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A new study on two different vaccinated fractional-order COVID-19 models via numerical algorithms

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ABSTRACT

The main purpose of this paper is to provide new vaccinated models of COVID-19 in the sense of Caputo-Fabrizio and new generalized Caputo-type fractional derivatives. The formulation of the given models is presented including an exhaustive study of the model dynamics such as positivity, boundedness of the solutions, and local stability analysis. Furthermore, the unique solution existence for the proposed fractional-order models is discussed via fixed point theory. Numerical solutions are also derived by using two-steps Adams-Bashforth algorithm for Caputo-Fabrizio operator, and modified Predictor-Corrector method for generalised Caputo fractional derivative. Our analysis allows to show that the given fractional-order models exemplify the dynamics of COVID-19 much better than the classical ones. Also, the analysis on the convergence and stability for the proposed methods are performed. By this study, we see that how vaccine availability plays an important role in the control of COVID-19 infection.

1. Introduction

Throughout this pandemic known as COVID-19, we have experimented a great expansion of cases throughout the world. This situation converts into solid actions that affect the population: social isolation, use of masks, etc. Mathematical models play a key role in describing infectious diseases such as COVID-19 expansion. The development and investigation of this type of models provide us tools for describing and characterizing its transmission, and thus, we are able to propose successful techniques to foresee, prevent, and control infections, also to ensure that the population is well-being. Till present time, numerous mathematical models see (Bekiros and Kouloumpou, 2020; Bocharov et al., 2018; Brauer and Driessche, 2008; Brauer, 2017; Zaman et al., 2017) have been considered and analyzed to ponder the spreading of infections. COVID-19, has affected nearly 90% of countries across the globe with the infection rate rising rapidly at almost 5% per day. However, the COVID-19 infection behavior is different from nation-to-nation, and is dependent on numerous factors. In South Africa, with no exception, almost half a million positive cases have been reported already and is currently one of the five most affected countries globally. To date, various mathematical models have been applied to predict infection rates based on only time-series modes (Higazy, 2020; Zeb et al., 2020). Very few studies attempted to include other related factors to enhancing the modeling process such as the influence of climatic factors for the disease rapid spread. In the last year, numerical models for the COVID-19 plague have been taken into consideration by many scientists concerning the different nature and its behavior by applying different controls to avoid the spread of this pandemic see (Zhang et al., 2020; Zhang et al., 2020; Atangana and Iğret, 2021; Mahrouf et al., 2021) and references therein. Nowadays, a number of mathematicians are giving priority to fractional derivatives (Oldham and Spanier, 1974; Podlubny, 1998; Rudolf, 2000; Kilbas and Srivastava, 2020)

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in the study of mathematical models. Recently, thousands of epidemic models like tuberculosis (Abboubakar et al., 2021), malaria (Abboubakar et al., 2020), COVID-19 (Gao et al., 2020; Kumar et al., 2021; Kumar and Erturk, 2021) have been analyzed by applying non-classical derivative operators. Authors in Erturk and Kumar (2020) solved a nonlinear system of COVID-19 by using a recent modification in the Caputo derivative. They used fixed point theory techniques to demonstrate solution existence and they also analyzed the stability of the aforementioned model. The dynamics of COVID-19 in Brazil were studied in Kumar et al. (2021), and in Cameron in Nabi et al. (2020). A new model of COVID-19 disease in integer and non-integer sense was provided in Ref. Nabi et al. (2021). Analysis on the fractional-order mathematical model to simulate the COVID-19 disease outbreaks in Pakistan are proposed in Naik et al. (2020). Authors in Yavuz et al. (2021) have proposed a new non-linear model for deriving the nature of 2019-nCoV. In Naik et al. (2020), chaotic dynamics of a mathematical model of HIV-1 in the sense of fractional-order operators is given. In Ref. Hammouch et al. (2021), the authors have simulated a fractional-order chaotic system. The study proposed in Yavuz and Sene (2020) is dedicated to the solution of a fractional-order predator-prey model. In Naik et al. (2020), researchers have justified the clear role of prostitutes in the HIV disease. Authors in Yavuz and Özdemir (2020) have analyzed an epidemic model with exponential decay law. In Bonyah et al. (2021), some novel analysis on the listeriosis epidemic are performed. In Odibat et al. (2021), a modified version of the Predictor-Corrector technique for the delay-type fractional differential equations has been proposed. Authors in Kumar et al. (2021) have analyzed the predictions of COVID-19 cases in Argentina by using a real-data. In Kalaiselvi et al. (2021), researchers have introduced a mathematical model to simulate a biological phenomena. Recently, some authors have also tried fractional derivatives in ecological problems. One of the most recent application is given in Kumar and Erturk (2021).

Our objective in this paper is to continue this research line by introducing a new fractional COVID-19 model that takes into account the existence of vaccines. Our paper is organized as follows: Section 2, is related to providing some well-known results that will be later needed. Section 3 is devoted to the description of fractional order models using Caputo-Fabrizio and generalized Caputo non-classical derivatives. Section 4 contains the basic analysis of the model, involving the positivity, boundness, and reproductive number with stability along with disease free-equilibrium points. Next, in Section 5 and Section 6 the existence of solutions for the models via Adams-Bashforth in CF sense and modified Predictor-Corrector in generalized Caputo derivative sense are provided, respectively. These sections also contain the numerical simulations and graphical results for both models. Finally, in Section 7, we present the concluding remarks.

2. Preliminaries

Here we mention some definitions and results for further uses.

Definition 1. Caputo and Mauro (2015) The CF (Caputo-Fabrizio) fractional-derivative of $\varkappa$ order for a function $\mathscr{G} \in H^1(c,d)$ and $0<\varkappa<1$, is given by:

$$ {}^{CF}{}_cD_t^{\varkappa}\mathscr{G}(t)=\frac{1}{1-\kappa}\int_c^t\frac{d\mathscr{G}(\lambda)}{d\lambda}exp[-\varpi(t-\lambda)]d\lambda $$

where $\varpi=\frac{\varkappa}{1-\kappa}$.

The respective CF fractional integral is defined by

$$ {}^{CF}{}_cI_t^{\varkappa}\mathscr{G}(t)=(1-\varkappa)\mathscr{G}(t)+\varkappa\int_c^t\mathscr{G}(\lambda)d\lambda. $$

Theorem 1. Verma and Kumar (2020) *Let $\mathscr{M}$ be a compact metric space and $C(\mathscr{M},\mathbb{R})$ denotes the space of continuous functions when endowed with the supremum norm metric. A set $\mathscr{E}\subset C(\mathscr{M})$ is compact if and only if $\mathscr{E}$ is bounded, closed, and equicontinous.*

Definition 2. Naik et al. (2020) The modified Caputo fractional derivative operator, $D_{d_+}^{\varkappa,\sigma}$, of order $\varkappa>0$ is given by:

$$ (D_{d_+}^{\varkappa,\sigma}\Psi)(\xi)=\frac{\sigma^{\varkappa-n+1}}{\Gamma(n-\varkappa)}\int_d^{\xi}s^{\sigma-1}(\xi^{\sigma}-s^{\sigma})^{n-\varkappa-1}\left(s^{1-\sigma}\frac{d}{ds}\right)^n\Psi(s)ds,\ \xi>d, \tag{1} $$

where $\sigma>0, d\geqslant 0, andn-1<\varkappa\leqslant n$.

Theorem 2. Naik et al. (2020) *Let $n-1<\varkappa\leqslant n, \sigma>0, a\geqslant 0$ and $g\in C^n[a,b]$. Then, for $a<t\leqslant b$,*

$$ I_{a+}^{\varkappa,\sigma}D_{a+}^{\varkappa,\sigma}g(t)=g(t)-\sum_{m=0}^{n-1}\frac{1}{\sigma^m m!}(t^{\sigma}-a^{\sigma})^m\left[\left(x^{1-\sigma}\frac{d}{dx}\right)^m g(x)\right]_{x=a}. \tag{2} $$

3. Formulation of fractional-order Covid-19 models

In order to formulate our COVID-19 model with the influence of quarantine class and vaccination, we split the whole population into four different compartments. The first of them is the class of susceptible to disease which is represented as $\mathscr{S}_t$, second one is infective or infectious $\mathscr{I}_t$, third one is quarantined $\mathscr{Q}_t$ (in which the infectious peoples are putting for isolation), and last one is the recovered class $\mathscr{R}_t$ with temporary immunity. The flow of the population is described in the following system of differential equations:

$$ \begin{aligned} \frac{d\mathscr{S}_t}{dt}&=(1-q)b-\beta\mathscr{S}_t\mathscr{I}_t-d\mathscr{S}_t+\delta\mathscr{R}_t,\\ \frac{d\mathscr{I}_t}{dt}&=\beta\mathscr{S}_t\mathscr{I}_t-(\eta+\gamma+d+\sigma_1)\mathscr{I}_t,\\ \frac{d\mathscr{Q}_t}{dt}&=\eta\mathscr{I}_t-(\rho+d+\sigma_2)\mathscr{Q}_t,\\ \frac{d\mathscr{R}_t}{dt}&=\gamma\mathscr{I}_t+\rho\mathscr{Q}_t-(d+\delta)\mathscr{R}_t+qb, \end{aligned} \tag{3} $$

where b describes the enroll rate of the population that directly joins the susceptible class $\mathscr{S}_t, \beta$ stands for the contact rate mainly incidence rate at which susceptible class joins infectious class $\mathscr{I}_t, d$ denotes the out going rate of each class in the form of natural death or migration rate from each class, γ is the recovered rate of infected class to join recovered class $\mathscr{R}_t$ and ρ is the recovered rate of quarantine class people. Moreover, σ_1 and σ_2 are the disease related deaths rates for infected class and quarantined class, δ shows the relapse rate at which the recovered class $\mathscr{R}_t$ moves to susceptible class and q represents the vaccine rate, that is, the proportion of the susceptible class that becomes vaccinated with $0\leqslant q\leq 1$. To simulate the past history or hereditary characteristics in the given model (3), we utilized the Caputo-Fabrizio (CF) fractional derivatives instead of classical derivatives. In this matter, we propose the following model of fractional order type

$$ \begin{aligned} {}^{CF}{}_cD_t^{\varkappa}\mathscr{S}_t&=(1-q)b-\beta\mathscr{S}_t\mathscr{I}_t-d\mathscr{S}_t+\delta\mathscr{R}_t,\\ {}^{CF}{}_cD_t^{\varkappa}\mathscr{I}_t&=\beta\mathscr{S}_t\mathscr{I}_t-(\eta+\gamma+d+\sigma_1)\mathscr{I}_t,\\ {}^{CF}{}_cD_t^{\varkappa}\mathscr{Q}_t&=\eta\mathscr{I}_t-(\rho+d+\sigma_2)\mathscr{Q}_t,\\ {}^{CF}{}_cD_t^{\varkappa}\mathscr{R}_t&=\gamma\mathscr{I}_t+\rho\mathscr{Q}_t-(d+\delta)\mathscr{R}_t+qb, \end{aligned} \tag{4} $$

where $0<\varkappa<1$ and ${}^{CF}{}_cD_t^{\varkappa}$ presents the fractional derivative in the Caputo-Fabrizio sense.

For generating more diversity in the fractional-order simulations, we propose another fractional order model in the sense of generalized version of Caputo-type fractional derivative as follows:

$$\begin{aligned}
&{}^{C}D_t^{\varkappa,\sigma}\mathscr{S}_t = (1-q)b - \beta\mathscr{S}_t\mathscr{I}_t - d\mathscr{S}_t + \delta\mathscr{R}_t,\\
&{}^{C}D_t^{\varkappa,\sigma}\mathscr{I}_t = \beta\mathscr{S}_t\mathscr{I}_t - (\eta+\gamma+d+\sigma_1)\mathscr{I}_t,\\
&{}^{C}D_t^{\varkappa,\sigma}\mathscr{Q}_t = \eta\mathscr{I}_t - (\rho+d+\sigma_2)\mathscr{Q}_t,\\
&{}^{C}D_t^{\varkappa,\sigma}\mathscr{R}_t = \gamma\mathscr{I}_t + \rho\mathscr{Q}_t - (d+\delta)\mathscr{R}_t + qb,
\end{aligned} \tag{5}$$

where $0<\varkappa<1, \sigma>0$, and ${}^{C}D_t^{\varkappa,\sigma}$ presents the fractional derivative in the generalized (or modified) Caputo sense.

