



## Original article

## Bayesian estimation for parameters and reliability characteristic of the Weibull Rayleigh distribution

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## ABSTRACT

In this paper we obtain Bayes' estimators under symmetric and asymmetric loss functions for the unknown parameters of Weibull Rayleigh distribution. When all the three parameters are unknown, the closed-form expressions of the Bayes estimators cannot be obtained. We use Lindley's approximation to compute the Bayes estimates. The estimators have been compared through their simulated risks. We also obtain the Bayes estimators of the reliability characteristic using both symmetric as well as asymmetric loss functions and compare its performance based on a Monte Carlo simulation study. Finally, a numerical study is provided to illustrate the results.

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## 1. Introduction

In literature Weibull Rayleigh distribution has its origin in the papers by Merovci and Elbatal (2015). They discussed statistical properties of Weibull Rayleigh distribution. They proposed maximum likelihood estimation and least square estimation procedures.

The paper is organized as follows: In Section 2, we introduce the model and discussed reliability characteristic of this distribution. The maximum likelihood estimates (MLEs) of unknown parameters and reliability characteristic are obtained in Section 3. In Section 4, Bayes estimators are derived for these parameters and also for reliability characteristic. Also, using the Lindley approximation approach we have obtained Bayes estimators. In Section 5, a numerical study is performed between proposed estimates in terms of their mean square error and bias values and one data sets are analyzed for the purpose of illustration in Section 6.

## 2. Model

The probability density function (PDF) and cumulative distribution function of a random variable  $X$  with three parameters Weibull Rayleigh distribution are given by

$$f_X(x) = \alpha\beta\theta x e^{\frac{\theta}{2}x^2} (e^{\frac{\theta}{2}x^2} - 1)^{(\beta-1)} e^{-\alpha(e^{\frac{\theta}{2}x^2} - 1)^\beta}, \quad (1)$$

$$F_X(x) = 1 - e^{-\alpha(e^{\frac{\theta}{2}x^2} - 1)^\beta}, \quad x > 0, \alpha > 0, \beta > 0, \theta > 0, \quad (2)$$

We use the notation  $X \sim WR(\alpha, \beta, \theta)$ .

The Reliability function of a  $WR(\alpha, \beta, \theta)$  distribution is of the form

$$R(t) = e^{-\alpha(e^{\frac{\theta}{2}t^2} - 1)^\beta}, \quad t > 0, \quad (3)$$

while its hazard function is given by

$$h(t) = \alpha\beta\theta t e^{\frac{\theta}{2}t^2} (e^{\frac{\theta}{2}t^2} - 1)^{(\beta-1)}, \quad t > 0. \quad (4)$$

In the next section we derive the maximum likelihood estimator for the parameters  $\alpha, \beta$  and  $\theta$  and for reliability function and hazard function.

## 3. The maximum likelihood estimator

Consider a complete sample  $(X_1, X_2, \dots, X_n)$  taken from the model (1) the likelihood function of  $\alpha, \beta$  and  $\theta$  is obtained as

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$$L(\alpha, \beta, \theta) = \prod_{i=1}^n \alpha \beta \theta x_i e^{\frac{\theta x_i^2}{2}} (e^{\frac{\theta x_i^2}{2}} - 1)^{(\beta-1)} e^{-\alpha(e^{\frac{\theta x_i^2}{2}} - 1)^\beta} \tag{5}$$

and corresponding log likelihood function is given by

$$\log L \propto n \log \alpha + n \log \beta + n \log \theta + \frac{\theta}{2} \sum_{i=1}^n x_i^2 + (\beta - 1) \sum_{i=1}^n \log(e^{\frac{\theta x_i^2}{2}} - 1) - \alpha \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta \tag{6}$$

Further, the likelihood equations of  $\alpha, \beta$  and  $\theta$  are given by

