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ORIGINAL ARTICLE

Solitons and other nonlinear waves for the perturbed Boussinesq equation with power law nonlinearity

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Abstract This paper studies the Boussinesq equation in the presence of a couple of perturbation terms. The traveling wave hypothesis is used to extract the soliton solution. Subsequently, other nonlinear wave solutions are also obtained by the aid of exponential function and G'/G methods. The constraint relations are also indicated for the existence of these wave solutions.

1. Introduction

The Boussinesq equation (BE) is one of the nonlinear evolution equations (NLEEs) that model the shallow water waves (Bruzon, 2009; Christov and Choudhury, 2011; Daripa, 2006; Dutykh and Dias, 2007; Hamdi et al., 2005; Hsu et al., 2005; Kordyukova, 2008; Liu and Sun, 2005; Wang et al., 2009; Wazwaz, 2010). In fact BE can be asymptotically reduced to the Korteweg–de Vries (KdV) equation that is more commonly studied in the context of shallow water waves (Kordyukova, 2008). There are various other NLEEs that model these shallow water waves. Some of them are the modified KdV equation, Peregrine equation, Kawahara equation, Benjamin–Bona–Mahoney equation, just to name a few. In this paper, the perturbed BE will be studied with power law nonlinearity.

The integration of the BE will be carried out in this paper. There are several approaches to integrate NLEEs. Some of them are the variational iteration method, Hirota's bilinear method, Adomian decomposition method, Fan's F-expansion method. In this paper, however, first, the simplest method, namely the traveling wave approach will be made to obtain the solitary wave solution to this equation. Subsequently, the exponential function method and the G'/G method will be used to carry out the integration of this equation to extract a few other solutions.

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2. Traveling wave hypothesis

The strongly perturbed BE is

$$q_{tt} - k^2 q_{xx} + a\left(q^{2n}\right)_{xx} + bq_{xxxx} = \beta q_{xx} + \rho q_{xxxx} \quad (1)$$

where β is the coefficient of dissipation and ρ represents the higher order stabilization term. Using the traveling wave assumption that some $g(z)$ satisfies (1) where

$$z = x - vt, \quad (2)$$

(1) transforms to

$$v^2 g'' - k^2 g'' + a(g^{2n})'' + bg'''' = \beta g'' + \rho g'''' \quad (3)$$

Integrating (3) twice and taking both constants of integration to be zero, since the search is for solitary waves, lead to

$$(v^2 - k^2 - \beta)g + ag^{2n} + (b - \rho)g'' = 0 \quad (4)$$

Multiplying (4) by g' and integrating a third time and taking the constant of integration to be zero, once again, yield

$$(b - \rho)(g')^2 + (v^2 - k^2 - \beta)g^2 + \frac{2a}{2n+1} g^{2n+1} = 0 \quad (5)$$

After solving for g' and integrating using separation of variables, (5) becomes

$$\frac{2}{2n-1} \tanh^{-1}\left\{\sqrt{1 - \frac{2ag^{2n-1}}{(2n+1)(k^2 - v^2 + \beta)}}\right\} = z\sqrt{\frac{k^2 - v^2 + \beta}{b - \rho}} \quad (6)$$

Solving (6) for $g(z)$ and substituting in $z = x - vt$ gives the exact traveling wave solution to (1) as

$$q(x,t) = \left[\frac{2n+1}{2a}(k^2 - v^2 + \beta)\,\mathrm{sech}^2 \times \left\{\sqrt{\frac{k^2 - v^2 + \beta}{b - \rho}}\left(n - \frac{1}{2}\right)(x - vt)\right\}\right]^{\frac{1}{2n-1}} \quad (7)$$

which can be rewritten as

$$q(x,t) = A\,\mathrm{sech}^{\frac{2}{2n-1}}[B(x - vt)] \quad (8)$$

where the amplitude A of the solitary wave is given by

$$A = \left[\frac{(2n+1)(k^2 - v^2 + \beta)}{2a}\right]^{\frac{1}{2n-1}} \quad (9)$$

and

$$B = \frac{2n-1}{2}\sqrt{\frac{k^2 - v^2 + \beta}{b - \rho}} \quad (10)$$

Since

$$0 \leqslant v^2 < k^2 + \beta, \quad (11)$$

it is necessary that

$$\beta > -k^2. \quad (12)$$

The fact that

$$k^2 - v^2 + \beta > 0 \quad (13)$$

implies

$$b > \rho. \quad (14)$$

It must also be assumed that

$$n > 1/2 \quad (15)$$

for the existence of the solitary waves.