4. Basic analysis of the model

4.1. Positivity and boundedness

Suppose that

$$\mathbb{R}_+^4 = \{(\mathscr{S},\mathscr{I},\mathscr{Q},\mathscr{R})|\mathscr{S},\mathscr{I},\mathscr{Q},\mathscr{R} \geqslant 0\}.$$

From Odibat and Shawagfeh (2007) and utilizing a generalized mean value theorem and a fractional comparison principle, the proof of the following theorem is achieved. We state the analysis for the Caputo-Fabrizio fractional model (4) and it is straightforward to obtain the corresponding analysis for the generalised Caputo one (5).

Theorem 3 (*Positivity and boundedness*). *Let $(\mathscr{S}_0,\mathscr{I}_0,\mathscr{Q}_0,\mathscr{R}_0)$ be any initial data belonging to $\mathbb{R}_+^4$ and $(\mathscr{S}_t,\mathscr{I}_t,\mathscr{Q}_t,\mathscr{R}_t,)$ the corresponding solution of model (4) to the given initial data. The set $\mathbb{R}_+^4$ is positively invariant. Furthermore, we have*

$$\begin{aligned}
&\limsup_{t\to\infty}\mathscr{S}_t \leqslant \mathscr{S}_\infty := \tfrac{(1-q)b+\delta\mathscr{R}_\infty}{d},\\
&\limsup_{t\to\infty}\mathscr{I}_t \leqslant \mathscr{I}_\infty := \tfrac{(1-q)b+\delta\mathscr{R}_\infty}{\eta+\gamma+d+\sigma_1},\\
&\limsup_{t\to\infty}\mathscr{Q}_t \leqslant \mathscr{Q}_\infty := \tfrac{\eta\mathscr{I}_\infty}{\rho+d+\sigma_2},\\
&\limsup_{t\to\infty}\mathscr{R}_t \leqslant \mathscr{R}_\infty := \tfrac{\gamma\mathscr{I}_\infty+\rho\mathscr{Q}_\infty+qb}{d+\delta}.
\end{aligned} \tag{6}$$

Proof. From model (4), we have

$$\begin{aligned}
&{}^{CF}{}_{c}D_t^{\varkappa}\mathscr{S}|_{\mathscr{S}=0} &&= (1-q)b + \delta\mathscr{R}_t > 0,\\
&{}^{CF}{}_{c}D_t^{\varkappa}\mathscr{I}|_{\mathscr{I}=0} &&= 0,\\
&{}^{CF}{}_{c}D_t^{\varkappa}\mathscr{Q}|_{\mathscr{Q}=0} &&= \eta\mathscr{I}_t \geqslant 0,\\
&{}^{CF}{}_{c}D_t^{\varkappa}\mathscr{R}|_{\mathscr{R}=0} &&= \gamma\mathscr{I}_t + \rho\mathscr{Q}_t + qb \geqslant 0.
\end{aligned} \tag{7}$$

For all $t \geqslant 0$, with the help of generalized mean value theorem (Odibat and Shawagfeh, 2007) and system (7), we can conclude that $\mathscr{S}_t,\mathscr{I}_t,\mathscr{Q}_t,\mathscr{R}_t \geqslant 0$. First equation of system (4) implies that

$${}^{CF}{}_{c}D_t^{\varkappa}\mathscr{S} \leqslant (1-q)b - d\mathscr{S}_t + \delta\mathscr{R}_t.$$

By utilizing the fractional comparison principle, it follows that

$$\limsup_{t\to\infty}\mathscr{S}_t \leqslant \mathscr{S}_\infty := \frac{(1-q)b+\delta\mathscr{R}_\infty}{d}.$$

The second equation of the system (4) implies that

$${}^{CF}{}_{c}D_t^{\varkappa}(\mathscr{S}+\mathscr{I}) \leqslant (1-q)b - d\mathscr{S}_t + \delta\mathscr{R}_t - (\eta+\gamma+d+\sigma_1)\mathscr{I}_t,$$

which implies that

$$\limsup_{t\to\infty}\mathscr{I}_t \leqslant \mathscr{I}_\infty.$$

As a result, the second estimate of (6) is obtained. While third equation of the system (4) gives us

$${}^{CF}{}_{c}D_t^{\varkappa}\mathscr{Q} \leqslant \eta\mathscr{I}_\infty - (\rho+d+\sigma_2)\mathscr{Q}_t,$$

for enough large value of t. This follows the third estimate of (6).
Finally, the fourth equation of system (4), implies that

$${}^{CF}{}_{c}D_t^{\varkappa}\mathscr{R} \leqslant \gamma\mathscr{I}_\infty + \rho\mathscr{Q}_\infty - (d+\delta)\mathscr{R}_t + qb,$$

for enough large value of t and the fourth estimate of (6) holds. □

4.2. Free virus equilibrium point and reproduction number

Diseases Free Equilibrium (DFE) point of system (3) is given by

$$\mathscr{E}_0 = (\mathscr{S}_0,\mathscr{I}_0,\mathscr{Q}_0,\mathscr{R}_0) = \left(\frac{b((1-q)d+\delta)}{d(\delta+d)}, 0, 0, \frac{qb}{\delta+d}\right). \tag{8}$$

For the reproductive number of model (3), suppose that $y=(\mathscr{I}_t,\mathscr{S}_t)$ and using next generation matrix approach (Brauer and Driessche, 2008), we have

$$\frac{dy}{dt} = \mathscr{F}(y) - \mathscr{V}(y), \tag{9}$$

where Jacobian of $\mathscr{F}$ and $\mathscr{V}$ are

$$\mathscr{F}(y) = \begin{pmatrix}\beta\mathscr{S}_t\mathscr{I}_t\\ 0\end{pmatrix} \quad \mathscr{V}(y) = \begin{pmatrix}(\eta+\gamma+d+\sigma_1)\mathscr{I}_t\\ -(1-q)b+\beta\mathscr{S}_t\mathscr{I}_t+d\mathscr{S}_t-\delta\mathscr{R}_t\end{pmatrix}. \tag{10}$$

At $\mathscr{E}_0$, we have

$$\mathbb{F}(\mathscr{E}_0) = \begin{pmatrix}\frac{\beta b((1-q)d+\delta)}{d(\delta+d)} & 0\\ 0 & 0\end{pmatrix}, \quad \mathbb{V}(\mathscr{E}_0) = \begin{pmatrix}\eta+\gamma+d+\sigma_1 & o\\ \frac{\beta b((1-q)d+\delta)}{d(\delta+d)} & d\end{pmatrix}.$$

Hence, the reproductive number for model (3) is

$$\psi_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\beta b((1-q)d+\delta)}{d(\delta+d)(\eta+\gamma+d+\sigma_1)}. \tag{11}$$

$$\psi_0 = \rho(\mathbb{F}\mathbb{V}^{-1}) = \frac{\beta b((1-q)d+\delta)}{d(\delta+d)(\eta+\gamma+d+\sigma_1)}. \tag{12}$$

The results about the positive endemic equilibrium point are contained in the next theorem.

Theorem 4. *There exists a unique positive endemic equilibrium point $\mathscr{E}^*$ for system (3) if $\psi_0 > 1$.*

Proof. Endemic equilibrium point (EEP) is obtained from the system (3), by putting right hand side of each equation equal to zero, we have

$$\begin{aligned}
&\mathscr{S}_t^* = \tfrac{\eta+\gamma+d+\sigma_1}{\beta},\\
&\mathscr{Q}_t^* = \tfrac{\eta}{\rho+d+\sigma_2}\mathscr{I}_t,\\
&\mathscr{R}_t^* = \tfrac{qb}{d+\delta} + \tfrac{\gamma(\rho+d+\sigma_2)+\rho\eta}{(d+\delta)(\rho+d+\sigma_2)}\mathscr{I}_t,\\
&(1-q)b - (\eta+\gamma+d+\sigma_1)\mathscr{I}_t^* - d\mathscr{S}_t^* + \delta\mathscr{R}_t^* = 0,
\end{aligned} \tag{13}$$

Now, from the last equation of system (13), we have

$$\begin{aligned}
\Phi(\mathscr{I}_t^*) = {}& \frac{b((1-q)d+\rho)}{d+\delta}\\
&+ \left(\frac{\gamma(\rho+d+\sigma_2)+\rho\eta}{(d+\delta)(\rho+d+\sigma_2)} - (\eta+\gamma+d+\sigma_1)\right)\mathscr{I}_t^*\\
&- \frac{d(\eta+\gamma+d+\sigma_1)}{\beta}.
\end{aligned} \tag{14}$$

By the values of $\mathscr{S}^*, \mathscr{I}^*, \mathscr{Q}^*$ and $\mathscr{R}^*$, it is clear that a unique EEP $\mathscr{E}^*$ exists, if $\psi_0 > 1$. □

Theorem 5. *The model (3) is locally stable at $\mathscr{E}_0$ for $\psi_0 < 1$ and unstable for $\psi_0 > 1$.*

Proof. The Jacobian of the model (3) is

$$J=\begin{pmatrix} -\beta\mathscr{I}_t-d & -\beta\mathscr{S}_t & 0 & \delta \\ \beta\mathscr{I}_t & \beta\mathscr{S}_t-(\eta+\gamma+d+\sigma_1) & 0 & 0 \\ 0 & \eta & -(\rho+d+\sigma_2) & 0 \\ 0 & \gamma & \rho & -(d+\delta) \end{pmatrix}. \tag{15}$$

Along $\mathscr{E}_0$, it implies that

$$J(\mathscr{E}_0)=\begin{pmatrix} -d & -\frac{\beta b((1-q)d+\delta)}{d(\delta+d)} & 0 & \delta \\ 0 & \frac{\beta b((1-q)d+\delta)}{d(\delta+d)}-(\eta+\gamma+d+\sigma_1) & 0 & 0 \\ 0 & \eta & -(\rho+d+\sigma_2) & 0 \\ 0 & \gamma & \rho & -(d+\delta) \end{pmatrix}, \tag{16}$$

which follows that all the eignvalues are negative if $\psi_0<1$ and eigenvalue λ_2 is positive for $\psi_0>1$. Hence, we conclude that the system (3) is locally stable under the condition $\psi_0<1$ and unstable for $\psi_0>1$. □

Theorem 6. *The model (3) is globally stable, if $\psi_0>1$ at $\mathscr{E}_0$.*

Proof. First, we define the Lyapunov function $\mathscr{V}(t)$, for the system as:

$$\mathscr{V}(t)=1+\mathscr{I}_t-\ln\frac{\mathscr{I}_t}{\mathscr{I}_0}. \tag{17}$$

Then differentiating the Eq. (17) with respect to time, we have

$$\begin{aligned}\frac{d}{dt}(\mathscr{V}(t)) &= \left(1-\frac{1}{\mathscr{I}_t}\right)\frac{d\mathscr{I}_t}{d(t)}\\ &= \frac{d\mathscr{I}_t}{dt}-\beta S_t+(\eta+\gamma+d+\sigma_1).\end{aligned}$$

By manipulating along the point $\mathscr{E}_0$, we get

$$\begin{aligned}\frac{d}{dt}(\mathscr{V}(t)) &= -(\beta\mathscr{S}_t-(\eta+\gamma+d+\sigma_1))\\ &= -\left(\beta\frac{b((1-q)d+\delta)}{d(\delta+d)}-(\eta+\gamma+d+\sigma_1)\right)\\ &= -(\eta+\gamma+d+\sigma_1)\left(\frac{\beta b((1-q)d+\delta)}{d(\delta+d)(\eta+\gamma+d+\sigma_1)}-1\right)\\ &\leqslant 0 \quad for\ \psi_0>1.\end{aligned}$$

Therefore, if $\psi_0>1$, then $\frac{d}{dt}(\mathscr{V}(t)<0$, which implies that the system (3) is globally stable for $\psi_0>1$ at $\mathscr{E}_0$. □

Remark 1. The simulations of stability of $\mathscr{E}^*$ is an important mathematical term, but in this paper, we particularly focus on the case $\psi_0<1$ to find effective manners to prevent the epidemic.