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta = 0, \tag{7}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \log(e^{\frac{\theta x_i^2}{2}} - 1) - \alpha \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta \log(e^{\frac{\theta x_i^2}{2}} - 1) = 0, \tag{8}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \frac{1}{2} \sum_{i=1}^n x_i^2 + \frac{(\beta - 1)}{2} \sum_{i=1}^n \frac{x_i^2 e^{\frac{\theta x_i^2}{2}}}{(e^{\frac{\theta x_i^2}{2}} - 1)} - \frac{\alpha \beta}{2} \sum_{i=1}^n x_i^2 e^{\frac{\theta x_i^2}{2}} (e^{\frac{\theta x_i^2}{2}} - 1)^{(\beta-1)} = 0. \tag{9}$$

The maximum likelihood estimates  $\hat{\alpha}, \hat{\beta}$  and  $\hat{\theta}$ , respectively of  $\alpha, \beta$  and  $\theta$ , are simultaneous solutions of the Eqs. (7)–(9). The system of Eqs. (7)–(9) can not be solved analytically. So we need to apply a suitable numerical method like Newton–Raphson to obtain the desired MLEs for the given equations. The R statistical software is used for all the computations. In particular, the nonlinear equations are solved using nleqslv package of this software.

In the next section, Bayes estimates of unknown parameters are obtained.

#### 4. The Bayes estimator

In this section, we obtain some Bayesian estimates of  $\alpha, \beta, \theta, R(t)$  and  $h(t)$  under symmetric as well as asymmetric loss functions. The most commonly used loss function is the squared error which is symmetric in the sense that underestimation and overestimation are equally penalized. In most estimation problems the incurred losses may not be symmetric and hence use of a symmetric loss function such as squared error can not be properly justified. In practice, such situations arise in life testing and reliability estimation where incurred losses are not symmetric in general and, hence, asymmetric loss functions would be more useful to develop Bayesian procedures. Varian (1975) introduced the LINEX loss function which is asymmetric. Zellner (1986) studied further properties of this loss function and mentioned that squared error loss is a particular case of the LINEX loss function. Another useful loss function is the entropy loss function.

Recently, many authors consider Bayesian estimates for univariate distributions like: Ahmed et al. (2010), Basu and Ebrahimi (1991), Nassar and Eissa (2005) Pandey (1997), Roio (1987), Soliman et al. (2002), Soliman et al. (2005), Soliman et al. (2006), Singh et al. (2002), Singh et al. (2013b), Singh et al. (2013a) etc.

We obtain desired Bayesian estimates under stated loss functions (squared error, LINEX and entropy) which are defined as, respectively,

$$L_S(\hat{d}(\theta), d(\theta)) = (\hat{d}(\theta) - d(\theta))^2, \\ L_L(\hat{d}(\theta), d(\theta)) = e^{m(\hat{d}(\theta) - d(\theta))} - m(\hat{d}(\theta) - d(\theta)) - 1, \quad m \neq 0, \\ L_E(\hat{d}(\theta), d(\theta)) \propto \left(\frac{\hat{d}(\theta)}{d(\theta)}\right)^w - w \log\left(\frac{\hat{d}(\theta)}{d(\theta)}\right) - 1, \quad w \neq 0,$$

where  $\hat{d}(\theta)$  is an estimate of  $d(\theta)$ . Further, under Bayesian phenomenon, an optimal estimate relative to a given loss function can easily be obtained following the usual procedure of minimizing the average risk of  $\hat{d}(\theta)$  with respect to a weight function usually known as the prior distribution of  $\theta$ . For example, the Bayesian estimate, say  $\hat{d}_{BS}$ , under the loss  $L_S$  is the posterior mean of  $d(\theta)$ . Next, the Bayesian estimate of  $d(\theta)$  for the loss function  $L_L$  is given by

$$\hat{d}_{BL} = -\frac{1}{m} \log\{E_\theta(e^{-m\theta} | \underline{x})\}$$

whereas under the loss function  $L_E$  the corresponding estimate is of the form

$$\hat{d}_{BE} = (E_\theta(\theta^{-w} | \underline{x}))^{-\frac{1}{w}}$$

given that corresponding  $E_\theta(\cdot)$  exist.