3. The $\frac{G'}{G}$ method

In this section, we first describe the $\frac{G'}{G}$-expansion method, then apply it to construct the traveling wave solutions for the perturbed Boussinesq equation.

3.1. Details of the method

Suppose that a non-linear partial differential equation is given by

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots) = 0, \quad (16)$$

where $u(x,t)$ is an unknown function, F is a polynomial in $u = u(x,t)$ and its partial derivatives, in which the highest order derivatives and non-linear terms are involved. In the following, we give the main steps of the $\frac{G'}{G}$-expansion method:

Step 1. The traveling wave variable

$$u(x,t) = u(\xi), \quad \xi = x - vt, \quad (17)$$

where v is constant, permits us to reduce Eq. (16) to an ODE for $u = u(\xi)$ in the form:

$$F(u, u', u'', \cdots) = 0. \quad (18)$$

Step 2. Suppose that the solution of (18) can be expressed as a polynomial in $\frac{G'}{G}$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \quad (19)$$

where $G = G(\xi)$ satisfies the second order linear differential equation in the form:

$$G'' + \lambda G' + \mu G = 0, \quad (20)$$

where α_i, v, λ and μ are constants to be determined later, $\alpha_n \neq 0$. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and non-linear terms appearing in (18).

Step 3. Substituting (19) into (18) and using (20), collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, and then equating each coefficient of the resulted polynomial to zero, yield a set of algebraic equations for α_i, v, λ and μ.

Step 4. Since the general solutions of (20) have been well known to us, then substituting α_i, v, λ and μ and the general solutions of (20) into (19) we have more traveling wave solutions of the non-linear partial differential Eq. (16).

3.2. Application to Boussinesq equation

To apply the $\frac{G'}{G}$-expansion method for the perturbed Boussinesq equation, we consider Eq. (5). Suppose that the solutions of the O.D.E (5) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$g(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{21}$$

where α_i is the arbitrary constant, while $G(\xi)$ satisfies the following second order linear O.D.E:

$$G'' + \lambda G' + \mu G = 0, \tag{22}$$

where λ and μ are constants. Eq. (22) can be changed into

$$\frac{d}{d\xi}\left(\frac{G'}{G}\right) = -\left(\frac{G'}{G}\right)^2 - \lambda\left(\frac{G'}{G}\right) - \mu. \tag{23}$$

By using Eq. (23), balancing $\left(\frac{dg(\xi)}{d\xi}\right)^2$ with $(g(\xi))^{2n+1}$ in Eq. (5) give

$$\left(\frac{G'}{G}\right)^{2m+2} = \left(\frac{G'}{G}\right)^{(2n+1)m} \tag{24}$$

so that

$$m = \frac{2}{2n-1}, \tag{25}$$

Thus we make the transformation

$$g = W^{\frac{2}{2n-1}}, \tag{26}$$

and transform Eq. (5) into the following ordinary differential equation

$$2a(2n-1)^2 W^4(\xi) - (2n+1)(2n-1)^2(-v^2 + k^2 + \beta) W^2(\xi) + 4(2n+1)(b-\rho)(W')^2(\xi) = 0 \tag{27}$$

Suppose that the solutions of Eq. (27) can be expressed by a polynomial in $\frac{G'}{G}$ as follows

$$W = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{28}$$

where $G = G(\xi)$ satisfies Eq. (22). Balancing W^4 with $(W')^2$ in Eq. (27) gives

$$\left(\frac{G'}{G}\right)^{4m} = \left(\frac{G'}{G}\right)^{2(m+1)} \tag{29}$$

so

$$m = 1. \tag{30}$$

Thus we can write Eq. (28) as

$$W = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right), \tag{31}$$

where α_0 and α_1 are constants to be determined.

