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Nonlinear third-order differential equations with distributed delay: Some new oscillatory solutions

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ABSTRACT

We consider a certain class of third order nonlinear delay differential equations in this work. The results that we obtained are an improvement and extension of some results mentioned in previous literature, as the criteria we obtained are less restrictive compared to the previous results reported in literature. An example is provided to illustrate new results.

1. Introduction

In this paper, we consider the third-order neutral nonlinear differential equation of the form

$$\left(\imath(\xi)(\psi''(\xi))^{\alpha}\right)' + \int_a^b \vartheta(\xi,\omega)\chi^{\alpha}(\tau(\xi,\omega))\mathrm{d}\omega = 0, \text{for } \xi \geqslant \xi_0, \quad (1)$$

where α is a ratio of positive odd integers and

$\psi(\xi) = \chi(\xi) + p_0\chi(\xi - \delta_0)$.

Throughout this work, we will assume the following:

($\mathbf{I}_1$) δ_0, p_0 are constants such that $p_0, \delta_0 \geqslant 0$;
($\mathbf{I}_2$) $\imath \in C^1([\xi_0,\infty),(0,\infty)), \vartheta, \tau \in C([\xi_0,\infty)\times[a,b],\mathbb{R}), \tau(\xi,s) < \xi$, $\lim_{\xi\to\infty}\tau(\xi,\omega) = \infty, \vartheta(\xi,\omega)$ don't not vanish identically, and

$$\int_{\xi_0}^{\infty} \imath^{-1/\alpha}(\xi)\mathrm{d}\xi = \infty.$$

When studying the behavior of positive solutions of (1), we note that there are only two cases for $\xi > \xi_1$ is sufficiently large:

Case (**i**) : $\psi(\xi) > 0, \psi'(\xi) > 0, \psi''(\xi) > 0$;
Case (**ii**) : $\psi(\xi) > 0, \psi'(\xi) < 0, \psi''(\xi) > 0$.

Definition 1. A solution of (1) means $\chi \in C([\xi_a,\infty),\mathbb{R})$ where $\xi_a := \min\{\xi_0 - \delta_0, \min_{\xi\in I}\tau(\xi)\}$ which satisfies (1) and the property $\imath(\psi'')^{\alpha} \in C^1(I,\mathbb{R})$ on I. We consider the nontrivial solutions of (1) that satisfy the condition $\sup\{|\chi(\xi)| : \xi \geqslant \xi_1\} > 0$ for all $\xi_1 \geqslant \xi_a$.

Definition 2. A solution χ of (1) is said to be *nonoscillatory if it is*neither positive nor negative eventually. Otherwise, it is *oscillatory*.

Definition 3. If (1) has property D, then we say that solution χ of (1) is either oscillatory or satisfies $\lim_{\xi\to\infty}\chi(\xi) = 0$.

A long time ago, third order differential equations have been involved in many mathematical models in various field of applied sciences where the famous isoperimetric problem was formulated. Later, a solution was found based on a third-order differential equation. Thus, third-order differential equations have become the target of researchers and those interested in their effectiveness in modeling many phenomena of economic and scientific life, especially physical, engineering and biological ones, we refer to Agarwal et al. (2000), Agarwal et al. (2001), Baculikova and Dzurina (2010), Baculikova and Dzurina (2011), Baculikova and Dzurina (2012), Baculikova and Dzurina (2014), Stability and Gorain (1998), Candan (2015), Dzurina et al. (2012), Erbe et al. (1995), Grace (1984), Győi et al. (1991), Hale (1977), Karpuz

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et al. (2010), Kitamura and Kusano (1980), Ladde et al. (1987), Li (1996), Li and Rogovchenko (2014), Li and Rogovchenko (2020), Li and Thandapani (2011), Li et al. (2010), Li et al. (2012), Liu and Triggiani (2013), Marchand et al. (2012), Philos (1981), Rath et al. (2004), Shang (2012), Tang (2002), Tekin (2021), Wu et al. (2016), Wu et al. (2018), Xing et al. (2011), Xu and Xia (2004), Yang and Xu (2014), Zhang and Wang (2010).