5. Solution of the variable Caputo-Fabrizio fractional order model (4)

5.1. Existence and uniqueness analysis

Since the last few years, a lot of work has been done in the field of the existence of solution for different types of fractional differential equations by using techniques from fixed point theory. In order to fulfill this requirement for the proposed model, we use the procedure which has been recently proposed by *Verma et al.* in Verma and Kumar (2020). For this purpose, we rewrite our model in a compact form given by:

$$\begin{cases} {}^{CF}D_t^{\varkappa}\mathscr{S}_t=\mathscr{G}_1(t,\mathscr{S}_t),\\ {}^{CF}D_t^{\varkappa}\mathscr{I}_t=\mathscr{G}_2(t,\mathscr{I}_t),\\ {}^{CF}D_t^{\varkappa}\mathscr{Q}_t=\mathscr{G}_3(t,\mathscr{Q}_t)\\ {}^{CF}D_t^{\varkappa}\mathscr{R}_t=\mathscr{G}_4(t,\mathscr{R}_t).\end{cases} \tag{18}$$

Now the above system (18) converts to the following fractional Volterra integral form when we apply CF integral operator on it of order $0<\varkappa<1$,

$$\begin{aligned}\mathscr{S}_t(t)-\mathscr{S}_t(0)&=(1-\varkappa)\mathscr{G}_1(t,\mathscr{S}_t)+\varkappa\int_0^t\mathscr{G}_1(\chi,\mathscr{S}_t)d\chi,\\ \mathscr{I}_t(t)-\mathscr{I}_t(0)&=(1-\varkappa)\mathscr{G}_2(t,\mathscr{I}_t)+\varkappa\int_0^t\mathscr{G}_2(\chi,\mathscr{I}_t)d\chi,\\ \mathscr{Q}_t(t)-\mathscr{Q}_t(0)&=(1-\varkappa)\mathscr{G}_3(t,\mathscr{Q}_t)+\varkappa\int_0^t\mathscr{G}_3(\chi,\mathscr{Q}_t)d\chi,\\ \mathscr{R}_t(t)-\mathscr{R}_t(0)&=(1-\varkappa)\mathscr{G}_4(t,\mathscr{R}_t)+\varkappa\int_0^t\mathscr{G}_4(\chi,\mathscr{R}_t)d\chi.\end{aligned} \tag{19}$$

Now we derive the analysis for $\mathscr{S}_t(t)$ and it is straightforward to mention that the given analysis will exist in a similar way for the other model equations of (18).

Consider the Banach space $\mathscr{B}=C([0,T])$ with the associated norm $\|\mathscr{S}_t\|=\max_{t\in[0,T]}\{|\mathscr{S}_t|,\forall\mathscr{S}_t\in\mathscr{B}\}$ and $\varkappa^*=\min_{t\in[0,T]}\{\varkappa\}$ and $\varkappa^{**}=\max_{t\in[0,T]}\{\varkappa\}$ be the minimum and maximum weight of the variable non-integer order $\varkappa$ on $[0,T]$. Now, we recall the following hypothesis to explore our main observations:

$[\mathscr{X}_1]$: There exist constants $\mathscr{G}_c,H_c>0$, and $k\in[0,1)$ such that $|\mathscr{G}_1(t,\mathscr{S}_t)|\leqslant\mathscr{G}_c|\mathscr{S}_t|^k+H_c$.

$[\mathscr{X}_2]$: There exists a constant $N_c>0$, such that $|\mathscr{G}_1(t,\mathscr{S}_{t_1})-\mathscr{G}_1(t,\mathscr{S}_{t_2})|\leqslant N_c|\mathscr{S}_{t_1}(t)-\mathscr{S}_{t_2}(t)|$.

Now, we define the operator $\mathscr{O}:\mathscr{B}\to\mathscr{B}$ as

$$\mathscr{O}(\mathscr{S}_t(t))=\mathscr{S}_t(0)+(1-\varkappa)\mathscr{G}_1(t,\mathscr{S}_t)+\varkappa\int_0^t\mathscr{G}_1(\chi,\mathscr{S}_t)d\chi. \tag{20}$$

It is clear that operator $\mathscr{O}(\mathscr{S}_t(t))=\mathscr{O}_1(\mathscr{S}_t(t))+\mathscr{O}_2(\mathscr{S}_t(t))$, where

$$\mathscr{O}_1(\mathscr{S}_t(t))=\mathscr{S}_t(0)+(1-\varkappa)G_1(t,\mathscr{S}_t). \tag{21}$$

$$\mathscr{O}_2(\mathscr{S}_t(t))=\varkappa\int_0^t\mathscr{G}_1(\chi,\mathscr{S}_t)d\chi. \tag{22}$$

Theorem 7. *Assume that hypothesis $[\mathscr{X}_2]$ holds and there exists $\mathscr{C}>0$ (constant) such that $\mathscr{C}=[(1-\varkappa^*)N_c+\varkappa^*N_cT]<1$. Then $\mathscr{O}$ has a unique fixed point for the model (18) on $\mathscr{B}$.*

Proof. Let consider $\mathscr{S}_{t_1},\mathscr{S}_{t_2}\in\mathscr{B}$. Then

$$\begin{aligned}&\|\mathscr{O}\mathscr{S}_{t_1}-\mathscr{O}\mathscr{S}_{t_2}\|\leqslant\|\mathscr{O}_1\mathscr{S}_{t_1}-\mathscr{O}_1\mathscr{S}_{t_2}\|+\|\mathscr{O}_2\mathscr{S}_{t_1}-\mathscr{O}_2\mathscr{S}_{t_2}\|\\ &\leqslant(1-\varkappa)\max_{t\in[0,T]}|\mathscr{G}_1(t,\mathscr{S}_{t_1})-G_1(t,\mathscr{S}_{t_2})|\\ &+\varkappa\max_{t\in[0,T]}\left|\int_0^t\mathscr{G}_1(\xi,\mathscr{S}_{t_1})d\xi-\int_0^t\mathscr{G}_1(\xi,\mathscr{S}_{t_1})d\xi\right|\\ &\leqslant[(1-\varkappa)N_c+\varkappa N_cT]\max_{t\in[0,T]}|\mathscr{S}_{t_1}-\mathscr{S}_{t_2}|\\ &\leqslant[(1-\varkappa^*)N_c+\varkappa^*N_cT]\|\mathscr{S}_{t_1}-\mathscr{S}_{t_2}\|\\ &\leqslant\mathscr{C}\|\mathscr{S}_{t_1}-\mathscr{S}_{t_2}\|.\end{aligned} \tag{23}$$

Since $\mathscr{C}=[(1-\varkappa^*)N_c+\varkappa^*N_cT]<1$, using Banach fixed point theorem, we conclude that the operator $\mathscr{O}$ has a unique fixed point. Then, the model (18) has a unique solution. □

Theorem 8. *Assume that statements $[\mathscr{X}_1]-[\mathscr{X}_2]$ hold and $0<(1-\varkappa^*)N_c<1$. Then the system (18) has at least one solution.*

Proof. First, we show the operator $\mathscr{O}_1$ is a contraction. Indeed, it is given $\mathscr{S}_t\in\mathscr{T}$ where $\mathscr{T}=\{\mathscr{S}_t\in\mathscr{B}:\|\mathscr{S}_t\|\leqslant w,w>0\}$ is a closed convex set it follows that

$$\begin{aligned}\|\mathscr{O}_1\mathscr{S}_{t_1}-\mathscr{O}_1\mathscr{S}_{t_2}\| &=(1-\varkappa)\max_{t\in[0,T]}|\mathscr{G}_1(t,\mathscr{S}_{t_1})-\mathscr{G}_1(t,\mathscr{S}_{t_2})|\\ &\leqslant[(1-\varkappa^*)N_c\|\mathscr{S}_{t_1}-\mathscr{S}_{t_2}\|.\end{aligned} \tag{24}$$

Hence $\mathcal{O}_1$ is a contraction. Now to demonstrate that the second operator O_2 is compact we can see that O_2 is continuous and compact for any $\mathscr{S}_t \in \mathscr{T}$, then $\mathcal{O}_2$ is contraction as $\mathscr{G}_1$ is continuous, then

$$\begin{aligned} \|\mathcal{O}_2\mathscr{S}_t(t)\| &= \max_{t\in[0,T]} |\varkappa \textstyle\int_0^t \mathscr{G}_1(\xi,\mathscr{S}_t)d\xi| \leqslant |\varkappa| \int_0^t |\mathscr{G}_1(\xi,\mathscr{S}_t)|d\xi \\ &\leqslant \varkappa^* T[\mathscr{G}_c|\mathscr{S}_t|^k + H_c]. \end{aligned} \tag{25}$$

So, $\mathcal{O}_2$ is bounded. Now, assume $t_1 > t_2 \in [0.T]$, such that

$$\begin{aligned} \|\mathcal{O}_2\mathscr{S}_t(t_1) - \mathcal{O}_2\mathscr{S}_t(t_2)\| &= \varkappa^* \max_{t\in[0,T]} |\textstyle\int_0^{t_1} \mathscr{G}_1(\xi,\mathscr{S}_t)d\xi - \int_0^{t_2} \mathscr{G}_1(\xi,\mathscr{S}_t)d\xi| \\ &\leqslant \varkappa^*[\mathscr{G}_c|\mathscr{S}_t|^k + H_c]|t_1 - t_2|. \end{aligned} \tag{26}$$

This yields $\|\mathcal{O}_2(\mathscr{S}_t(t_1)) - \mathcal{O}_2(\mathscr{S}_t(t_2))\| \to 0$ as $t_1 \to t_2$. Hence, the operator $\mathcal{O}_2$ is equicontinuous. As a consequence of Theorem 1, $\mathcal{O}_2$ is compact. Now by referring to the analysis given in Section 5 of Verma and Kumar (2020), we conclude that the given system has at least one solution. □

5.2. Numerical solution of CF system

Now we write the solution of the proposed system in CF sense applying two-step Adams-Bashforth algorithm. Our time interval is $[a, T]$ with the step width $h = \frac{T-a}{N}$, where N is the sample size.

Let $\mathscr{S}_{t_j}$ be the numerical approximation of $\mathscr{S}_t(t)$ at $t = t_j$, where $t_j = 0 + jh$ and, $j = 0, 1, \ldots, N$. Writing the equations of $\mathscr{S}_t(t)$ at the uniform grid points $(t_0, t_1, t_2, \ldots, t_{j-1}, t_j, t_{j+1})$, we get the estimations at distinct grid point values. For doing it, first we consider the equivalent Volterra CF integral equation for $\mathscr{S}_t(t)$ which is,

$$\mathscr{S}_t(t) = \mathscr{S}_t(0) + (1-\varkappa)\mathscr{G}_1(t,\mathscr{S}_t(t)) + \varkappa \int_0^t \mathscr{G}_1(s,\mathscr{S}_t(s))ds. \tag{27}$$

So the estimations at t_j are

$$\begin{aligned} \mathscr{S}_t(t_j) &= \mathscr{S}_{t_0} + (1-\varkappa)\mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1})) \\ &\quad + \varkappa \int_0^{t_j} \mathscr{G}_1(t,\mathscr{S}_t(t))dt, \end{aligned} \tag{28}$$

and at t_{j+1}

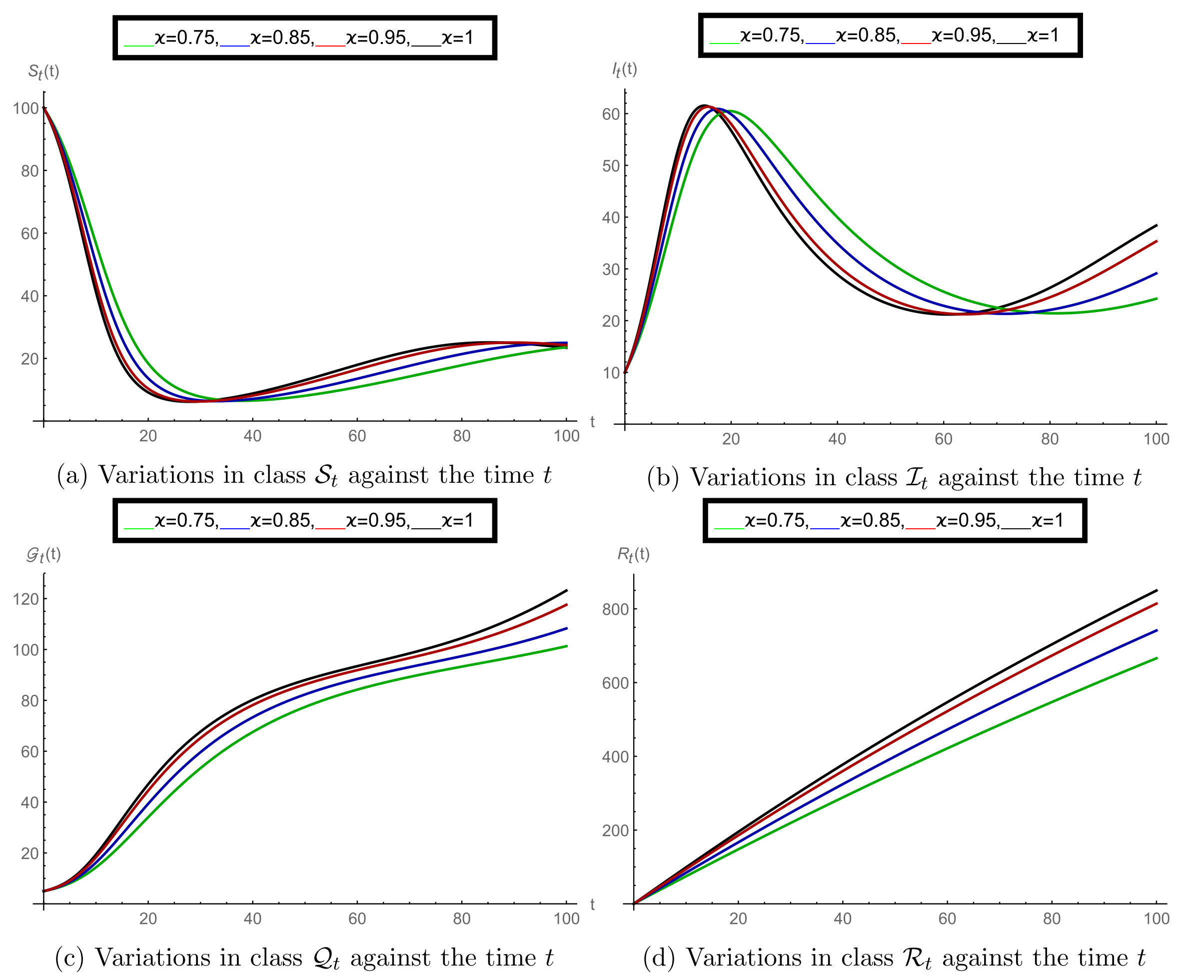
$$\begin{aligned} \mathscr{S}_t(t_{j+1}) &= \mathscr{S}_{t_0} + (1-\varkappa)\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) \\ &\quad + \varkappa \int_0^{t_{j+1}} \mathscr{G}_1(t,\mathscr{S}_t(t))dt. \end{aligned} \tag{29}$$