With the help of the symbolic software Maple, substitution of Eq. (31) with Eq. (22) into Eq. (27) shows that the set of algebraic equation (collecting the coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, \cdots, 4)$ and setting it to zero) possesses the solutions:

$$\alpha_1 = \pm\frac{2}{2n-1}\sqrt{\frac{(\rho-b)(1+2n)}{2a}}, \quad \mu = \frac{a\alpha_0^2(1-2n)^2}{2(\rho-b)(1+2n)},$$
$$\lambda = \pm\frac{2a\alpha_0(2n-1)}{(\rho-b)(1+2n)}\sqrt{\frac{(\rho-b)(1+2n)}{2a}}, \quad v = \pm\sqrt{\beta+k^2}, \tag{32}$$

where α_0 is an arbitrary constant.

Eq. (32) can be written by using Eq. (31) as

$$W = \alpha_0 \pm \frac{\sqrt{2}}{2n-1}\sqrt{\frac{(\rho-b)(1+2n)}{2a}}\left(\frac{G'}{G}\right). \tag{33}$$

Since in (32), $\delta = \lambda^2 - 4\mu = 0$ so we obtain only rational function solution

$$W = \alpha_0 + w_0\left(\frac{-a\alpha_0(2n-1)w_0}{2(\rho-b)(1+2n)} + \frac{c_2}{c_1 + c_2\xi}\right). \tag{34}$$

Using Eq. (26), the perturbed Boussinesq equation with any-order nonlinear terms have the solutions

$$g = \left(\alpha_0 + w_0\left(\frac{-a\alpha_0(2n-1)w_0}{2(\rho-b)(1+2n)} + \frac{c_2}{c_1 + c_2\xi}\right)\right)^{\frac{2}{2n-1}}. \tag{35}$$

where $w_0 = \pm\sqrt{\frac{2(\rho-b)(1+2n)}{a}}$, $\xi = x - vt$, $v = \pm\sqrt{\beta + k^2}$ and α_0, c_1, c_2 are free parameters.

4. Exponential function method

In this section, we first give the details of the exponential function method, then apply it to the perturbed Boussinesq equation.

4.1. Details of the method

We now present briefly the main steps of the Exp-function method that will be applied. A traveling wave transformation $u = u(\xi), \xi = x - vt$ converts a partial differential equation

$$\Psi(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{36}$$

into an ordinary differential equation

$$\Phi(u, -vu', u', u'', -vu'', v^2u'', \cdots) = 0. \tag{37}$$

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{38}$$

where c, d, p and q are positive integers which are known to be determined further, a_n and b_m are unknown constants. Eq. (38) can be rewritten in an alternative form

$$u(\xi) = \frac{a_{-c}\exp(-c\xi) + \cdots + a_d \exp(d\xi)}{b_{-p}\exp(-p\xi) + \cdots + b_q \exp(q\xi)}. \tag{39}$$

To determine the values of d and q, we balance the linear term of the highest order in Eq. (37) with the highest order nonlinear term. Similarly, to determine the values of c and p, we balance the linear term of the lowest order in Eq. (37) with the lowest order nonlinear term.

4.2. Application to Boussinesq equation

We consider the perturbed Boussinesq Eq. (1). For getting the traveling wave solutions of (1) using the Exp-function method, we consider Eq. (5) as the converted form of (1). We make the transformation

$$g = W^{\frac{2}{2n-1}}, \tag{40}$$

and transform Eq. (5) into the following ordinary differential equation

$$2a(2n-1)^2W^4(\xi)-(2n+1)(2n-1)^2(-v^2+k^2+\beta)W^2(\xi)+4(2n+1)(b-\rho)(W')^2(\xi)=0 \quad (41)$$