We derive Bayesian estimates of  $\alpha, \beta, \theta$ , the reliability function  $R(t)$  and the hazard function  $h(t)$  under loss functions  $L_S, L_L$  and  $L_E$  respectively. The likelihood function of  $\alpha, \beta$  and  $\theta$  is as defined in (5). Suppose that  $X_1, X_2, \dots, X_n$  is complete random sample taken from a  $WR(\alpha, \beta, \theta)$  distribution. Based on this complete sample we derive corresponding Bayesian estimates of all unknowns. We assume that  $\alpha, \beta$  and  $\theta$  are statistically independent and take  $Gamma(a, b)$ ,  $Gamma(c, d)$  and  $Gamma(p, q)$  distributions as priors for these parameters. Thus, the prior distribution of  $\alpha, \beta$  and  $\theta$  is of the form

$$\pi(\alpha, \beta, \theta) \propto \alpha^{a-1} e^{-b\alpha} \beta^{c-1} e^{-d\beta} \theta^{p-1} e^{-q\theta}, \tag{10}$$

all  $\alpha > 0, a > 0, b > 0, \beta > 0, c > 0, \theta > 0, d > 0, p > 0, q > 0$ .

Then the posterior distribution of  $\alpha, \beta$  and  $\theta$  is given by

$$\pi(\alpha, \beta, \theta | \underline{x}) = \frac{1}{k} \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b + \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta \right\}} e^{-\theta \left\{ q - \frac{1}{2} \sum_{i=1}^n x_i^2 \right\}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta x_i^2}{2}} - 1)}, \tag{11}$$

where  $\underline{x} = (x_1, x_2, \dots, x_r)$  and

$$k = \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b + \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta \right\}} e^{-\theta \left\{ q - \frac{1}{2} \sum_{i=1}^n x_i^2 \right\}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta x_i^2}{2}} - 1)} d\alpha d\beta d\theta.$$

First we obtain Bayesian estimate of  $\alpha$  under the loss function  $L_S$  using the posterior distribution  $\pi(\alpha, \beta, \theta | \underline{x})$ . The estimate is obtained as

$$\tilde{\alpha}_{BS} = \frac{1}{k} \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+a)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-\alpha \left\{ b + \sum_{i=1}^n (e^{\frac{\theta x_i^2}{2}} - 1)^\beta \right\}} e^{-\theta \left\{ q - \frac{1}{2} \sum_{i=1}^n x_i^2 \right\}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta x_i^2}{2}} - 1)} d\alpha d\beta d\theta.$$

Next, when the loss function is  $L_L$  we have the Bayesian estimate of  $\alpha$  as

$$\tilde{\alpha}_{BL} = -\frac{1}{m} \log\{E(e^{-m\alpha} | \underline{x})\}, \quad m \neq 0,$$

where

$$E[e^{-m\alpha}|\mathbf{X}] = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-\alpha \left\{ m+b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{-d\beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

Finally, for the loss function  $L_E$  we obtained that

$$\tilde{\alpha}_{BE} = \{E(\alpha^{-w}|\mathbf{X})\}^{-\frac{1}{w}},$$

where

$$E[\alpha^{-w}|\mathbf{X}] = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{(n+a-w-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

In a similar manner we consider deriving Bayesian estimates of  $\beta$  and  $\theta$  under stated loss functions.

Under the assumption that both  $\alpha, \beta$  and  $\theta$  are unknown, expressions for Bayesian estimates of reliability function  $R(t)$  are obtained in a similar manner. In fact, under the loss function  $L_S$  it is given as

$$\tilde{R}(t)_{BS} = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{-\alpha (e^{\frac{\theta}{2^2}-1})^\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

When the loss function is  $L_L$  we have

$$\tilde{R}(t)_{BL} = -\frac{1}{m} \log \{E(e^{-mR(t)}|\mathbf{X})\}, \quad m \neq 0,$$

where

$$E[e^{-mR(t)}|\mathbf{X}] = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{-m e^{-\alpha (e^{\frac{\theta}{2^2}-1})^\beta}} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta,$$

and finally under the loss function  $L_E$  the Bayesian estimate of  $R(t)$  is obtained to be

$$\tilde{R}(t)_{BE} = [E((R(t))^{-w}|\mathbf{X})]^{-\frac{1}{w}}$$

where

$$E[(R(t))^{-w}|\mathbf{X}] = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-d\beta} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{w\alpha (e^{\frac{\theta}{2^2}-1})^\beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