When considering partial differential equations and their applications, Adeye- mo and Khalique studied an extended Kadomtsev–Petviashvili-like equation (Adeyemo and Khalique, 2022a), and a higher-dimensional soliton equation (Adeyemo and Khalique, 2022b) achieving analytic solutions. They emphasized the importance of these solutions in scientific fields. The authors of Adeyemo et al. (2022) obtained travelling wave solutions for a (3 + 1)-dimensional generalized Zakharov-Kuznetsov equation. In Adeyemo (2022), a generalized extended (2 + 1)-D quantum Zakharov-Kuznetsov equation was analytically studied, where the author outlined the applications of cnoidal and snoidal waves of the model in ocean engineering and oceanography. A (2 + 1)-D generalized Bogoyavlensky-Konopelchenko equation was investigated in Adeyemo et al. (2022).

In a bounded domain in $\mathbb{R}^n$ with a smooth boundary, the boundary stabilization of the problem satisfying the differential equation $\theta'' + \lambda\theta''' = k^2(\Delta\theta + \mu\Delta\theta')$ is investigated in Stability and Gorain (1998). As modeled by the Standard linear model of viscoelasticity, these equations appear in the vibrations of flexible structures with internal material damping. Under mixed boundary conditions, the authors examined an exponential energy decay for their stated problem. An inverse problem for the linearized Jordan-Moore-Gibson-Thompson equation, a third-order in time partial differential equation that appears in nonlinear acoustic waves modeling high-intensity ultrasound, is introduced in Liu and Triggiani (2013). By using only one boundary measurement, the two canonical recovery issues of uniqueness and stability are examined. The Jordan-Moore-Gibson-Thompson equation's dynamical decomposition is the foundation of the suggested method. The authors of Marchand et al. (2012) investigated the Moore-Gibson-Thompson equation, which arises in high-intensity ultrasound. They presented an abstract third-order equation in a Hilbert space. The authors provided that this third-order abstract equation with unbounded free dynamical operator is not well-posed in its simplest form, with a particular set of parameter values. In Tekin (2021), the authors examined the inverse problem of recovering a time-dependent coefficient of a nonlinear third order in time partial differential equation, also known as the Moore-Gibson-Thompson equation, from a single boundary measurement.

Despite the importance of third-order differential equations, the literature that has appeared so far is few compared to second-order differential equations, we recommend recent monographs to the reader (Baculikova and Dzurina, 2011; Wu et al., 2018; Wu et al., 2016 and Zhang and Wang, 2010).

It is worth noting that we find that third-order delay differential equations may have only oscillatory solutions, while they may have oscillatory and non-oscillatory solutions; for instant, the solutions for the eqaution

$$\psi'''(\xi) + \psi(\xi - \sigma) = 0,$$

are oscillatory if and only if, $\sigma > 3/e$. On the other hand, the equation of the form

$$\psi'''(\xi) + 2z'(\xi) - \psi\left(\xi - \frac{1}{2}3\pi\right) = 0$$

has an oscillatory and a nonoscillatory solution $(sin\xi, e^{\lambda\xi})$ such that $\lambda^3 + 2\lambda = e^{-(3\pi l2)\lambda}, \lambda > 0$.

Recently, Baculikova and Dzurina (2010) and Yang and Xu (2014) established some different sufficient criteria which ensure that all nonoscillatory solutions to the equation

$$\left(\imath(\xi)\left((\chi(\xi) + p_0\chi(\sigma(\xi)))''\right)^{\alpha}\right)' + \vartheta(\xi)\chi^{\alpha}(\tau(\xi))\mathrm{d}s = 0 \tag{2}$$

tend to zero. Also, Li et al. (2012) and Baculikova and Dzurina (2010) investigated Eq. (2) for $\alpha = 1$ under the condition

$$0 \leqslant p_0 < 1. \tag{3}$$

Assuming $\tau(\xi) = \xi - \delta_0$, Li and Rogovchenko (2014) studied asymptotics of Eq. (2) with the condition

$$0 \leqslant p_0 < \infty.$$

In the present paper, by using different techniques (comparison with first order delay equations and the technique of Riccati transformation), we obtain the conditions that ensure the oscillation of the solutions of this equation. Moreover, we extend and improve previous results.

Furthermore, we present new criteria that ensure the oscillation of all the solutions of Eq. (1), these criteria are an improvement of previous results, as the conditions mentioned are less restrictive and easier to apply. By taking advantage of the results obtained recently and the current results, the conditions achieved ensure the oscillation of all solutions of Eq. (1).

We state the following lemma, which we will need to prove our results later.