Subtracting Eq. (29) from (28), we get

$$\begin{aligned} \mathscr{S}_t(t_{j+1}) - \mathscr{S}_t(t_j) &= (1-\varkappa)\big(\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1}))\big) \\ &\quad + \varkappa \int_{t_j}^{t_{j+1}} \mathscr{G}_1(t,\mathscr{S}_t(t))dt. \end{aligned} \tag{30}$$

Now, by applying linear interpolation to $\mathscr{G}_1(t,\mathscr{S}_t(t))$ and employing trapezoid rule on the integral part, we obtain

$$\int_{t_j}^{t_{j+1}} \mathscr{G}_1(t,\mathscr{S}_t(t))dt \approxeq \frac{3\Delta t}{2}\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \frac{\Delta t}{2}\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)), \tag{31}$$



(a) Variations in class $\mathcal{S}_t$ against the time t
(b) Variations in class $\mathcal{I}_t$ against the time t
(c) Variations in class $\mathcal{Q}_t$ against the time t
(d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 1. Structure of the model classes in CF sense at various values of order $\varkappa$, when vaccination fraction $q = 1$.

where $\Delta t = t_j - t_{j-1}$. Hence, we have established the numerical approximation for $\mathscr{S}_t(t)$ as

$$\mathscr{S}_t(t_{j+1}) = \mathscr{S}_t(t_j) + \left(1 - \varkappa + \frac{3\varkappa\Delta t}{2}\right)\mathscr{G}_1(t_j, \mathscr{S}_t(t_j)) - \left(1 - \varkappa + \frac{\varkappa\Delta t}{2}\right)\mathscr{G}_1(t_{j-1}, \mathscr{S}_t(t_{j-1})). \tag{32}$$

As a consequence, the solution of the proposed CF model (18) states as follows:

$$\begin{aligned}
\mathscr{S}_t(t_{j+1}) &= \mathscr{S}_t(t_j) + \left(1-\varkappa+\tfrac{3\varkappa\Delta t}{2}\right)\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \left(1-\varkappa+\tfrac{\varkappa\Delta t}{2}\right)\mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1})),\\
\mathscr{I}_t(t_{j+1}) &= \mathscr{I}_t(t_j) + \left(1-\varkappa+\tfrac{3\varkappa\Delta t}{2}\right)\mathscr{G}_2(t_j,\mathscr{I}_t(t_j)) - \left(1-\varkappa+\tfrac{\varkappa\Delta t}{2}\right)\mathscr{G}_2(t_{j-1},\mathscr{I}_t(t_{j-1})),\\
\mathscr{Q}_t(t_{j+1}) &= \mathscr{Q}_t(t_j) + \left(1-\varkappa+\tfrac{3\varkappa\Delta t}{2}\right)\mathscr{G}_3(t_j,\mathscr{Q}_t(t_j)) - \left(1-\varkappa+\tfrac{\varkappa\Delta t}{2}\right)\mathscr{G}_3(t_{j-1},\mathscr{Q}_t(t_{j-1})),\\
\mathscr{R}_t(t_{j+1}) &= \mathscr{R}_t(t_j) + \left(1-\varkappa+\tfrac{3\varkappa\Delta t}{2}\right)\mathscr{G}_4(t_j,\mathscr{R}_t(t_j)) - \left(1-\varkappa+\tfrac{\varkappa\Delta t}{2}\right)\mathscr{G}_4(t_{j-1},\mathscr{R}_t(t_{j-1})).
\end{aligned} \tag{33}$$

Theorem 9. *The proposed numerical scheme (32) is unconditionally stable if (particularly for the first model equation)*

$\|\mathscr{G}_1(t_{j+1}, \mathscr{S}_t(t_{j+1})) - \mathscr{G}_1(t_j, \mathscr{S}_t(t_j))\| \to 0.$

Proof. Given $\mathscr{S}_t(t)$ the solution of (27), we have that:

$$\begin{aligned}
\|\mathscr{S}_t(t_{j+1}) - \mathscr{S}_t(t_j)\| &= \Big\|(1-\varkappa)\big(\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1}))\big)\\
&\quad + \varkappa\int_{t_j}^{t_{j+1}}\mathscr{G}_1(\eta,\mathscr{S}_t(\eta))d\eta\Big\| \leqslant (1-\varkappa)\big\|\big(\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1}))\big)\big\| + \varkappa\\
&\quad \left\|\int_{t_j}^{t_{j+1}}\mathscr{G}_1(\eta,\mathscr{S}_t(\eta))d\eta\right\|.
\end{aligned} \tag{34}$$

Making $j \to \infty$, we get

$$\begin{aligned}
\|\mathscr{S}_t(t_{j+1}) - \mathscr{S}_t(t_j)\|_\infty &\leqslant \lim_{j\to\infty}(1-\varkappa)\big\|\big(\mathscr{G}_1(t_j,\mathscr{S}_t(t_j)) - \mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1}))\big)\big\|\\
&\quad + \lim_{j\to\infty}\varpi\left\|\int_{t_j}^{t_{j+1}}\mathscr{G}_1(\eta,\mathscr{S}_t(\eta))d\eta\right\|\\
&< \lim_{j\to\infty}(1-\varkappa)\big\|\big(\mathscr{G}_1(t_j,P(t_j)) - \mathscr{G}_1(t_{j-1},\mathscr{S}_t(t_{j-1}))\big)\big\|\\
&\quad + \lim_{j\to\infty}\varkappa\int_{t_j}^{t_{j+1}}\left|\sum_{j=0}^{j}\prod_{j=0}^{j}\frac{\eta-t_j}{\Delta t}\mathscr{G}_1(t_j,\mathscr{S}_t(t_j))\right|d\eta
\end{aligned} \tag{35}$$

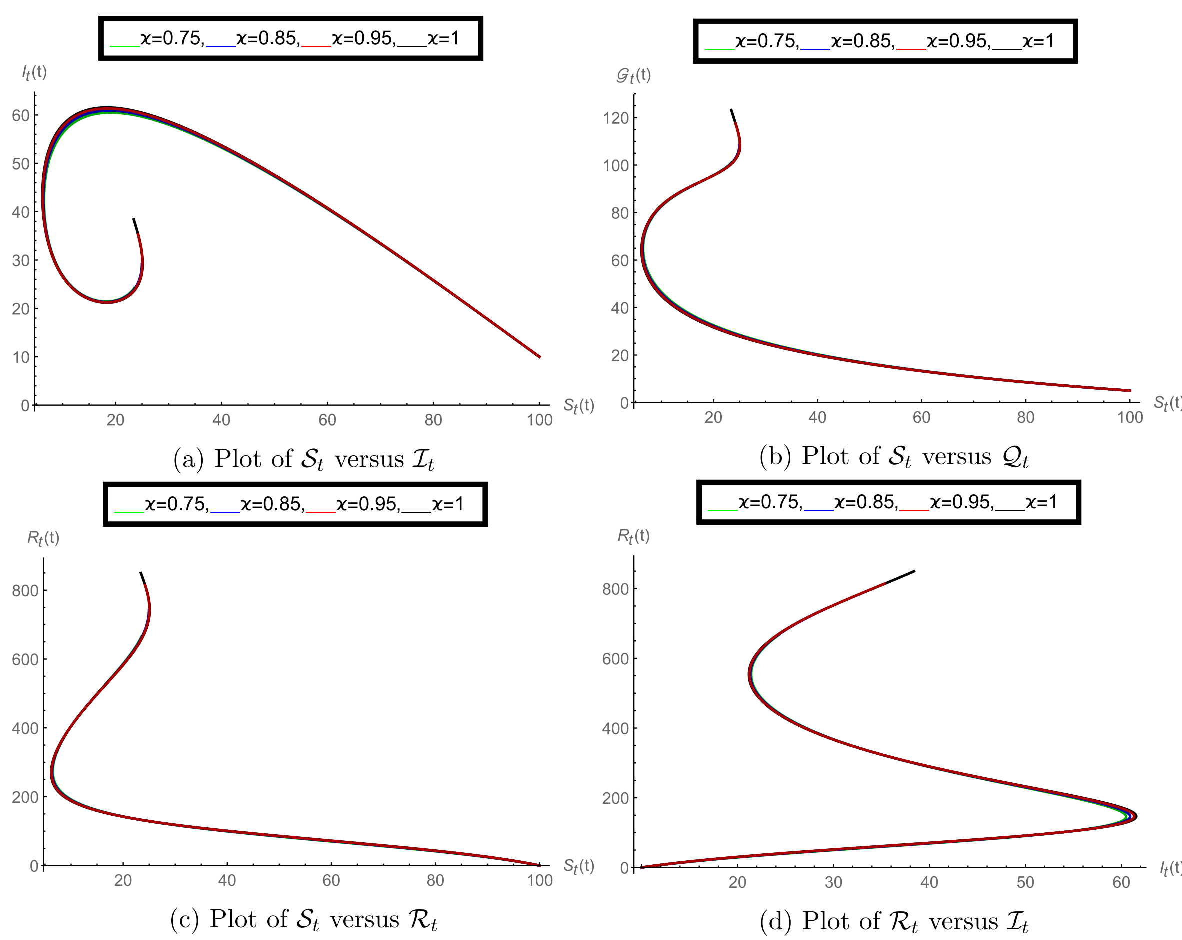
Clearly, the second part of the above inequality goes to zero when $j \to \infty$. Now, if $\|\mathscr{G}_1(t_{j+1}, \mathscr{S}_t(t_{j+1})) - \mathscr{G}_1(t_j, \mathscr{S}_t(t_j))\| \to 0$ as $j \to \infty$, we conclude that the given scheme is stable. □

Theorem 10 (*Convergence*). *Let the solution of* ${}^{CF}D_t^{\varkappa}S_t(t)$ *be* $S_t(t)$. *Then there exist* Γ, *such that* $\left\|\mathbb{O}_{\varkappa}^{k}\right\| \leqslant \Gamma$.