Further, the Bayesian estimate of the hazard function  $h(t)$  under the loss function  $L_S$  is obtained as

$$\tilde{h}(t)_{BS} = \int_0^\infty \int_0^\infty \alpha^{(n+a)} \beta^{(n+c)} \theta^{(n+p)} t e^{\frac{\theta}{2} t^2} (e^{\frac{\theta}{2} t^2} - 1)^{(\beta-1)} e^{-d\beta} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

For the loss function  $L_L$  the corresponding Bayesian estimate is

$$\tilde{h}(t)_{BL} = -\frac{1}{m} \log \{E(e^{-mh(t)}|\mathbf{X})\}, \quad m \neq 0,$$

where

$$E[e^{-mh(t)}|\mathbf{X}] = \int_0^\infty \int_0^\infty \alpha^{(n+a-1)} \beta^{(n+c-1)} \theta^{(n+p-1)} e^{-m\alpha\beta\theta t e^{\frac{\theta}{2} t^2}} (e^{\frac{\theta}{2} t^2} - 1)^{(\beta-1)} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta,$$

and for the loss function  $L_3$  the corresponding estimate is

$$\tilde{h}(t)_{BE} = [E((h(t))^{-w}|\mathbf{X})]^{-\frac{1}{w}}$$

where

$$E[(h(t))^{-w}|\mathbf{X}] = \int_0^\infty \int_0^\infty \alpha^{(n+a-w-1)} \beta^{(n+c-w-1)} \theta^{(n+p-w-1)} t^{-w} e^{-w\frac{\theta}{2} t^2} (e^{\frac{\theta}{2} t^2} - 1)^{-w(\beta-1)} e^{-\theta \left\{ q^{-\frac{1}{2}} \sum_{i=1}^n x_i^2 \right\}} e^{-\alpha \left\{ b+\sum_{i=1}^n (e^{\frac{\theta}{2^i}-1)} \right\}^\beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2^i}-1})} d\alpha d\beta.$$

#### 4.1. Lindley approximation

In previous section, we obtained Bayes estimators of  $\alpha, \beta$  and  $\theta$  under various loss functions such as squared error, linex and entropy. Note that these estimators are of the form of ratio of two integrals which can not be simplified into a closed form. However, using the approach developed by Lindley (1980), one can approximate these Bayes estimators into a form containing no integrals. From practical point at view, this method provides a simplified form of Bayes estimator which is easy to apply in practice. Consider the ratio of integral  $I(X)$ , where

$$I(X) = \frac{\int_{(\delta_1, \delta_2, \delta_3)} u(\delta_1, \delta_2, \delta_3) e^{L(\delta_1, \delta_2, \delta_3) + \rho(\delta_1, \delta_2, \delta_3)} d(\delta_1, \delta_2, \delta_3)}{\int_{(\delta_1, \delta_2, \delta_3)} e^{L(\delta_1, \delta_2, \delta_3) + \rho(\delta_1, \delta_2, \delta_3)} d(\delta_1, \delta_2, \delta_3)} \quad (1)$$

where  $u(\delta_1, \delta_2, \delta_3)$  is function of  $\delta_1, \delta_2$  and  $\delta_3$  only and  $L(\delta_1, \delta_2, \delta_3)$  is the log-likelihood and  $\rho(\delta_1, \delta_2, \delta_3) = \log \rho(\delta_1, \delta_2, \delta_3)$ . Let  $(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3)$  denote the MLE of  $(\delta_1, \delta_2, \delta_3)$ . For sufficiently large sample size  $n$ , using the approach developed in Lindley (1980), the ratio of integral  $I(X)$  as defined in (1) can be written as

$$I(X) = u(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3) + (u_1 v_1 + u_2 v_2 + u_3 v_3 + v_4 + v_5) + 0.5[A(u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}) + B(u_1 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}) + C(u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33})]$$

where  $\hat{\delta}_1, \hat{\delta}_2$  and  $\hat{\delta}_3$  are the MLE of  $\delta_1, \delta_2$  and  $\delta_3$  respectively.

