the physical problem with an IVP such that we can determine *the replacement of mass y* in the *spring-mass-damper systems.*

Consider the vibration system characterized by fractional-order linear viscous damping, associated with three elements, i.e., *mass*, *spring-pot* and *dash-pot* see Fig. 1.

We study an IVP for the generalized form of FVE with a constant delay in the special case where fractional orders $1 < \alpha \leqslant 2$, $0 < \beta \leqslant 1$:

$$\begin{cases} \lambda_3\left({}^{C}D^{\alpha}_{0^+}y\right)(r) + \lambda_2\left({}^{C}D^{\beta}_{0^+}y\right)(r) + \lambda_1 y(r-\tau) = f(r), & r > 0, \quad \tau > 0, \\ y(r) = \varphi(r), & -\tau \leqslant r \leqslant 0, \end{cases} \tag{8.5}$$

where $\lambda_i, \quad i = 1, 2, 3$ are positive real numbers.

To find *the displacement* $y(r)$, we apply Laplace transform technique to both sides of (8.5):

$$\begin{aligned} &\lambda_3\left[s^{\alpha}Y(s) - s^{\alpha-1}\varphi_0 - s^{\alpha-2}\varphi'_0\right] + \lambda_2\left[s^{\beta}Y(s) - s^{\beta-1}\varphi_0\right] \\ &\quad + \lambda_1\left[Y(s)e^{-s\tau} + \mathcal{L}\{\hat{\varphi}(r-\tau)\}(s)\right] \\ &\quad = F(s). \end{aligned}$$

From above equality, we find $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{\lambda_3 s^{\alpha-1} + \lambda_2 s^{\beta-1}}{\lambda_3 s^{\alpha} + \lambda_2 s^{\beta} + \lambda_1 e^{-s\tau}}\varphi_0 + \lambda_3\frac{s^{\alpha-2}}{\lambda_3 s^{\alpha} + \lambda_2 s^{\beta} + \lambda_1 e^{-s\tau}}\varphi'_0 \\ &+ \lambda_1\frac{\mathcal{L}\{\hat{\varphi}(r-\tau)\}(s)}{\lambda_3 s^{\alpha} + \lambda_2 s^{\beta} + \lambda_1 e^{-s\tau}} + \frac{F(s)}{\lambda_3 s^{\alpha} + \lambda_2 s^{\beta} + \lambda_1 e^{-s\tau}} \\ &= \frac{s^{\alpha-1} + \frac{\lambda_2}{\lambda_3}s^{\beta-1}}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}}\varphi_0 + \frac{s^{\alpha-2}}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}}\varphi'_0 \\ &+ \frac{\lambda_1}{\lambda_3}\frac{\mathcal{L}\{\hat{\varphi}(r-\tau)\}(s)}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}} + \frac{1}{\lambda_3}\frac{F(s)}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}} \\ &= s^{-1}\left[1 - \frac{\lambda_1}{\lambda_3}\frac{e^{-s\tau}}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}}\right]\varphi_0 + \frac{s^{\alpha-2}}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}}\varphi'_0 \\ &+ \frac{\lambda_1}{\lambda_3}\frac{\mathcal{L}\{\hat{\varphi}(r-\tau)\}(s)}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}} + \frac{1}{\lambda_3}\frac{F(s)}{s^{\alpha} + \frac{\lambda_2}{\lambda_3}s^{\beta} + \frac{\lambda_1}{\lambda_3}e^{-s\tau}}. \end{aligned} \tag{8.6}$$