Lemma 1. *Let $h_1, h_2 \in [0, \infty)$ and $m > 0$. Then*

$$(h_1 + h_2)^m \leqslant \mu(h_1^m + h_2^m). \tag{4}$$

where

$$\mu = 1 \text{ when } m \leqslant 1;$$
$$\mu = 2^{m-1} \text{ when } m > 1.$$

Lemma 2. *Let $H, F \in C(I, \mathbb{R}), F_* \in \mathbb{R}$ and $H(\xi) = F(\xi) + aF(\xi - k), \xi \geqslant \xi_0 + \max\{0, k\}$, where a and k are constants and $a \neq 1$. Assume that $\exists l \in \mathbb{R}$ such that $\lim_{\xi\to\infty} H(\xi) = l$, moreover,*

$$l = \begin{cases} (1 + a)F_* & \text{if} \quad \liminf_{\xi\to\infty} F(\xi) = F_* \\ (1 + a)F^* & \text{if} \quad \limsup_{\xi\to\infty} F(\xi) = F^*. \end{cases}$$

where $F_*, F_* \in \mathbb{R}$.

2. Main results

Lemma 3. *Let $\chi(\xi) > 0$ be a solution of Eq. (1). Assume that $\psi(\xi)$ satisfies case (**ii**). If*

$$\int_{\xi_2}^{\infty} \int_{\upsilon}^{\infty} \left(\frac{1}{\imath(u)} \int_{u}^{\infty} \int_{a}^{b} \vartheta(\kappa, \omega)\mathrm{d}\omega\mathrm{d}\kappa\right)^{1/\alpha} \mathrm{d}u\mathrm{d}\upsilon = \infty, \tag{5}$$

then,

$$\lim_{\xi\to\infty} \chi(\xi) = 0. \tag{6}$$

Proof. $\psi(\xi)$ is a nonincreasing positive function, then there exists a $\psi_0 \geqslant 0$ such that

$$\lim_{\xi\to\infty} \psi(\xi) = \psi_0 \geqslant 0.$$

We claim that $\psi_0 = 0$. Otherwise, by the above Lemma, we get

$$\lim_{\xi\to\infty}\chi(\xi) = \psi_0/(1+p_0) > 0.$$

Therefore, there exists a $\xi_2 \geqslant \xi_0$ such that, for all $\xi \geqslant \xi_2$

$$\chi(\tau(\xi,s)) > \frac{\psi_0}{2(1+p_0)} > 0. \tag{7}$$

From (1) and (7), it follows that

$$\left(\imath(\xi)(\psi''(\xi))^\alpha\right)' \leqslant -\int_a^b \vartheta(\xi,\omega)\left(\frac{\psi_0}{2(1+p_0)}\right)^\alpha d\omega.$$

Integrating the above inequality from ξ to ∞, we get

$$\imath(\xi)(\psi''(\xi))^\alpha \geqslant \left(\frac{\psi_0}{2(1+P_0)}\right)^\alpha \int_\xi^\infty \int_a^b \vartheta(\kappa,\omega)d\omega d\kappa.$$

It follows that

$$\psi''(\xi) \geqslant \frac{\psi_0}{2(1+P_0)}\left(\frac{1}{\imath(\xi)}\int_\xi^\infty \int_a^b \vartheta(\kappa,\omega)d\omega d\kappa\right)^{\frac{1}{\alpha}}. \tag{8}$$

Integrating (8) from ξ to ∞, yields

$$-\psi'(\xi) \geqslant \frac{\psi_0}{2(1+P_0)}\int_\xi^\infty \left(\frac{1}{\imath(u)}\int_u^\infty \int_a^b \vartheta(\kappa,\omega)d\omega d\kappa\right)^{1/\alpha} du.$$

Integrating again from ξ_2 to ∞, gives

$$\psi(\xi_2) \geqslant \frac{\psi_0}{2(1+P_0)}\int_{\xi_2}^\infty \int_v^\infty \left(\frac{1}{\imath(u)}\int_u^\infty \int_a^b \vartheta(\kappa,\omega)d\omega d\kappa\right)^{1/\alpha} du dv,$$

this contradicts (5). Therefore, $\lim_{\xi\to\infty}\psi(\xi) = 0$, and from $0 < \chi(\xi) \leqslant \psi(\xi)$, we have (6).