$$v_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}, \quad i = 1, 2, 3 \quad v_4 = u_{12} \sigma_{12} + u_{13} \sigma_{13} + u_{23} \sigma_{23} \\ v_5 = 0.5(u_{11} \sigma_{11} + u_{22} \sigma_{22} + u_{33} \sigma_{33}) \\ A = \sigma_{11} L_{111} + 2\sigma_{12} L_{121} + 2\sigma_{13} L_{131} + 2\sigma_{23} L_{231} + \sigma_{22} L_{221} + \sigma_{33} L_{331} \\ B = \sigma_{11} L_{112} + 2\sigma_{12} L_{122} + 2\sigma_{13} L_{132} + 2\sigma_{23} L_{232} + \sigma_{22} L_{222} + \sigma_{33} L_{332} \\ C = \sigma_{11} L_{113} + 2\sigma_{12} L_{123} + 2\sigma_{13} L_{133} + 2\sigma_{23} L_{233} + \sigma_{22} L_{223} + \sigma_{33} L_{333}$$

and subscripts 1,2,3 on the right-hand sides refer to  $\delta_1, \delta_2, \delta_3$  respectively and

$$\rho_i = \frac{\partial \rho}{\partial \delta_i}, u_i = \frac{\partial u(\delta_1, \delta_2, \delta_3)}{\partial \delta_i}, u_{ij} = \frac{\partial^2 u(\delta_1, \delta_2, \delta_3)}{\partial \delta_i \partial \delta_j}, L_{ij} = \frac{\partial^2 L}{\partial \delta_i \partial \delta_j},$$

$$L_{ijk} = \frac{\partial^3 L}{\partial \delta_i \partial \delta_j \partial \delta_k},$$

$i, j, k = 1, 2, 3$  and also,  $\sigma_{ij} = (i, j)^{th}$  elements of the inverse of the matrix  $[-\frac{\partial^2 l(\alpha, \beta, \theta | x)}{\partial \alpha \partial \beta \partial \theta}]^{-1}$  evaluated at  $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ . Other expressions are obtained as,

$$L_{11} = -\frac{n}{\alpha^2}, L_{12} = L_{21} = -\sum_{i=1}^n (e^{\frac{\beta}{\alpha^2} x_i^2} - 1) \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1), L_{111} = \frac{2n}{\alpha^3},$$

$$L_{22} = \frac{-n}{\beta^2} - \alpha \sum_{i=1}^n (e^{\frac{\beta}{\alpha^2} x_i^2} - 1) (\log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1))^2, L_{121} = L_{131} = 0,$$

$$L_{23} = L_{32} = \frac{1}{2} \sum_{i=1}^n \frac{x_i^2 e^{\frac{\beta}{\alpha^2} x_i^2}}{(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)} - \frac{\alpha}{2} \sum_{i=1}^n x_i^2 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-1)} \{1 + \beta \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)\},$$

$$L_{33} = -\frac{n}{\theta^2} - \frac{(\beta-1)}{4} \sum_{i=1}^n \frac{x_i^4 e^{\frac{\beta}{\alpha^2} x_i^2}}{(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^2} - \frac{\alpha \beta}{4} \sum_{i=1}^n x_i^4 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-2)} (\beta e^{\frac{\beta}{\alpha^2} x_i^2} - 1),$$

$$L_{122} = -\sum_{i=1}^n (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^\beta (\log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1))^2,$$

$$L_{222} = \frac{2n}{\beta^3} - \alpha \sum_{i=1}^n (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^\beta (\log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1))^3,$$

$$L_{123} = -\frac{1}{2} \sum_{i=1}^n x_i^2 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-1)} \{1 + \beta \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)\},$$

$$L_{133} = -\frac{\beta}{4} \sum_{i=1}^n x_i^4 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-2)} (\beta e^{\frac{\beta}{\alpha^2} x_i^2} - 1),$$

$$L_{223} = -\frac{\alpha}{2} \sum_{i=1}^n x_i^2 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-1)} \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1) \{2 + \beta \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)\},$$