Proof. Starting from Eq. (30) and performing linear interpolation, we have

$$\begin{aligned}
S_t(t_{j+1}) - S_t(t_j) = {} & (1-\varkappa)\big(G_1(t_j,S_t(t_j)) - G_1(t_{j-1},S_t(t_{j-1}))\big) + \varkappa\int_{t_j}^{t_{j+1}}G_1(\eta,S_t(\eta))d\eta\\
= {} & (1-\varkappa)\big(G_1(t_j,S_t(t_j)) - G_1(t_{j-1},S_t(t_{j-1}))\big)\\
& + \varkappa\int_{t_j}^{t_{j+1}}\left\{G_1(t_j,S_t(t_j))\left(\frac{\eta-t_{j-1}}{\Delta t}\right) - G_1(t_{j-1},S_t(t_{j-1}))\left(\frac{\eta-t_j}{\Delta t}\right)\right\}d\eta\\
& + \varkappa\int_{t_j}^{t_{j+1}}\left\{\sum_{a=0}^{j-2}\prod_{a=0}^{j-2}G_1(t_a,S_t(t_a))\left(\frac{\eta-t_a}{(-1)^a\Delta_a}\right)\right\}d\eta
\end{aligned} \tag{36}$$

Simplifying further, we arrive at the numerical solution with the truncation term



(a) Plot of $\mathcal{S}_t$ versus $\mathcal{I}_t$ (b) Plot of $\mathcal{S}_t$ versus $\mathcal{Q}_t$ (c) Plot of $\mathcal{S}_t$ versus $\mathcal{R}_t$ (d) Plot of $\mathcal{R}_t$ versus $\mathcal{I}_t$

Fig. 2. Variations in the model classes compare to each other, when vaccination fraction $q = 1$.

$$
\begin{aligned}
S_t(t_{j+1}) &= S_t(t_j) + \left(1 - \varkappa + \tfrac{3\varkappa\Delta t}{2}\right)G_1(t_j, S_t(t_j)) \\
&\quad + \left(1 - \varkappa + \tfrac{\varkappa\Delta t}{2}\right)G_1(t_{j-1}, S_t(t_{j-1})) + \mathbb{O}^j_{\varkappa}
\end{aligned} \tag{37}
$$

where the truncation term is written as

$$
\mathbb{O}^j_{\varkappa} = \varkappa \int_{t_j}^{t_{j+1}} \left\{ \sum_{a=0}^{j-2} \prod_{a=0}^{j-2} G_1(t_a, S_t(t_a)) \left(\frac{\eta - t_a}{(-1)^a \Delta t_a} \right) \right\} d\eta \tag{38}
$$

Then taking the norm, we have

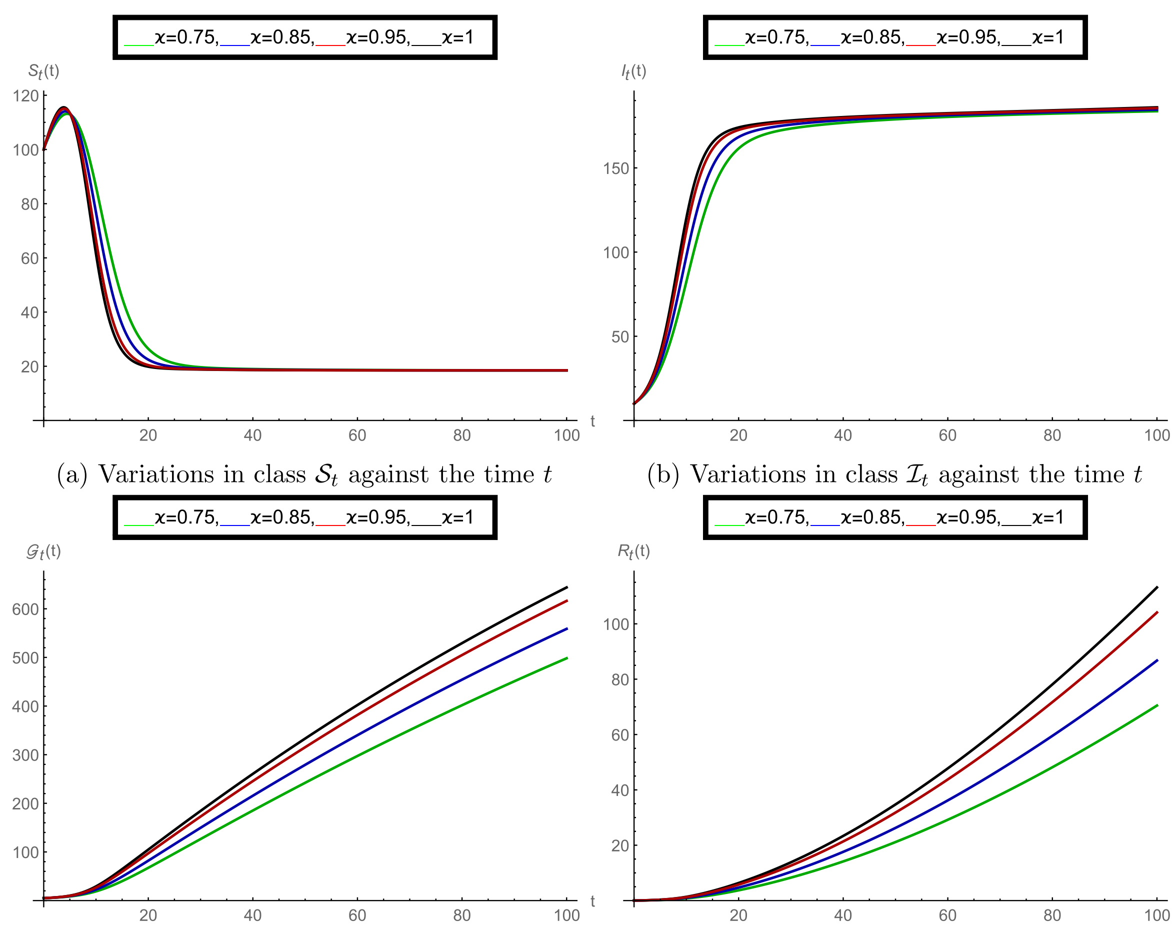
$$
\begin{aligned}
\|\mathbb{O}^j_{\varkappa}\| &= \left\| \varkappa \int_{t_j}^{t_{j+1}} \left\{ \sum_{a=0}^{j-2} \prod_{a=0}^{j-2} G_1(t_a, S_t(t_a)) \left(\tfrac{\eta - t_a}{(-1)^a \Delta t_a} \right) \right\} d\eta \right\| \\
&\leqslant \varkappa \int_{t_j}^{t_{j+1}} \left\| \left\{ \sum_{a=0}^{j-2} \prod_{a=0}^{j-2} G_1(t_a, S_t(t_a)) \left(\tfrac{\eta - t_a}{(-1)^a \Delta t_a} \right) \right\} d\eta \right\| \\
&< \varkappa \int_{t_j}^{t_{j+1}} \sum_{a=0}^{j-2} \prod_{a=0}^{j-2} \left| \tfrac{\eta - t_a}{\Delta t_a} \right| \sup_a \left(\max_a |G_1(t_a, S_t(t_a))| \right) d\eta \\
&< \varkappa (j-1)! \Delta t^{j-1} \Gamma
\end{aligned} \tag{39}
$$

Hence, the solution has a convergence result. □

5.3. Graphical simulations

In this section, we derive the all necessary plots by using the above given scheme. We use the initial populations $\mathscr{S}_t(0) = 100, \mathscr{I}_t(0) = 10, \mathscr{Q}_t(0) = 5, \mathscr{R}_t(0) = 0$, and parameter values $d = 0.001, \beta = 0.003, \delta = 0.003, \gamma = 0.002, b = 10, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002, q = 1$ (this value is just an assumption) which are taken from the literature of COVID-19 cases in China (Gao et al., 2020; Erturk and Kumar, 2020). In the collection of Fig. 1, the subFigs. 1a, 1b, 1c, 1d are devoted to showing the variations in $\mathscr{S}_t, \mathscr{I}_t, \mathscr{Q}_t$, and $\mathscr{R}_t$ against the time variable t. Here, the variations in the dynamics of the model can be clearly explored at different derivative order values. We can observe that when the fractional order values changes then the differences between phases of the plot lines increases. Fig. 2 reflects the relations between the given classes. SubFig. 2a plots the variations of $\mathscr{S}_t$ verses $\mathscr{I}_t$, subFig. 2b plots the corresponding ones for $\mathscr{S}_t$ versus $\mathscr{Q}_t$ and 2c plots the variations of $\mathscr{S}_t$ versus $\mathscr{R}_t$. Finally, subFig. 2d plots $\mathscr{R}_t$ against $\mathscr{I}_t$. The fractional order values which have been considered are $\varkappa = 0.75, 0.85, 0.95, 1$.

Now we intend to explore the role of the vaccines in the given model classes. For this purpose, we change the value of the vaccination fraction q to simulate the model structure. Here, in the family of Fig. 3, the subFigs. 3a, 3b, 3c, 3d demonstrate the variations in $\mathscr{S}_t, \mathscr{I}_t, \mathscr{Q}_t$, and $\mathscr{R}_t$ against the time variable t at the vaccination fraction $q = 0$, where all other values are same as used above. Similarly, Fig. 4 shows the corresponding ones when the vaccination fraction $q = 0.5$. By the comparison of these figures, we can easily observe that when the value of vaccination fraction q increases then the population of infectious humans decreases. So, vaccine availability is one of the most important control measures to reduce the infection of COVID-19.



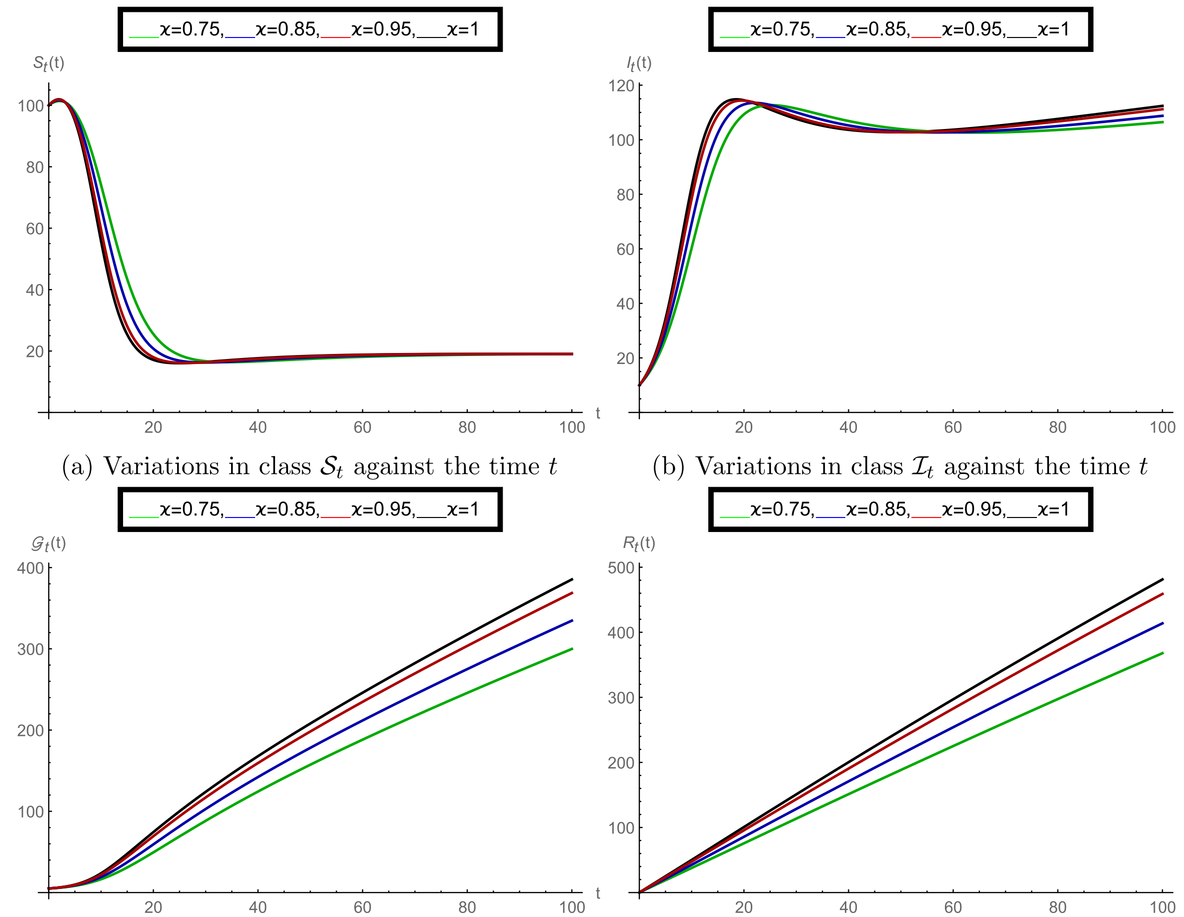
(a) Variations in class $\mathcal{S}_t$ against the time t

(b) Variations in class $\mathcal{I}_t$ against the time t

(c) Variations in class $\mathcal{Q}_t$ against the time t

(d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 3. Structure of the model classes in CF sense at various values of order $\varkappa$, when vaccination fraction $q = 0$.



(a) Variations in class $\mathcal{S}_t$ against the time t

(b) Variations in class $\mathcal{I}_t$ against the time t

(c) Variations in class $\mathcal{Q}_t$ against the time t

(d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 4. Structure of the model classes in CF sense at various values of order $\varkappa$, when vaccination fraction $q = 0.5$.