According to the Exp-function method, we assume that the solution of Eq. (41) can be expressed in the following form

$$W(\xi)=\frac{\sum_{n=-c}^{d}a_n\exp(n\xi)}{\sum_{m=-p}^{q}b_m\exp(m\xi)}, \quad (42)$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. Eq. (42) can be re-written in an alternative form as follows

$$W(\xi)=\frac{a_{-c}\exp(-c\xi)+\cdots+a_d\exp(d\xi)}{b_{-p}\exp(-p\xi)+\cdots+b_q\exp(q\xi)}. \quad (43)$$

In order to determine values of c and p, we balance the linear term of the highest order in Eq. (41) with the highest order nonlinear term. By simple calculation, we have

$$W^4=\frac{c_1\exp[4d\xi]+\cdots}{c_2\exp[4q\xi]+\cdots}, \quad (44)$$

and

$$(W')^2=\frac{c_3\exp[(2q+2d)\xi]+\cdots}{c_4\exp[4q\xi]+\cdots}, \quad (45)$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of Exp-function in Eqs. (44) and (45) we have

$$4d=2q+2d, \quad (46)$$

which leads to the result

$$d=q. \quad (47)$$

Similarly to determine values of c and p, we balance the linear term of the lowest order in Eq. (41)

$$W^4=\frac{d_1\exp[-4c\xi]+\cdots}{d_2\exp[-4p\xi]+\cdots}, \quad (48)$$

and

$$(W')^2=\frac{d_3\exp[-(2p+2c)\xi]+\cdots}{d_4\exp[-4p\xi]+\cdots}, \quad (49)$$

where d_i are determined coefficients only for simplicity. From (48) and (49), we obtain

$$-4c=-(2p+2c), \quad (50)$$

which leads to the result

$$p=c. \quad (51)$$

We can freely choose the values of c and d, but the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p=c=1$ and $d=q=1$, then Eq. (43) becomes

$$W(\xi)=\frac{a_{-1}\exp(-\xi)+a_0+a_1\exp(\xi)}{b_{-1}\exp(-\xi)+b_0+b_1\exp(\xi)}. \quad (52)$$

Substituting Eq. (52) into Eq. (41), and equating to zero the coefficients of all powers of $exp(n\xi)$ yield a set of algebraic equations for a_0, b_0, a_1, a_{-1}, b_{-1}, b_1 and v. Solving the system of algebraic equations by the help of Maple, we obtain

Case 1

$$a_1=0,\ a_{-1}=0,\ b_0=0,\ b_1=\frac{aa_0^2(2n-1)^2}{8b_{-1}(1+2n)(b-\rho)}, \quad (53)$$

$$v=\pm\frac{\sqrt{(2n-1)^2(k^2+\beta)+4(\rho-b)}}{2n-1}. \quad (54)$$

Substituting Eqs. (53) and (54) into Eq. (52) yields

$$W(\xi)=\frac{a_0}{b_{-1}e^{-\xi}+\frac{aa_0^2(2n-1)^2}{8b_{-1}(1+2n)(b-\rho)}e^{\xi}}, \quad (55)$$

where $\xi=x-vt$. Thus, from transformation (40), the exact traveling wave solution to (1) is

$$q(x,t)=\left(\frac{a_0}{b_{-1}e^{vt-x}+\frac{aa_0^2(2n-1)^2}{8b_{-1}(1+2n)(b-\rho)}e^{x-vt}}\right)^{\frac{2}{2n-1}}. \quad (56)$$

where a_0 and b_{-1} are arbitrary constants.

Case 2

$$a_0=0,\ a_{-1}=0,\ b_1=0,\ b_{-1}=0, \quad (57)$$

$$v=\pm\frac{\sqrt{(2n-1)^2(k^2+\beta)+4(\rho-b)}}{2n-1}. \quad (58)$$

Substituting Eqs. (57) and (58) into Eq. (52) yields

$$W(\xi)=\frac{a_1}{b_0}e^{\xi}, \quad (59)$$

where $\xi=x-vt$. Thus, from transformation (40), the exact traveling wave solution to (1) is

$$q(x,t)=\left(\frac{a_1}{b_0}e^{x-vt}\right)^{\frac{2}{2n-1}}. \quad (60)$$

where a_1 and b_0 are arbitrary constants.