$$L_{233} = -\frac{1}{4} \sum_{i=1}^n \frac{x_i^4 e^{\frac{\beta}{\alpha^2} x_i^2}}{(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^2}$$

$$-\frac{\alpha}{4} \sum_{i=1}^n x_i^4 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-2)} \{2\beta e^{\frac{\beta}{\alpha^2} x_i^2} - 1 + \beta(\beta e^{\frac{\beta}{\alpha^2} x_i^2} - 1) \log(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)\},$$

$$L_{333} = \frac{2n}{\theta^3} + \frac{(\beta-1)}{8} \sum_{i=1}^n \frac{x_i^6 e^{\frac{\beta}{\alpha^2} x_i^2} (1 + e^{\frac{\beta}{\alpha^2} x_i^2})}{(e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^3}$$

$$-\frac{\alpha \beta}{8} \sum_{i=1}^n x_i^6 e^{\frac{\beta}{\alpha^2} x_i^2} (e^{\frac{\beta}{\alpha^2} x_i^2} - 1)^{(\beta-3)} \{e^{\frac{\beta}{\alpha^2} x_i^2} (\beta^2 e^{\frac{\beta}{\alpha^2} x_i^2} - 3\beta + 1) + 1\},$$

$$\rho_1 = \frac{a-1}{\alpha} - b, \rho_2 = \frac{c-1}{\beta} - d, \rho_3 = \frac{p-1}{\theta} - q$$

Now we can obtain the values of the Bayes estimates of various parameters under different loss function. (a) Case of the squared error loss function

$$\tilde{\alpha}_{BS} = \hat{\alpha} + \delta_1 + 0.5(\sigma_{11}A + \sigma_{21}B + \sigma_{31}C),$$

$$\tilde{\beta}_{BS} = \hat{\beta} + \delta_2 + 0.5(\sigma_{12}A + \sigma_{22}B + \sigma_{32}C),$$

$$\tilde{\theta}_{BS} = \hat{\theta} + \delta_3 + 0.5(\sigma_{13}A + \sigma_{23}B + \sigma_{33}C).$$

(b) Case of the Linex loss function

$$\tilde{\alpha}_{BL} = -\frac{1}{m} \log \{ e^{-m\hat{\alpha}} + 0.5u_{11}\sigma_{11} + u_1\delta_1 + 0.5u_1(\sigma_{11}A + \sigma_{21}B + \sigma_{31}C) \},$$

$$\tilde{\beta}_{BL} = -\frac{1}{m} \log \{ e^{-m\hat{\beta}} + 0.5u_{22}\sigma_{22} + u_2\delta_2 + 0.5u_2(\sigma_{12}A + \sigma_{22}B + \sigma_{32}C) \},$$

$$\tilde{\theta}_{BL} = -\frac{1}{m} \log \{ e^{-m\hat{\theta}} + 0.5u_{33}\sigma_{33} + u_3\delta_3 + 0.5u_3(\sigma_{13}A + \sigma_{23}B + \sigma_{33}C) \}.$$

(c) Case of the Entropy loss function

$$\hat{\alpha}_{BE} = \{ \hat{\alpha}^{-w} + 0.5u_{11}\sigma_{11} + u_1\delta_1 + 0.5u_1(\sigma_{11}A + \sigma_{21}B + \sigma_{31}C) \}^{\frac{1}{w}},$$

$$\hat{\beta}_{BE} = \{ \hat{\beta}^{-w} + 0.5u_{22}\sigma_{22} + u_2\delta_2 + 0.5u_2(\sigma_{12}A + \sigma_{22}B + \sigma_{32}C) \}^{\frac{1}{w}},$$

$$\hat{\theta}_{BE} = \{ \hat{\theta}^{-w} + 0.5u_{33}\sigma_{33} + u_3\delta_3 + 0.5u_3(\sigma_{13}A + \sigma_{23}B + \sigma_{33}C) \}^{\frac{1}{w}}.$$

Proceeding similarly the Bayes estimators for the reliability function and hazard rate function can be evaluated under all three loss functions.