6. Solution of the generalised Caputo fractional model (5)

6.1. Existence and uniqueness analysis

In this concern, to prove the existence of a unique solution of the proposed modified Caputo type fractional order model, we again write the given model into compact form as

$$\begin{cases} {}^{C}D_t^{\varkappa,\sigma}\mathscr{S}_t = \mathscr{G}_1(t,\mathscr{S}_t), \\ {}^{C}D_t^{\varkappa,\sigma}\mathscr{I}_t = \mathscr{G}_2(t,\mathscr{I}_t), \\ {}^{C}D_t^{\varkappa,\sigma}\mathscr{Q}_t = \mathscr{G}_3(t,\mathscr{Q}_t) \\ {}^{C}D_t^{\varkappa,\sigma}\mathscr{R}_t = \mathscr{G}_4(t,\mathscr{R}_t). \end{cases} \tag{40}$$

Now we just adopt the first equation of the above system to derive the necessary results.

$$ {}^{C}D_t^{\varkappa,\sigma}\mathscr{S}_t(t) = \mathscr{G}_1(t,\mathscr{S}_t), \tag{40a}$$
$$ \mathscr{S}_t(0) = \mathscr{S}_{t_0}. \tag{40b}$$

The equivalent Volterra integral equation of the proposed IVP is

$$\mathscr{S}_t(t) = \mathscr{S}_t(0) + \frac{\sigma^{1-\varkappa}}{\Gamma(\varkappa)}\int_0^t \xi^{\sigma-1}(t^\sigma - \xi^\sigma)^{\varkappa-1}\mathscr{G}_1(\xi,\mathscr{S}_t)d\xi. \tag{41}$$

Theorem 11. Erturk and Kumar (2020) (Existence). *Let $0<\varkappa\leqslant 1, N_0\in\mathbb{R}, K>0$ and $T^*>0$. Let $\mathscr{G} := \{(t,\mathscr{S}_t) : t\in[0,T^*], |\mathscr{S}_t - \mathscr{S}_{t_0}| \leqslant K\}$ and take the function $\mathscr{G}_1 : \mathscr{G} \to \mathbb{R}$ be continuous. Further, describe $M := \sup_{(t,\mathscr{S}_t)\in\mathscr{G}}|\mathscr{G}_1(t,\mathscr{S}_t)|$ and*

$$T = \begin{cases} T^*, & \text{if} M = 0, \\ min\left\{T^*, \left(\frac{K\Gamma(\varkappa+1)\sigma^\varkappa}{M}\right)^{\frac{1}{\varkappa}}\right\} & otherwise. \end{cases} \tag{42}$$

Then, there exists a function $\mathscr{S}_t \in C[0,T]$ that satisfies the IVP (40a) and (40b).

Lemma 1. Erturk and Kumar (2020) *If the assumptions of the statement of Theorem 1 hold, the function $\mathscr{S}_t \in C[0,T]$ satisfies the IVP (40a) and (40b) if and only if it satisfies the non-linear Volterra integral Eq. (41).*

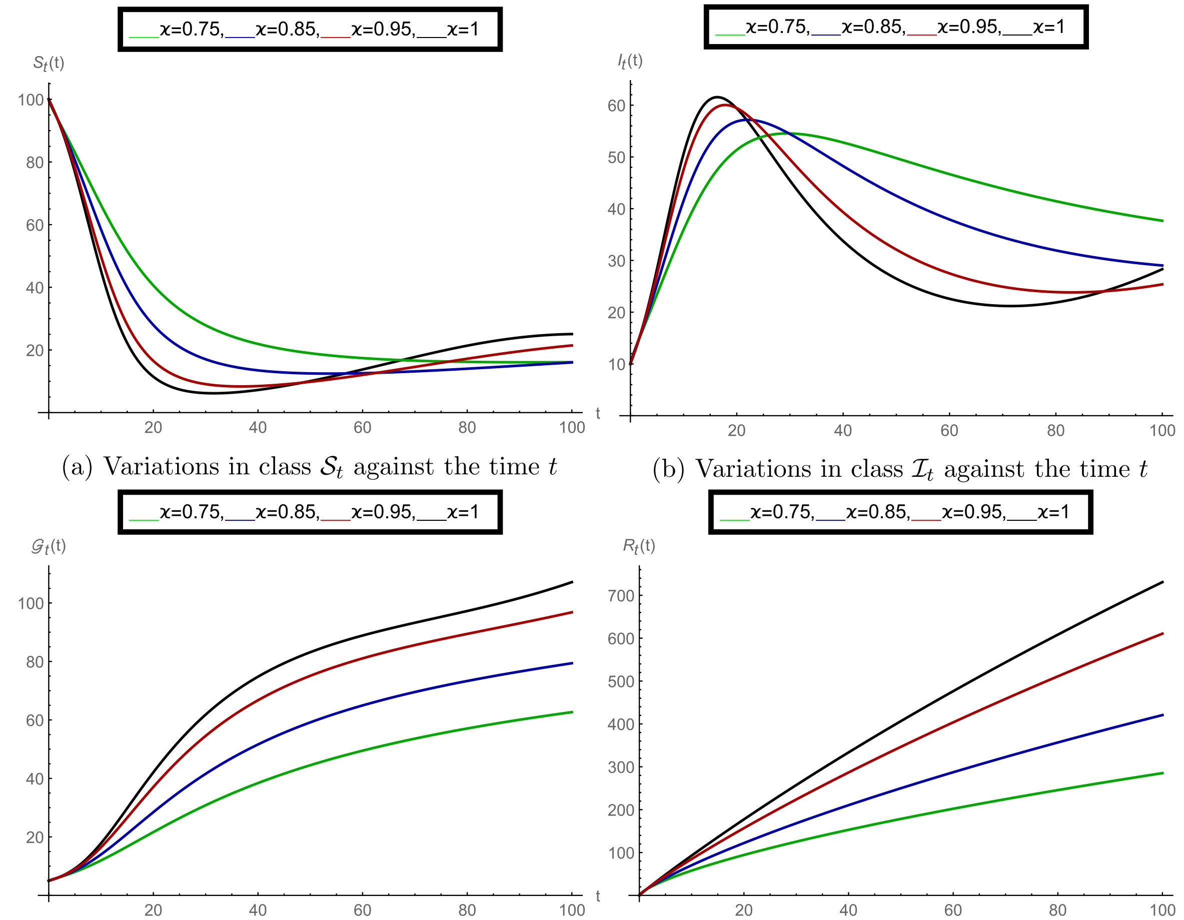
Theorem 12. Erturk and Kumar (2020) (Uniqueness). *Consider $\mathscr{S}_t(0)\in\mathbb{R}, K>0 and T^*>0$. Also, let $0<\varkappa\leqslant 1 and m = \lceil\varkappa\rceil$. For the set $\mathscr{G}$ as given in Theorem 9 and assume $\mathscr{G}_1 : \mathscr{G}\to\mathbb{R}$ be continuous. Assume that $\mathscr{G}_1$ agrees to the Lipschitz condition with respect to the second variable, i.e.*

$$|\mathscr{G}_1(t,\mathscr{S}_{t_1}) - \mathscr{G}_1(t,\mathscr{S}_{t_2})| \leqslant V|\mathscr{S}_{t_1} - \mathscr{S}_{t_2}|,$$

for some constant $V>0$ which does not dependent to $t, \mathscr{S}_{t_1}, and \mathscr{S}_{t_2}$. Then, a unique solution $\mathscr{S}_t \in C[0,T]$ exists for the IVP (40a) and (40b).

6.2. Derivation of the solution via modified Predictor-Corrector algorithm

Now we construct the numerical solution of the proposed Caputo fractional model using a modified form of the PC algorithm as mentioned in Naik et al. (2020) with some appropriate



(a) Variations in class $\mathcal{S}_t$ against the time t (b) Variations in class $\mathcal{I}_t$ against the time t

(c) Variations in class $\mathcal{Q}_t$ against the time t (d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 5. Structure of the model classes in modified Caputo sense at various values of order $\varkappa$, when vaccination fraction $q = 1$.

changes. Here we start with Volterra integral Eq. (41), which provides

$$\mathscr{S}_t(t) = \mathscr{S}_t(0) + \frac{\sigma^{1-\varkappa}}{\Gamma(\varkappa)} \int_0^t \xi^{\sigma-1}(t^\sigma - \xi^\sigma)^{\varkappa-1} \mathscr{G}_1(\xi, \mathscr{S}_t) d\xi, \tag{43}$$

Here, first we recall that a unique solution of the proposed model exists under suitable conditions on the function $\mathscr{G}_1$ on some interval $[0, T]$. We divide the interval $[0, T]$ into N non-uniform subintervals $\{[t_k, t_{k+1}], k = 0, 1, \ldots, N-1\}$ taking the mesh points

$$\begin{cases} t_0 = 0, \\ t_{k+1} = (t_k^\sigma + h)^{1/\sigma}, \ k = 0, 1, \ldots, \mathbb{N} - 1, \end{cases} \tag{44}$$

here $h = \frac{T^\sigma}{\mathbb{N}}$. We now analyse the approximations $\mathscr{S}_{t_k}, k = 0, 1, \ldots, \mathbb{N}$, to solve numerically the proposed IVP. First of all, assuming the approximation $\mathscr{S}_{t_j} \approx \mathscr{S}_t(t_j) (j = 1, 2, \ldots, k)$, we estimate $\mathscr{S}_{t_{k+1}} \approx \mathscr{S}_t(t_{k+1})$ by means of the integral equation

$$\mathscr{S}_t(t_{k+1}) = \mathscr{S}_t(0) + \frac{\sigma^{1-\varkappa}}{\Gamma(\varkappa)} \int_0^{t_{k+1}} \xi^{\sigma-1}(t_{k+1}^\sigma - \xi^\sigma)^{\varkappa-1} \mathscr{G}_1(\xi, \mathscr{S}_t) d\xi. \tag{45}$$

Substituting $z = \xi^\sigma$, we get

$$\mathscr{S}_t(t_{k+1}) = \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa}}{\Gamma(\varkappa)} \int_0^{t_{k+1}^\sigma} (t_{k+1}^\sigma - z)^{\varkappa-1} \mathscr{G}_1(z^{1/\sigma}, \mathscr{S}_t(z^{1/\sigma})) dz, \tag{46}$$

equivalently

$$\mathscr{S}_t(t_{k+1}) = \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa}}{\Gamma(\varkappa)} \sum_{j=0}^{k} \int_{t_j^\sigma}^{t_{k+1}^\sigma} (t_{k+1}^\sigma - z)^{\varkappa-1} \mathscr{G}_1(z^{1/\sigma}, \mathscr{S}_t(z^{1/\sigma})) dz. \tag{47}$$

Here, to simulate the integrals from the right-side of Eq. (47), we apply the trapezoidal quadrature rule for the weight function $(t_{k+1}^\sigma - z)^{\varkappa-1}$. We shift the function $\mathscr{G}_1(z^{1/\sigma}, \mathscr{S}_t(z^{1/\sigma}))$ by its piecewise linear interpolants with choosing nodes at the $t_j^\sigma (j = 0, 1, \ldots, k+1)$, and then we get

$$\begin{aligned} &\int_{t_j^\sigma}^{t_{k+1}^\sigma} (t_{k+1}^\sigma - z)^{\varkappa-1} \mathscr{G}_1(z^{1/\sigma}, \mathscr{S}_t(z^{1/\sigma})) dz \approx \tfrac{h^\varkappa}{\varkappa(\varkappa+1)} \\ &\quad \Big[\Big((k-j)^{\varkappa+1} - (k-j-\varkappa)(k-j+1)^\varkappa\Big) \\ &\times \mathscr{G}_1(t_j, \mathscr{S}_t(t_j)) + \Big((k-j+1)^{\varkappa+1} - (k-j+\varkappa+1)(k-j)^\varkappa\Big) \\ &\mathscr{G}_1(t_{j+1}, \mathscr{S}_t(t_{j+1}))\Big]. \end{aligned} \tag{48}$$

Substituting the above approximation into Eq. (47), we get the corrector formula for $\mathscr{S}_t(t_{k+1}), k = 0, 1, \ldots, \mathbb{N} - 1$,

$$\begin{aligned} \mathscr{S}_t(t_{k+1}) &\approx \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa} h^\varkappa}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_1(t_j, \mathscr{S}_t(t_j)) \\ &\quad + \frac{\sigma^{-\varkappa} h^\varkappa}{\Gamma(\varkappa+2)} \mathscr{G}_1(t_{k+1}, \mathscr{S}_t(t_{k+1})), \end{aligned} \tag{49}$$

where

$$a_{j,k+1} = \begin{cases} k^{\varkappa+1} - (k-\varkappa)(k+1)^{\varkappa} \ if \ j=0, \\ (k-j+2)^{\varkappa+1} + (k-j)^{\varkappa+1} - 2(k-j+1)^{\varkappa+1} \ if 1 \leqslant j \leqslant k. \end{cases} \tag{50}$$