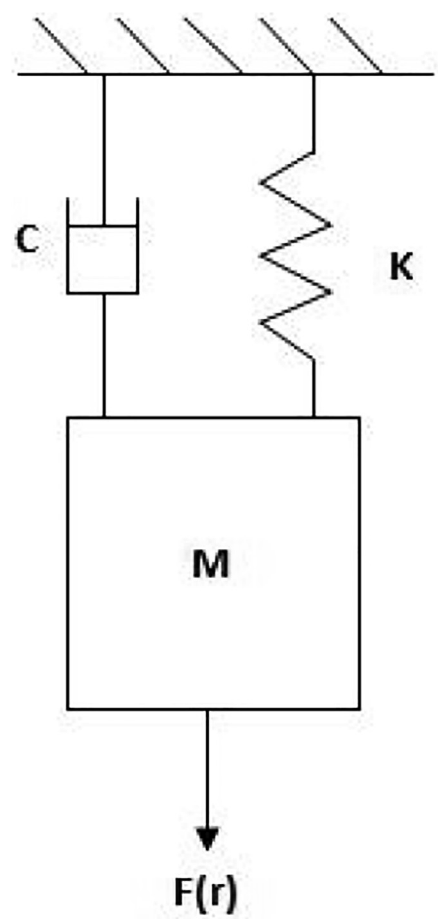

Fig. 1. Single of freedom vibration system.

Since Lemma 2.2, we have

$$\begin{aligned}\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta+\tfrac{\lambda_1}{\lambda_3}e^{-s\tau}\right)^{-1} &= \left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-1}\left(1+\tfrac{\lambda_1}{\lambda_3}(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta)^{-1}e^{-s\tau}\right)^{-1}\\ &= \left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-1}\left(1-\tfrac{\lambda_1}{\lambda_3}\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-1}e^{-s\tau}\right.\\ &\left.+\left(\tfrac{\lambda_1}{\lambda_3}\right)^2\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-2}e^{-2s\tau}+\ldots+\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-l}e^{-ls\tau}+\ldots\right)\\ &= \left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-1}\sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-l}e^{-ls\tau}\\ &= \sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-(l+1)}e^{-ls\tau}.\end{aligned} \tag{8.7}$$

Plugging the above relation (8.7) in (8.6), we attain that

$$\begin{aligned}Y(s) &= \left[s^{-1}+s^{-1}\sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^{l+1}\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-(l+1)}e^{-(l+1)s\tau}\right]\varphi_0\\ &+s^{\alpha-2}\sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-(l+1)}e^{-ls\tau}\varphi_0'\\ &+\tfrac{\lambda_1}{\lambda_3}\mathcal{L}\{\hat{\varphi}(r-\tau)\}(s)\sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-(l+1)}e^{-ls\tau}\\ &+\tfrac{1}{\lambda_3}F(s)\sum_{l=0}^{\infty}\left(-\tfrac{\lambda_1}{\lambda_3}\right)^l\left(s^\alpha+\tfrac{\lambda_2}{\lambda_3}s^\beta\right)^{-(l+1)}e^{-ls\tau}.\end{aligned} \tag{8.8}$$

Then applying inverse Laplace transform, Lemma 3.2 and Pascal's rule, we acquire

$$\begin{aligned}y(r) &= \left[1+\left(-\tfrac{\lambda_1}{\lambda_3}\right)\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(-\frac{\lambda_1}{\lambda_3})^l(-\frac{\lambda_2}{\lambda_3})^p(r-(l+1)\tau)^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\mathcal{H}(r-(l+1)\tau)\right]\varphi_0\\ &+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(-\frac{\lambda_1}{\lambda_3})^l(-\frac{\lambda_2}{\lambda_3})^p(r-l\tau)^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\mathcal{H}(r-l\tau)\varphi_0'\\ &+\tfrac{\lambda_1}{\lambda_3}\int_{-\tau}^{r-\tau}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(-\frac{\lambda_1}{\lambda_3})^l(-\frac{\lambda_2}{\lambda_3})^p(r-(l+1)\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\mathcal{H}(r-(l+1)\tau-s)\tilde{\varphi}(s)ds\\ &+\tfrac{1}{\lambda_3}\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(-\frac{\lambda_1}{\lambda_3})^l(-\frac{\lambda_2}{\lambda_3})^p(r-l\tau-s)^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\mathcal{H}(r-l\tau-s)f(s)ds\\ &= \left[1-\tfrac{\lambda_1}{\lambda_3}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha+1}(-\tfrac{\lambda_1}{\lambda_3},-\tfrac{\lambda_2}{\lambda_3};r-\tau)\right]\varphi_0+\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,2}(-\tfrac{\lambda_1}{\lambda_3},-\tfrac{\lambda_2}{\lambda_3};r)\varphi_0'\\ &+\tfrac{\lambda_1}{\lambda_3}\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(-\tfrac{\lambda_1}{\lambda_3},-\tfrac{\lambda_2}{\lambda_3};r-\tau-s)\varphi(s)ds+\tfrac{1}{\lambda_3}\int_0^r\mathbb{E}^{\tau}_{\alpha,\alpha-\beta,\alpha}(-\tfrac{\lambda_1}{\lambda_3},-\tfrac{\lambda_2}{\lambda_3};r-s)f(s)ds,\end{aligned}$$