Case 3

$$a_0=0,\ a_1=0,\ b_1=0,\ b_{-1}=0, \quad (61)$$

$$v=\pm\frac{\sqrt{(2n-1)^2(k^2+\beta)+4(\rho-b)}}{2n-1}. \quad (62)$$

Substituting Eqs. (61) and (62) into Eq. (52) yields

$$W(\xi)=\frac{a_{-1}}{b_0}e^{-\xi}, \quad (63)$$

where $\xi=x-vt$. Thus, from transformation (40), the exact traveling wave solution to (1) is

$$q(x,t)=\left(\frac{a_{-1}}{b_0e^{x-vt}}\right)^{\frac{2}{2n-1}}. \quad (64)$$

where a_{-1} and b_0 are arbitrary constants.

Case 4

$$a_0 = 0,\ b_1 = 0,\ b_{-1} = 0,\ a_1 = \frac{-4b_0^2(2n+1)(b-\rho)}{aa_{-1}(1-2n)^2} \tag{65}$$

$$v = \pm\frac{\sqrt{(2n-1)^2(k^2+\beta)+28(b-\rho)}}{2n-1}. \tag{66}$$

Substituting Eqs. (65) and (66) into Eq. (52) yields

$$W(\xi) = \frac{a_{-1}e^{-\xi} + \frac{-4b_0^2(2n+1)(b-\rho)}{aa_{-1}(1-2n)^2}e^{\xi}}{b_0}, \tag{67}$$

where $\xi = x - vt$. Thus, from transformation (40), the exact traveling wave solution to (1) is

$$q(x,t) = \left(\frac{a_{-1}e^{vt-x} + \frac{-4b_0^2(2n+1)(b-\rho)}{aa_{-1}(1-2n)^2}e^{x-vt}}{b_0}\right)^{\frac{2}{2n-1}}. \tag{68}$$

where a_{-1} and b_0 are arbitrary constants.

Case 5

$$b_1 = 0,\ b_{-1} = 0,\ a_0 = \frac{2z_1b_0}{2n-1},\ v = \frac{2z_2}{2n-1}, \tag{69}$$

$$a_1 = \frac{b_0^2(8az_1^2+(2n+1)(b-\rho))}{aa_{-1}(1-2n)^2}. \tag{70}$$

Substituting Eqs. (69) and (70) into Eq. (52) yields

$$W(\xi) = \frac{a_{-1}e^{-\xi} + \frac{2z_1b_0}{2n-1} + \frac{b_0^2(8az_1^2+(2n+1)(b-\rho))}{aa_{-1}(1-2n)^2}e^{\xi}}{b_0}, \tag{71}$$

where $\xi = x - vt$. Thus, from transformation (40), the exact traveling wave solution to (1) is

$$q(x,t) = \left(\frac{a_{-1}e^{vt-x} + \frac{2z_1b_0}{2n-1} + \frac{b_0^2(8az_1^2+(2n+1)(b-\rho))}{aa_{-1}(1-2n)^2}e^{x-vt}}{b_0}\right)^{\frac{2}{2n-1}}. \tag{72}$$

where z_1 and z_2 are respectively roots of the equations

$$168a^2z_1^4 + 60a(1+2n)(b-\rho)z_1^2 + 5(1+2n)^2(b-\rho)^2 = 0 \tag{73}$$

$$(1+2n)((1-2n)^2(k^2+\beta)+12(b-\rho)) + 112az_1^2 + (8n+4)z_2^2 = 0. \tag{74}$$

where a_{-1}, b_0 are arbitrary constants.

5. Conclusions

In this paper, the perturbed BE is studied by the traveling wave hypothesis. The solitary wave solution is obtained. There are the parameter restrictions that fell out while conducting the analysis of the traveling wave solutions. The perturbations are taken to be strong perturbations. Subsequently, the G'/G method and the exponential method are employed to integrate this perturbed BE with power law nonlinearity. Several other solutions are obtained.

In future several more solutions will be retrieved including the cnoidal and snoidal waves as well as the quasi-stationary solutions in the presence of such perturbation terms when they are weak. Those results will be reported elsewhere.

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