Theorem 1. *Let (5) be satisfied and assume that there exists a function $\varrho \in C(I,\mathbb{R})$ where $\varrho(\xi) \leqslant \tau(\xi,\omega), \varrho(\xi) < \xi$ and $\lim_{\xi\to\infty}\varrho(\xi) = \infty$. If the first-order delay differential equation*

$$y'(\xi) + \frac{1}{(1+p_0)^\alpha}\int_a^b R(\varrho)\vartheta(\xi,\omega)d\omega y(\varrho(\xi)) = 0, \tag{9}$$

where

$$R(\xi) := \left(\int_{\xi_2}^{\varrho(\xi)} \int_{\xi_1}^{\rho} \imath^{-1/\alpha}(\omega)d\omega d\rho\right)^\alpha,$$

is oscillatory; eventually, then (1) has property D.

Proof. Suppose that $\chi(\xi)$ is a positive solution of (1); there exists a $\xi_1 \geqslant \xi_0$ such that either (**i**) or (**ii**) holds for all $\xi \geqslant \xi_1$. Let ψ satisfies case (**ii**), by Lemma 3, we see that (6) holds. Assume that ψ satisfies case (**i**), Since $\imath(\xi)(\psi''(\xi))^\alpha$ is nonincreasing, we have

$$\begin{aligned}\psi'(\xi) &\geqslant \int_{\xi_1}^{\xi} \tfrac{1}{\imath^{1/\alpha}(\omega)}\imath^{1/\alpha}(\omega)(\psi''(\omega))d\omega \\ &\geqslant \imath^{1/\alpha}(\xi)(\psi''(\xi))\int_{\xi_1}^{\xi}\tfrac{1}{\imath^{1/\alpha}(\omega)}d\omega,\end{aligned} \tag{10}$$

Integrating (10) from ξ_2 to ξ, where $\xi_2 > \xi_1$, we get

$$\psi(\xi) \geqslant \imath(\xi)^{1/\alpha}(\psi''(\xi))\int_{\xi_2}^{\xi}\left(\int_{\xi_1}^{\rho}\frac{1}{\imath^{1/\alpha}(\omega)}d\omega\right)d\rho. \tag{11}$$

Now, Since $\psi'(\xi)$ is a nondecreasing positive function. There exists a constant c_0 such that $\lim_{\xi\to\infty}\psi'(\xi) = c_0 > 0$ (or $c_0 = \infty$). By Lemma 2, we have

$$\lim_{\xi\to\infty}\chi'(\xi) = c_0/(1+p_0) > 0,$$

this implies that $\chi(\xi)$ is a nondecreasing function,we get

$$\psi(\xi) = \chi(\xi) + p_0\chi(\xi-\delta_0) \leqslant (1+p_0)\chi(\xi).$$

Therefore,

$$\chi(\xi) \geqslant \frac{1}{1+p_0}\psi(\xi).$$

Since $\tau(\xi,s) \geqslant \varrho(\xi)$, we obtain

$$\chi(\tau(\xi,s)) \geqslant \frac{1}{1+p_0}\psi(\varrho(\xi)).$$

From (1), we have

$$\left(\imath(\xi)(\psi''(\xi))^\alpha\right)' + \frac{\psi^\alpha(\varrho(\xi))}{(1+p_0)^\alpha}\int_a^b \vartheta(\xi,\omega)d\omega \leqslant 0. \tag{12}$$

Using (12) and (11), we arrive at

$$\begin{aligned}&\left(\imath(\xi)(\psi''(\xi))^\alpha\right)' \\ &+ \frac{\int_a^b \vartheta(\xi,\omega)d\omega}{(1+p_0)^\alpha}\left(\int_{\xi_2}^{\varrho(\xi)}\int_{\xi_1}^{\rho}\imath^{-1/\alpha}(\omega)d\omega d\rho\right)^\alpha\left(\imath(\varrho(\xi))(\psi''(\varrho(\xi)))^\alpha\right) \\ &\leqslant 0.\end{aligned}$$

Therefore, we have $y(\xi) = \imath(\xi)(\psi''(\xi))^\alpha$ is a positive solution of (9).

Corollary 1. *Let (5) holds, and suppose that there exists a function $\varrho \in C(I,\mathbb{R})$ such that $\varrho(\xi) \leqslant \tau(\xi,\omega), \varrho(\xi) < \xi$ and $\lim_{\xi\to\infty}\varrho(\xi) = \infty$. If*

$$\liminf_{\xi\to\infty}\int_{\varrho(\xi)}^{\xi} R(u)\left(\int_a^b \vartheta(\xi,\omega)d\omega\right)du > \frac{(1+p_0)^\alpha}{e}, \tag{13}$$

then Eq. (1) has property D.