where $\tilde{\varphi}(\cdot):\mathbb{R}\to\mathbb{R}$ is the unit-step function defined as follows:

$$\tilde{\varphi}(r)=\begin{cases}\varphi(r), & -\tau\leqslant r\leqslant 0,\\ 0, & r>0.\end{cases}$$

In the particular case, if we take $\alpha=2$, $\beta\in(0,1]$ and $\lambda_3=M$, $\lambda_2=C$, $\lambda_1=K$, we derive vibration time-delay equation with single fractional-order damping term:

$$\begin{cases}My''(r)+C\left({}^{C}D^{\beta}_{0^+}y\right)(r)+Ky(r-\tau)=\mathbb{F}_e(r), & r>0,\quad \tau>0,\\ y(r)=\varphi(r), & -\tau\leqslant r\leqslant 0,\end{cases} \tag{8.9}$$

with its exact analytical representation of solution

$$\begin{aligned}y(r) &= \left[1-\tfrac{K}{M}\mathbb{E}^{\tau}_{2,2-\beta,3}(-\tfrac{K}{M},-\tfrac{C}{M};r-\tau)\right]\varphi_0+\mathbb{E}^{\tau}_{2,2-\beta,2}(-\tfrac{K}{M},-\tfrac{C}{M};r)\varphi_0'\\ &+\tfrac{K}{M}\int_{-\tau}^{\min\{r-\tau,0\}}\mathbb{E}^{\tau}_{2,2-\beta,2}(-\tfrac{K}{M},-\tfrac{C}{M};r-\tau-s)\varphi(s)ds\\ &+\tfrac{1}{M}\int_0^r\mathbb{E}^{\tau}_{2,2-\beta,2}(-\tfrac{K}{M},-\tfrac{C}{M};r-s)\mathbb{F}_e(s)ds,\quad r>0.\end{aligned} \tag{8.10}$$

We consider the values of each parameter such as $\alpha=2,\beta=0.7,M=2,C=10,K=50$ in Fig. 2 and 3. Since *a natural frequency* of fractional-order vibration system defined by $\nu=\sqrt{\frac{K}{M}}$, then for the given values of K and M, we find $\nu=5$. To compare results under *harmonic excitation* in the form of *a exponential function* i.e., $\mathbb{F}_e(r)=F_0\exp(\nu r)$; two kinds of *periodic external forces* $\mathbb{F}_e(r)=F_0\cos(\nu r)$ and $\mathbb{F}_e(r)=F_0\sin(\nu r)$ of *disturbing force* $\mathbb{F}_e(r)$, we provide three interesting cases for *a replacement of mass* $y(r)$ with respect to time r.

The plot of *displacement* $y(r)$ will be demonstrated with $F_0=25$ and $\nu=5$ in the following cases:

Case 1: If $\mathbb{F}_e(r)=F_0\exp(\nu r)$, then see Fig. 2 (a);
Case 2: If $\mathbb{F}_e(r)=F_0\cos(\nu r)$, then see Fig. 2 (b);
Case 3: If $\mathbb{F}_e(r)=F_0\sin(\nu r)$, then see Fig. 2 (c).

To see comparison clearly in Fig. 2 [(b) and (c)], we describe two plots (b and c) in the same graph in Fig. 3.