### 5. Numerical comparisons

In Sections 3 and 4 several estimates of  $\alpha, \beta, \theta$  and  $R(t), h(t)$  of a  $WR(\alpha, \beta, \theta)$  distribution are obtained using complete sample. The Bayesian estimates of Section 4 are obtained against different symmetric and asymmetric loss functions such as squared error, linex and general entropy. In this section performance of all these estimates have been compared numerically in terms of their mean square errors (MSEs) and average values. For our purpose we have evaluated MSEs and average values based on 5000 generations of random sample of size  $n$  from the  $WR(\alpha, \beta, \theta)$  distribution. The true value of  $(\alpha, \beta, \theta)$  is taken as (0.2, 0.4, 0.5) and hyperparameters are assigned values as  $a = 1, b = 5, c = 2, d = 5, p = 4, q = 8$ . In each case, the Bayes estimates with respect to the loss function  $L_L$  is computed for three distinct values of  $m$ , namely,  $-0.1, 0.5, 1$ . Also under  $L_E$  loss associated estimates are obtained for  $w = -0.5, 0.5, 1$ . In addition, MSEs and average values of  $R(t)$  and  $h(t)$  are computed for two distinct values of  $t$ , namely, 2, 3.5.

In Table 51 the MSEs and average values of estimators  $\hat{\alpha}, \tilde{\alpha}_{SB}, \tilde{\alpha}_{LB}, \tilde{\alpha}_{EB}, \hat{\beta}, \tilde{\beta}_{SB}, \tilde{\beta}_{LB}, \tilde{\beta}_{EB}$  and  $\hat{\theta}, \tilde{\theta}_{SB}, \tilde{\theta}_{LB}, \tilde{\theta}_{EB}$  are presented for different choices of  $n$ . In each cell, all values for a particular value of  $n$  there are six rows, the first row values represent estimate of  $\alpha$  and the second row values represent the corresponding MSE value. The third and fourth values are similarly associated with the parameter  $\beta$  and the fifth and sixth values are similarly associated with the parameter  $\theta$ . In general, we observed that Bayes estimates are superior than the respective MLEs under MSE criterion. For estimating the unknown parameters the choice  $m = 1$  seems to be reasonable under  $L_L$  loss. In case of  $L_E$  loss, the choice  $w = 1$  seems a reasonable choice.

In Tables 52 and 53 the MSEs and average values of the estimates  $\hat{R}(t), \tilde{R}_{SB}(t), \tilde{R}_{LB}(t)$  and  $\tilde{R}_{EB}(t)$  of reliability function  $R(t)$  are presented for different choices of  $t$ . In this case also we find that Bayes estimators show superior performance compare to the MLE. Furthermore, the squared error Bayes estimates of  $R(t)$  has minimum MSE values among all other estimates of  $R(t)$ . Among estimators obtained from linex loss function  $m = -0.5$  gives a better choice.

The MSEs and average values of all estimates of  $h(t)$  are presented in Tables 54 and 55. We again notice that Bayes estimates are quite superior than the MLE. The choice  $m = 1$  with respect to the  $L_L$  loss produce best estimate of  $h(t)$ .

In general mean squared error values all estimates decreases as  $n$  increases.

### 6. Data analysis

Two examples are presented in this section for the purpose of illustration. First, we treat a real data set.

**Example 1 (real data):** This data set was originally discussed by Meeker and Escobar (2014). Merovci and Elbatal (2015) fitted this data set to WR with complete sample. In this data, the times of failure and running times for a sample of devices from a eld-tracking