At the end, we aim to change the quantity $\mathscr{S}_t(t_{k+1})$ from the right-side of Eq. (49) with the predictor term $\mathscr{S}_t^P(t_{k+1})$ that can be calculated by applying the one-step Adams-Bashforth rule to the integral Eqn. (46). We then substitute $\mathscr{G}_1(z^{1/\sigma}, \mathscr{S}_t(z^{1/\sigma}))$ by $\mathscr{G}_1(t_j, \mathscr{S}_t(t_j))$ at each integral in Eq. (47), obtaining

$$\begin{aligned} \mathscr{S}_t^P(t_{k+1}) &\approx \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa}}{\Gamma(\varkappa)} \sum_{j=0}^{k} \int_{t_j^\sigma}^{t_{j+1}^\sigma} (t_{k+1}^\sigma - z)^{\varkappa-1} \mathscr{G}_1(t_j, \mathscr{S}_t(t_j)) dz \\ &= \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+1)} \sum_{j=0}^{k} [(k+1-j)^{\varkappa} - (k-j)^{\varkappa}] \mathscr{G}_1(t_j, \mathscr{S}_t(t_j)). \end{aligned} \tag{51}$$

Therefore, our P-C scheme, for approximating $\mathscr{S}_{t_{k+1}} \approx \mathscr{S}_t(t_{k+1})$, is given by

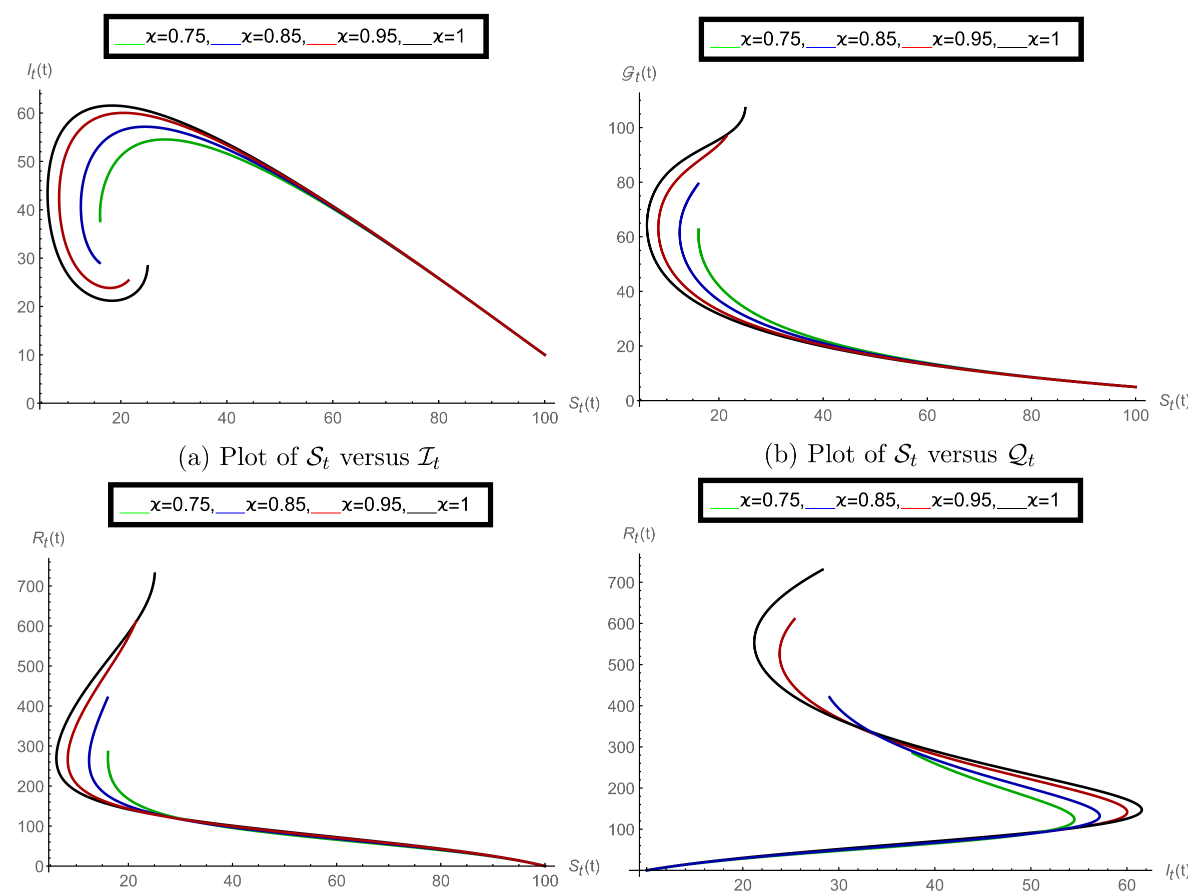
$$\begin{aligned} \mathscr{S}_{t_{k+1}} &\approx \mathscr{S}_t(0) + \frac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_1(t_j, \mathscr{S}_{t_j}) \\ &\quad + \frac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \mathscr{G}_1(t_{k+1}, \mathscr{S}_{t_{k+1}}^P), \end{aligned} \tag{52}$$

where $\mathscr{S}_{t_j} \approx \mathscr{S}_t(t_j), j = 0, 1, \ldots, k,$ and the predicted value $\mathscr{S}_{t_{k+1}}^P \approx \mathscr{S}_t^P(t_{k+1})$ can be simulated as shown in Eq. (51) with the quantities $a_{j,k+1}$ given in (50). We can repeat this procedure to approximate all equations of the system (40). So, the numerical solution formulae for the adopted model (40) can be written as:

$$\begin{aligned} \mathscr{S}_{t_{k+1}} &\approx \mathscr{S}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_1(t_j, \mathscr{S}_{t_j}) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \mathscr{G}_1(t_{k+1}, \mathscr{S}_{t_{k+1}}^P), \\ \mathscr{I}_{t_{k+1}} &\approx \mathscr{I}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_2(t_j, \mathscr{I}_{t_j}) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \mathscr{G}_2(t_{k+1}, \mathscr{I}_{t_{k+1}}^P), \\ \mathscr{Q}_{t_{k+1}} &\approx \mathscr{Q}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_3(t_j, \mathscr{Q}_{t_j}) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \mathscr{G}_3(t_{k+1}, \mathscr{Q}_{t_{k+1}}^P), \\ \mathscr{R}_{t_{k+1}} &\approx \mathscr{R}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \sum_{j=0}^{k} a_{j,k+1} \mathscr{G}_4(t_j, \mathscr{R}_{t_j}) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+2)} \mathscr{G}_4(t_{k+1}, \mathscr{R}_{t_{k+1}}^P), \end{aligned} \tag{53}$$

where

$$\begin{aligned} \mathscr{S}_t^P(t_{k+1}) &\approx \mathscr{S}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+1)} \sum_{j=0}^{k} [(k+1-j)^{\varkappa} - (k-j)^{\varkappa}] \mathscr{G}_1(t_j, \mathscr{S}_t(t_j)), \\ \mathscr{I}_t^P(t_{k+1}) &\approx \mathscr{I}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+1)} \sum_{j=0}^{k} [(k+1-j)^{\varkappa} - (k-j)^{\varkappa}] \mathscr{G}_2(t_j, \mathscr{I}_t(t_j)), \\ \mathscr{Q}_t^P(t_{k+1}) &\approx \mathscr{Q}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+1)} \sum_{j=0}^{k} [(k+1-j)^{\varkappa} - (k-j)^{\varkappa}] \mathscr{G}_3(t_j, \mathscr{Q}_t(t_j)), \\ \mathscr{R}_t^P(t_{k+1}) &\approx \mathscr{R}_t(0) + \tfrac{\sigma^{-\varkappa} h^{\varkappa}}{\Gamma(\varkappa+1)} \sum_{j=0}^{k} [(k+1-j)^{\varkappa} - (k-j)^{\varkappa}] \mathscr{G}_4(t_j, \mathscr{R}_t(t_j)). \end{aligned} \tag{54}$$



(a) Plot of $\mathcal{S}_t$ versus $\mathcal{I}_t$

(b) Plot of $\mathcal{S}_t$ versus $\mathcal{Q}_t$

(c) Plot of $\mathcal{S}_t$ versus $\mathcal{R}_t$

(d) Plot of $\mathcal{R}_t$ versus $\mathcal{I}_t$

Fig. 6. Variations in the model classes compare to each other, when vaccination fraction $q = 1$.

6.2.1. Stability analysis

Theorem 13. *If $\mathscr{G}_1(t,\mathscr{S}_t)$ satisfies a Lipschitz condition on the second variable and $\mathscr{S}_{t_j}(j=1,\ldots,k+1)$ are the solutions of the above approximations (53) and (54). Then, the proposed scheme (53) and (54) are conditionally stable.*

Proof. Let $\tilde{S}_{t_0},\tilde{S}_{t_j}(j=0,\ldots,r+1)$ and $\tilde{S}^P_{t_{r+1}}(r=0,\ldots,N-1)$ be perturbations of S_{t_0},S_{t_j} and $S^P_{t_{r+1}}$, respectively. Then, the proposed approximation equations are received by analysing Eqs. (53) and (54)

$$\tilde{S}^P_{t_{r+1}}=\tilde{S_{t_0}}+\frac{\theta^{-\varkappa}h^{\varkappa}}{\Gamma(\varkappa+1)}\sum_{j=0}^{r}b_{j,r+1}(\mathscr{G}_1(t_j,S_{t_j}+\tilde{S_{t_j}})-\mathscr{G}_1(t_j,S_{t_j})),\tag{55}$$

here $b_{j,r+1}=[(r+1-j)^{\varkappa}-(r-j)^{\varkappa}]$

$$\begin{aligned}\tilde{S_{t_{r+1}}}&=\tilde{S_{t_0}}+\tfrac{\theta^{-\varkappa}h^{\varkappa}}{\Gamma(\varkappa+2)}(\mathscr{G}_1(t_{r+1},S^P_{t_{r+1}}+\tilde{S}^P_{t_{r+1}})-\mathscr{G}_1(t_{r+1},S^P_{t_{r+1}}))+\\&\tfrac{\theta^{-\varkappa}h^{\varkappa}}{\Gamma(\varkappa+2)}\sum_{j=0}^{r}a_{j,r+1}(\mathscr{G}_1(t_j,S_{t_j}+\tilde{S_{t_j}})-\mathscr{G}_1(t_j,S_{t_j})),\end{aligned}\tag{56}$$

Using the Lipschitz condition, we simulate

$$|\tilde{S_{t_{r+1}}}|\leqslant\zeta_0+\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+2)}\left(|\tilde{M}^P_{r_{r+1}}|+\sum_{j=1}^{r}a_{j,r+1}|\tilde{S_{t_j}}|\right),\tag{57}$$

where $\zeta_0=\max_{0\leqslant k\leqslant N}\{|\tilde{S_{t_0}}|+\frac{\theta^{-\varkappa}h^{\varkappa}m_1a_{r,0}}{\Gamma(\varkappa+2)}|\tilde{S_{t_0}}|\}$. Also, as used in Erturk and Kumar (2020), we derive

$$|\tilde{S}^P_{t_{r+1}}|\leqslant\eta_0+\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+1)}\sum_{j=1}^{r}b_{j,r+1}|\tilde{S_{t_j}}|,\tag{58}$$