Proof. In view of Győi et al. (1991); Erbe et al. (1995) condition (13) implies the oscillation of the delay differential Eq. (1).

Theorem 2. *If a function $\theta \in C([\xi_0,\infty),(0,\infty))$ exists, where $\theta(\xi) \leqslant \xi, \tau(\xi,\omega) - \delta_0 = \tau(\xi-\delta_0,\omega), \tau(\xi,\omega) \leqslant (\theta(\xi) - \delta_0)$ and*

$$\begin{aligned}&\limsup_{\xi\to\infty}\left(\int_{\xi_2}^{\xi}\left(\int_{\xi_1}^{\rho}\frac{1}{\imath^{1/\alpha}(\omega)}d\omega\right)d\rho\right)\int_{\theta(\xi)}^{\xi}\int_a^b \widetilde{\vartheta}(u,\omega)d\omega du. \\ &> \mu(1+p_0^\alpha),\end{aligned} \tag{14}$$

where,

$$\widetilde{\vartheta}(\xi,\omega) := min\{\vartheta(\xi,\omega), \vartheta(\xi-\delta_0,\omega)\} \tag{15}$$

*then case(**ii**) is impossible to satisfy.*

Proof. Let $\chi > 0$ be a solution of (1). Then, $\chi(\xi), \chi(\tau(\xi))$ and $\chi(\xi-\delta_0)$ are positive functions for $\xi \geqslant \xi_1$ is sufficiently large. By using Lemma 1, we obtain

$$\psi^\alpha(\xi) \leqslant \mu\left(\chi^\alpha(\xi) + p_0^\alpha\chi^\alpha(\xi-\delta_0)\right),$$

and

$$\psi^\alpha(\tau(\xi,\omega)) \leqslant \mu\left(\chi^\alpha(\tau(\xi,\omega)) + p_0^\alpha\chi^\alpha(\tau(\xi,\omega)-\delta_0)\right). \tag{16}$$

Now, from (1) we have

$$\left(\imath(\xi-\delta_0)(\psi''(\xi-\delta_0))^{\alpha}\right)' + \int_a^b \vartheta(\xi-\delta_0,\omega)\chi^{\alpha}(\tau(\xi-\delta_0,\omega))\mathrm{d}\omega = 0. \tag{17}$$

Using (1), (16) and (17), we have

$$\begin{aligned} 0 \geqslant\ & \left(\imath(\xi)(\psi''(\xi))^{\alpha}\right)' + \int_a^b \vartheta(\xi,s)\chi^{\alpha}(\tau(\xi,s))\mathrm{d}s \\ & + p_0^{\alpha}\left(\imath(\xi-\delta_0)(\psi''(\xi-\delta_0))^{\alpha}\right)' \\ & + p_0^{\alpha}\int_a^b \vartheta(\xi-\delta_0,\omega)\chi^{\alpha}(\tau(\xi,\omega)-\delta_0)\mathrm{d}\omega \\ \geqslant\ & \left(\imath(\xi)(\psi''(\xi))^{\alpha}\right)' + p_0^{\alpha}\left(\imath(\xi-\delta_0)(\psi''(\xi-\delta_0))^{\alpha}\right)' \\ & + \int_a^b \widetilde{\vartheta}(\xi,\omega)\left(\chi^{\alpha}(\tau(\xi,\omega)) + p_0^{\alpha}\chi^{\alpha}(\tau(\xi)-\delta_0,\omega)\right)\mathrm{d}\omega. \end{aligned} \tag{18}$$

Thus

$$\left(\imath(\xi)(\psi''(\xi))^{\alpha} + p_0^{\alpha}\left(\imath(\xi-\delta_0)(\psi''(\xi-\delta_0))^{\alpha}\right)\right)' + \frac{1}{\mu}\int_a^b \widetilde{\vartheta}(\xi,\omega)\psi^{\alpha}(\tau(\xi,\omega))\mathrm{d}\omega \leqslant 0. \tag{19}$$