8. An example

In this section, we provide an example to verify our major theoretical results stated in Section 5 and 6. To show the existence and uniqueness and stability analysis of solutions in the following example, we need to apply Theorem 6.1.

Let $\alpha=1.2$, $\beta=0.8$, $\tau=2$, $m=2$ and $T=2$. Consider the following IVP for fractional Langevin DE with a constant delay:

$$\begin{cases}\left({}^{C}D^{1.2}_{0^+}y\right)(r)-3\left({}^{C}D^{1.8}_{0^+}y\right)(r)-5y(r-2)=\frac{\cos(y(r))}{r^2+1}, & 0<r\leqslant 2,\\ y(r)=r+5, & -2\leqslant r\leqslant 0,\end{cases} \tag{8.11}$$

with constants $\lambda=5,\mu=3$ and $\varphi(r)=r+5$ is continuously differentiable function for $r\in[-2,0]$ and nonlinear perturbation $g(r,y(r))=\frac{\cos(y(r))}{r^2+1}$ is continuous on a Cartesian product $[0,2]\times\mathbb{R}$.

Since $y(0)=6$, and $y'(0)=1$, the exact analytical representation of solution of (8.11) can be represented as follows:

$$\begin{aligned}y(r) &= \left[5+25\mathbb{E}^{2}_{1.2,0.4,2.2}(5,3;r-2)\right]+\mathbb{E}^{2}_{1.2,0.4,2}(5,3;r)\\ &+5\int_{-2}^{\min\{r-2,0\}}\mathbb{E}^{2}_{1.2,0.4,1.2}(5,3;r-2-s)\varphi(s)ds\\ &+\int_0^r\mathbb{E}^{2}_{1.2,0.4,1.2}(5,3;r-s)g(s,y(s))ds,\quad r>0.\end{aligned}$$

It is not difficult to see that condition (H_2) holds. By mean value theorem, for $\forall y,z\in\mathbb{R}$, there exists $\xi\in(y,z)$ such that

$$\begin{aligned}|g(r,y)-g(r,z)| &= \left|\frac{\cos(y(r))-\cos(z(r))}{r^2+1}\right|\leqslant\frac{\sin\xi}{r^2+1}|y-z|\\ &\leqslant|y-z|,\quad \forall r\in[0,2].\end{aligned}$$

Then (H_2) holds with $L_g\equiv 1$. By Theorem 6.1 and 6.1, the nonlinear FLE with a constant delay (8.11) has a unique solution which is stable in Ulam-Hyers sense on $[0,2]$.

9. Conclusions and future work

In recent years, the time-delay theory for Langevin equations in the fractional sense has not been able to get substantial development. As an urgent problem to be solved, we have investigated explicit analytical solutions for linear homogeneous and inhomogeneous Langevin time-delay DEs with general fractional orders in general and special cases via a newly defined delayed analogue of bivariate M-L functions. In the application of vibration theory, we acquire the solution of Langevin type DDE with two fractional orders of $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$ and compare results of *displacement of mass* y under various kinds of *external forces* $\mathbb{F}_e$. The main contributions of our research work are as below:

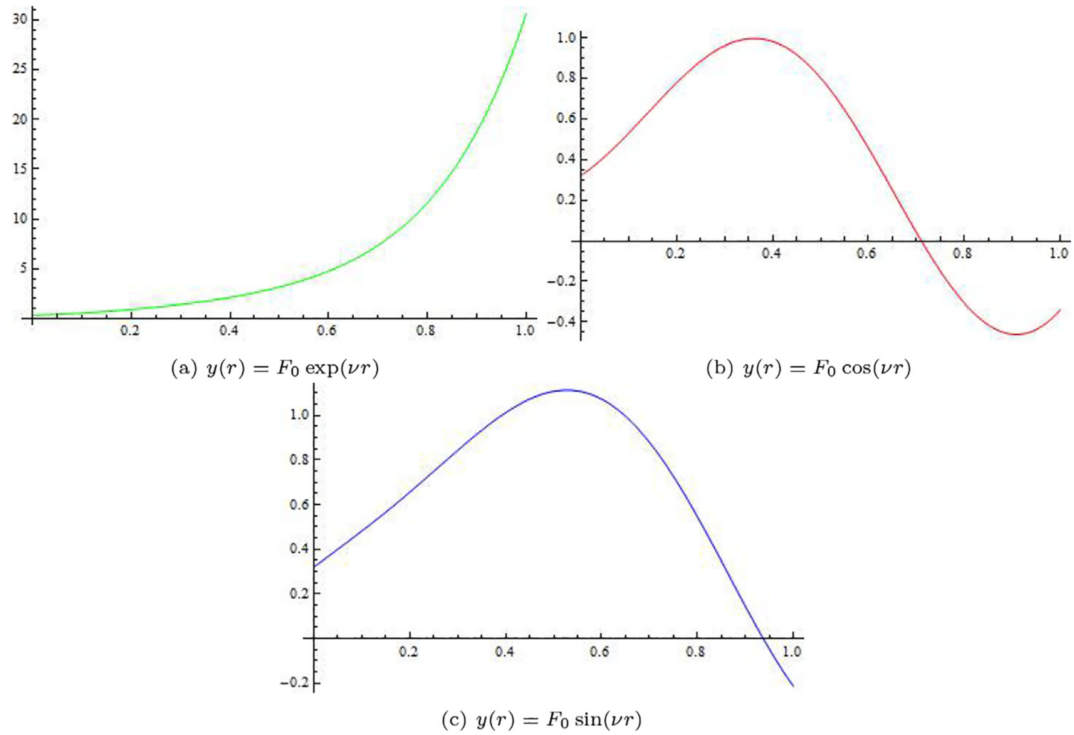

Fig. 2. The plots of displacement of mass $y(r)$ involving varying functions $F(r)$.

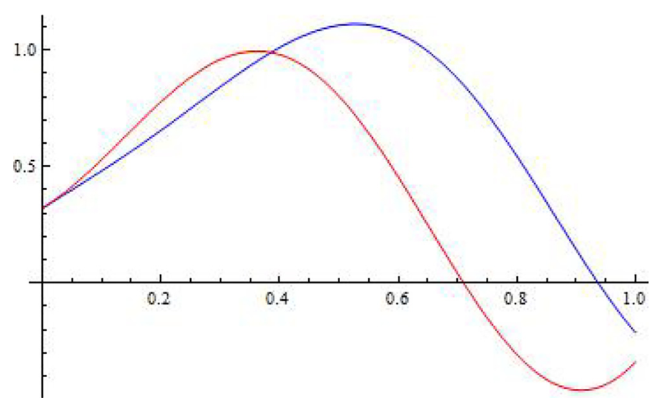

Fig. 3. Displacement of mass ($y(r)$) vs time (r) graph involving two various functions for $F(r)$.

- we introduce the exact analytical representation of homogeneous and inhomogeneous Langevin type time-delay equations with general fractional orders by means of a newly defined delayed M-L type function via double infinite series;
- we estimate a new delayed M-L type function with respect to exponential function;
- we introduce a new weighted maximum norm with regard to exponential function in $C^m([0,T],\mathbb{R}), m \geqslant 2$ and prove sufficient conditions to provide the existence and uniqueness of global solution on $[0,T]$ for nonlinear Langevin equations with a constant delay and general fractional orders in Caputo sense;
- we verify that our solutions with regard to M-L type functions are identical with the results by means of generalized Wright functions for delay-free systems;
- we study stability problem for the solutions of time-delay FLEs in Ulam-Hyers sense in a weighted space of continuous functions;
- we propose a new representation of solutions to the fractional-order vibration equations with a constant delay.

There are a number of potential directions in which the results acquired here can be extended. Our future work will proceed to study the asymptotic stability of the trivial solution with the help of the Lyapunov methods and relative controllability results of solutions with the aid of Gramian matrix and rank criterion to the FLEs with a constant delay.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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