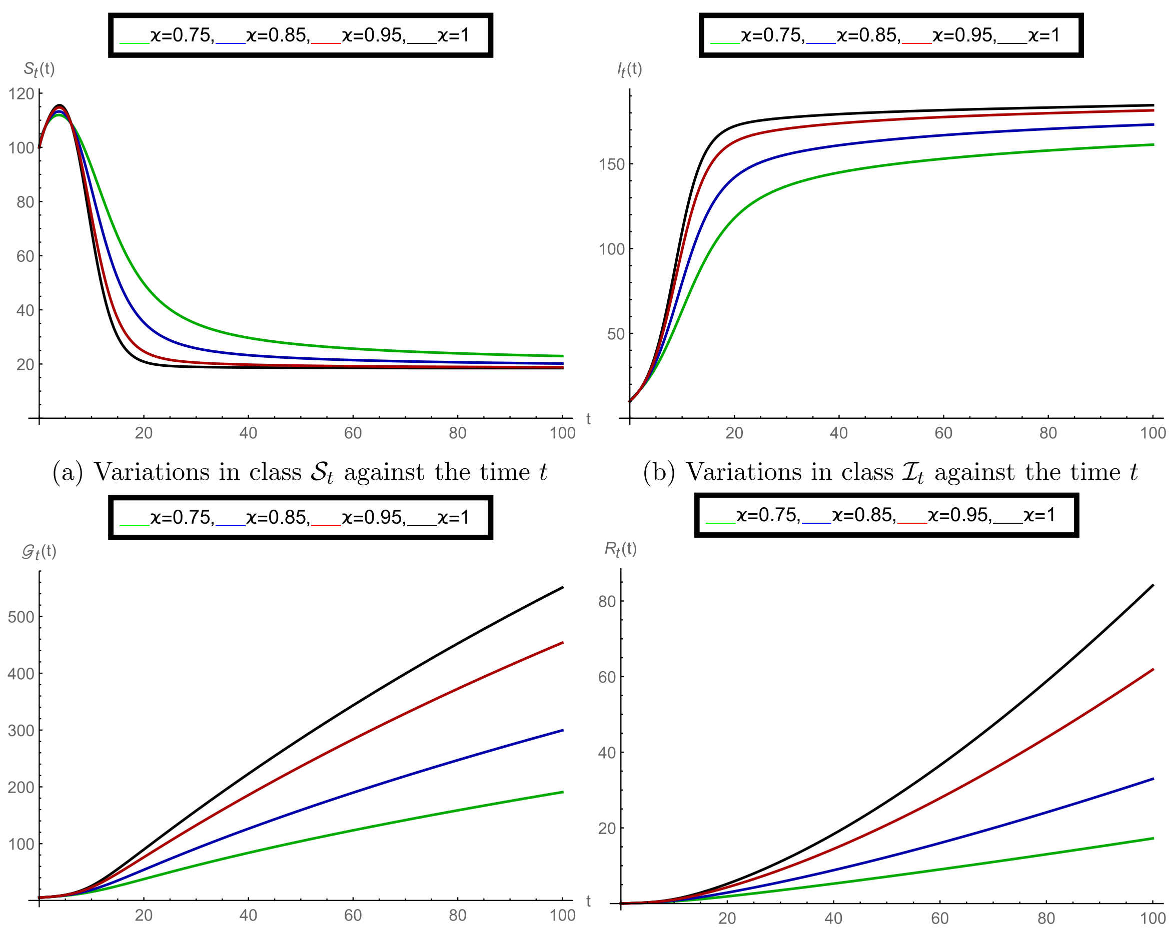
where $\eta_0=\max_{0\leqslant r\leqslant N}\{|\tilde{S_{t_0}}|+\frac{\theta^{-\varkappa}h^{\beta}m_1b_{r,0}}{\Gamma(\varkappa+1)}|\tilde{S_{t_0}}|\}$. Substituting $|\tilde{S}^P_{t_{r+1}}|$ from Eq. (58) into Eq. (57) results

$$\begin{aligned}|\tilde{S_{t_{r+1}}}|&\leqslant\gamma_0\\&+\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+2)}\left(\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+1)}\sum_{j=1}^{r}b_{j,r+1}|\tilde{S_{t_j}}|+\sum_{j=1}^{r}a_{j,r+1}|\tilde{S_{t_j}}|\right),\end{aligned}\tag{59}$$

$$\leqslant\gamma_0+\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+2)}\sum_{j=1}^{r}\left(\frac{\theta^{-\varkappa}h^{\varkappa}m_1}{\Gamma(\varkappa+1)}b_{j,r+1}+a_{j,r+1}\right)|\tilde{S_{t_j}}|,\tag{60}$$

$$\leqslant\gamma_0+\frac{\theta^{-\varkappa}h^{\varkappa}m_1\mathscr{C}_{\varkappa,2}}{\Gamma(\varkappa+2)}\sum_{j=1}^{r}(r+1-j)^{\varkappa-1}|\tilde{S_{t_j}}|,\tag{61}$$

where $\gamma_0=\max\{\zeta_0+\frac{\theta^{-\varkappa}h^{\varkappa}m_1a_{r+1,r+1}}{\Gamma(\varkappa+2)}\eta_0\}$. $C_{\varkappa,2}$ is a $+ve$ constant only depends on $\varkappa$ (Lemma 1 used in Erturk and Kumar (2020)) and h



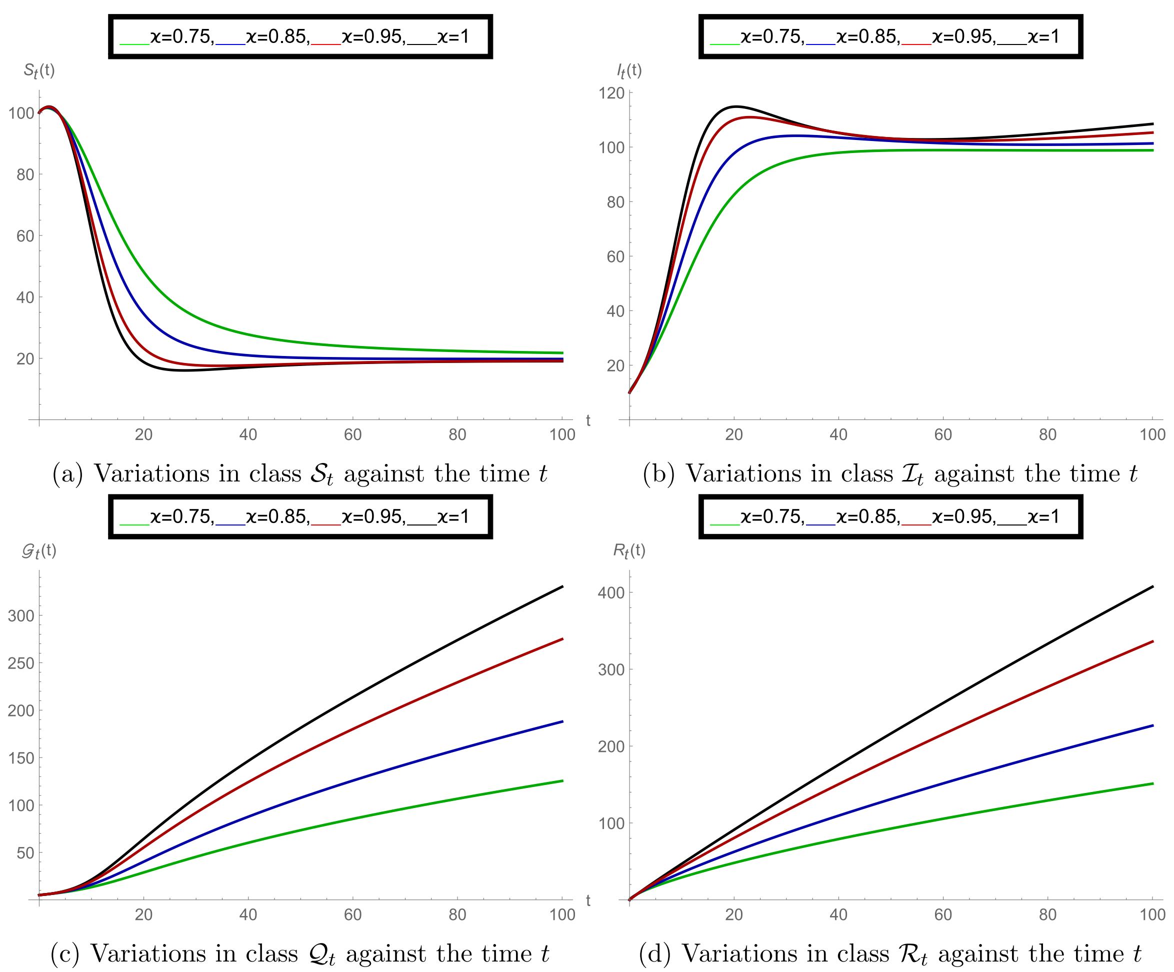
(a) Variations in class $\mathcal{S}_t$ against the time t
(b) Variations in class $\mathcal{I}_t$ against the time t
(c) Variations in class $\mathcal{Q}_t$ against the time t
(d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 7. Structure of the model classes in CF sense at various values of order $\varkappa$, when vaccination fraction $q=0$.

is supposed to be small enough. Lemma 2 as mentioned in Erturk and Kumar (2020) gives $|\tilde{S}_{t_{r+1}}| \leqslant \mathscr{C}\gamma_0$. which finishes the proof. □

6.3. Graphical results

In this section, we check the correctness of our numerical algorithm by simulating number of graphs at different fractional order values $\varkappa$. Here, we have considered the same initial populations $\mathscr{S}_t(0) = 100, \mathscr{I}_t(0) = 10, \mathscr{Q}_t(0) = 5, \mathscr{R}_t(0) = 0$, and parameter values $d = 0.001, \beta = 0.003, \delta = 0.003, \gamma = 0.002, b = 10, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002, q = 1$ as in the CF sense simulations. In the subFigs. 5a, 5b, 5c, 5d, we show the variations in $\mathscr{S}_t, \mathscr{I}_t, \mathscr{Q}_t$, and $\mathscr{R}_t$ against the time variable t. Here the variations in the dynamics of the model can be clearly explored at the various derivative order values. We observe that when the fractional order value changes then the differences between phases of the plot lines increase. Also, Fig. 6 shows the relations between the given classes at various values of $\varkappa$. More concretely, in subFig. 6a we plot the variations $\mathscr{S}_t$ versus $\mathscr{I}_t$, and in 6b we graph the variations $\mathscr{S}_t$ versus $\mathscr{Q}_t$. Meanwhile, in subfigure 6c we plot the variations $\mathscr{S}_t$ versus $\mathscr{R}_t$, and in 6d we plot the variations $\mathscr{R}_t$ versus $\mathscr{I}_t$. The fractional order values which we used here are $\varkappa = 0.75, 0.85, 0.95, 1$ as in the case of CF. Now, to simulate the role of vaccines on the proposed modified Caputo model classes, we change the value of the vaccination fraction q. Here, in the family of Fig. 7, the subFigs. 7a, 7b, 7c, 7d demonstrate the variations in $\mathscr{S}_t, \mathscr{I}_t, \mathscr{Q}_t$, and $\mathscr{R}_t$ against the time variable t at the vaccination fraction $q = 0$, where all other values are the same as used above. Similarly, Fig. 8 demonstrates the changes in the model classes when the vaccination rate $q = 0.5$. By the comparison of Fig. 7 and 8, we can easily observe that when the value of vaccination fraction q increases then the population of infectious humans decreases. This clearly means that a high vaccine rate gives much safety and becomes the only way to control the COVID-19. Now, as many countries like India, USA, UK, Spain, and Brazil have a good rate of vaccination which is a strong answer against COVID-19 infection. Vaccine availability alongwith quarantine and other optimal control facilities makes these countries much stronger to fight against this virus. From the given graphical observations, we can observe that the both kernel properties (exponential decay kernel in CF sense and singular kernel in modified Caputo sense) work well to study the given COVID-19 epidemic dynamics. All graphs are performed by using *Mathematica* software. The variations in the separate classes for both derivatives which are given in Figs. 1 and 5 are probably same but the dynamics of the given classes slightly change. This fact can be observed comparing the group of Figs. 2 and 6. It is clearly observed that vaccination fraction q plays a very important role in the given dynamics and increment in the vaccine rate can decrease the Covid-19 infection.



(a) Variations in class $\mathcal{S}_t$ against the time t
(b) Variations in class $\mathcal{I}_t$ against the time t
(c) Variations in class $\mathcal{Q}_t$ against the time t
(d) Variations in class $\mathcal{R}_t$ against the time t

Fig. 8. Structure of the model classes in CF sense at various values of order $\varkappa$, when vaccination fraction $q = 0.5$.

7. Conclusion

In this study, two new non-classical COVID-19 epidemic models have been proposed. As a novelty, we include vaccine rate. Firstly, we have proposed a classical order model and then we have justified the fractional-order models by analysing the positivity and boundedness of solutions. The disease-free and endemic equilibrium points are calculated along with basic reproductive number. We have satisfied the existence of unique solution for both variable order Caputo-Fabrizio and generalised Caputo-type fractional models. We used two different fractional numerical algorithms along with their stability analysis to solve the proposed models. A deep and long discussion on graphical simulations is given making use of *Mathematica* software. The current study provides a description of the propagation of COVID-19 disease and supporting analysis proves the correctness of our results. In future, the current model can be validated by using real data from different countries. Also, some other fractional derivatives can be used to solve the current dynamical model.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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