Integrating (19) from $\theta(\xi)$ to ξ, we see that

$$\begin{aligned} & \imath(\theta(\xi))(\psi''(\theta(\xi)))^{\alpha} + p_0^{\alpha}\imath(\theta(\xi)-\delta_0)(\psi''(\theta(\xi)-\delta_0))^{\alpha} \\ \leqslant\ & \imath(\xi)(\psi''(\xi))^{\alpha} + p_0^{\alpha}\imath(\xi-\delta_0)(\psi''(\xi-\delta_0))^{\alpha} \\ & -\tfrac{1}{\mu}\int_{\theta(\xi)}^{\xi}\int_a^b \widetilde{\vartheta}(u,\omega)\psi^{\alpha}(\tau(u,\omega))\mathrm{d}\omega\mathrm{d}u, \end{aligned}$$

Since $\imath(\xi)(\psi''(\xi))^{\alpha}$ is nonincreasing, we have

$$\left(1+p_0^{\alpha}\right)\imath(\theta(\xi)-\delta_0)(\psi''(\theta(\xi)-\delta_0))^{\alpha} \geqslant \frac{1}{\mu}\psi^{\alpha}(\tau(\xi,s))\int_{\theta(\xi)}^{\xi}\int_a^b \widetilde{\vartheta}(u,\omega)\mathrm{d}\omega\mathrm{d}u.$$

From (11), we obtain

$$\begin{aligned} & (1+p_0^{\alpha})\imath(\theta(\xi)-\delta_0)(\psi''(\theta(\xi)-\delta_0))^{\alpha} \\ \geqslant\ & \tfrac{1}{\mu}\imath(\tau(\xi,\omega))\left(\psi''(\tau((\xi,\omega)))^{\alpha} \times \left(\int_{\xi_2}^{\xi}\left(\int_{\xi_1}^{\rho}\frac{1}{\imath^{1/\alpha}(\omega)}\mathrm{d}\omega\right)\mathrm{d}\rho\right)\int_{\theta(\xi)}^{\xi}\int_a^b \widetilde{\vartheta}(u,\omega)\mathrm{d}\omega\mathrm{d}u. \right. \end{aligned}$$

This gives

$$\left(1+p_0^{\alpha}\right) \geqslant \frac{1}{\mu}\left(\int_{\xi_2}^{\xi}\left(\int_{\xi_1}^{\rho}\frac{1}{\imath^{1/\alpha}(\omega)}\mathrm{d}\omega\right)\mathrm{d}\rho\right)\int_{\theta(\xi)}^{\xi}\int_a^b \widetilde{\vartheta}(u,\omega)\mathrm{d}\omega\mathrm{d}u.$$

Applying the lim sup on both sides of the previous inequality, we gain a contradiction to (14).

Combining Corollary 1 with Theorem 2, we get the Theorem of oscillation for (1) as follows.

Theorem 3. *Assume that there exists a functions $\varrho \in C(I,\mathbb{R})$ and $\theta \in C([\xi_0,\infty),(0,\infty))$ such that $\varrho(\xi) \leqslant \tau(\xi,\omega), \varrho(\xi) < \xi, \lim_{\xi\to\infty}\varrho(\xi) = \infty, \theta(\xi) \leqslant \xi, \tau(\xi,\omega)-\delta_0 = \tau(\xi-\delta_0,\omega), \tau(\xi,\omega) \leqslant (\theta(\xi)-\delta_0)$. If (13) and (14) hold, then Eq. (1) is oscillatory.*

Example 1. Consider the following third-order neutral differential equation

$$\left(\chi(\xi)+p_0\chi(\xi-\delta_0)\right)''' + \left(1+p_0e^{\delta_0}\right)\chi(\xi) = 0. \tag{20}$$

Choose $\varrho(\xi) = \xi - 1$, by Corollary 1, we see that Eq. (20) has property D. Note that the solution $\chi(\xi) = e^{-\xi}$ is satisfying (6).

3. Conclusions

In this paper, we consider the oscillatory behavior of third-order neutral differential equation with distributed deviating arguments which is commonly used in the engineering and natural sciences for modeling various problems. Through this investigation, we were able to improve and extend upon previous results in the literature. In contrast to previous results, we obtained less restrictive conditions as where we do not need the conditions

$0 \leqslant p_0 < 1$

and

either $\imath'(\xi) \geqslant 0$ or $\imath'(\xi) \leqslant 0$,

which is an improvement compared to Baculikova and Dzurina (2010); Baculikova and Dzurina (2012); Candan (2015); Dzurina et al. (2012), and can be applied more widely in this field of study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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