**Table 51**  
MSEs and average values of  $\alpha, \beta$  and  $\theta$ .

n	$\hat{\alpha}$ $\hat{\beta}$ $\hat{\theta}$	$\tilde{\alpha}_{BS}$ $\tilde{\beta}_{BS}$ $\tilde{\theta}_{BS}$	$\tilde{\alpha}_{BL}$ $\tilde{\beta}_{BL}$ $\tilde{\theta}_{BL}$		$\tilde{\alpha}_{BE}$ $\tilde{\beta}_{BE}$ $\tilde{\theta}_{BE}$			
			m=-0.1	m = 0.5	m = 1.0	w=-0.5	w = 0.5	w = 1.0
20	0.243512	0.243613	0.244362	0.241534	0.23919	0.219603	0.189498	0.192441
	0.044964	0.015166	0.017509	0.016438	0.014956	0.025301	0.017199	0.018149
	0.473687	0.474434	0.476464	0.464162	0.453933	0.457477	0.388467	0.386258
	0.070325	0.080547	0.081027	0.080182	0.077959	0.084633	0.082489	0.077869
	0.602365	0.661007	0.660377	0.675875	0.665816	0.657956	0.572576	0.566286
40	0.21749	0.213914	0.214141	0.212144	0.210483	0.202679	0.191074	0.187867
	0.011084	0.004917	0.005942	0.005291	0.005144	0.00414	0.003865	0.003683
	0.439309	0.412514	0.412613	0.412196	0.413513	0.407034	0.405132	0.403436
	0.028522	0.010853	0.011614	0.010997	0.009302	0.02961	0.025197	0.024295
	0.548463	0.536169	0.538372	0.533912	0.526807	0.52273	0.527999	0.517651
60	0.211552	0.212817	0.213004	0.211375	0.209918	0.206993	0.197858	0.194267
	0.005376	0.003457	0.003651	0.003567	0.002792	0.004006	0.002362	0.002305
	0.427756	0.413517	0.413777	0.411974	0.410557	0.40735	0.401147	0.398316
	0.017371	0.003516	0.004497	0.003903	0.003863	0.009532	0.008658	0.007815
	0.524519	0.523591	0.524866	0.518613	0.51333	0.51215	0.497651	0.490168
80	0.209081	0.211491	0.211654	0.210605	0.209806	0.207755	0.200819	0.197606
	0.00349	0.002209	0.002222	0.002147	0.002091	0.002043	0.001877	0.001857
	0.423614	0.413096	0.413503	0.410973	0.409023	0.407593	0.398995	0.394991
	0.012366	0.003346	0.003363	0.003301	0.003258	0.004573	0.004458	0.004175
	0.510467	0.521683	0.522871	0.517392	0.512341	0.512888	0.494542	0.486171
100	0.206402	0.210299	0.210428	0.209651	0.209005	0.207351	0.201477	0.198742
	0.002802	0.001977	0.001986	0.001933	0.001892	0.001859	0.001726	0.001702
	0.415734	0.410755	0.411123	0.408934	0.407146	0.406002	0.397555	0.393695
	0.009354	0.003561	0.003579	0.003480	0.003418	0.003553	0.003549	0.0034581
	0.514015	0.523033	0.523891	0.519088	0.514802	0.514811	0.498813	0.490491
	0.022751	0.007816	0.008110	0.008053	0.007902	0.007376	0.008246	0.006788

**Table 52**  
MSEs and average values of all estimates of  $R(t)$  for different choices of  $n$  and  $t = 2$ .

n	$\hat{R}(t)$	$\tilde{R}(t)_{BS}$	$\tilde{R}(t)_{BL}$		$\tilde{R}(t)_{BE}$			
			m=-0.1	m = 0.5	m = 1.0	w=-0.5	w = 0.5	w = 1.0
20	0.791871	0.767433	0.767933	0.766425	0.766102	0.766646	0.763043	0.76292
	0.00578	0.003975	0.004049	0.004465	0.004585	0.004521	0.005382	0.00555
40	0.788253	0.779303	0.779643	0.779114	0.778108	0.778708	0.776708	0.775966
	0.002947	0.002108	0.002433	0.002485	0.002521	0.002291	0.002337	0.002349
60	0.785072	0.780397	0.780463	0.779957	0.779517	0.779657	0.778389	0.777841
	0.001958	0.001567	0.001561	0.001570	0.001575	0.001685	0.00177	0.001763
80	0.783489	0.780571	0.78064	0.780226	0.779882	0.780032	0.779087	0.778663
	0.001428	0.001170	0.001177	0.001181	0.001185	0.00121	0.001265	0.001273
100	0.78325	0.781134	0.78119	0.780855	0.780576	0.780774	0.780059	0.779703
	0.001262	0.001011	0.001017	0.001021	0.001022	0.001021	0.001030	0.001035

**Table 53**  
MSEs and average values of all estimates of  $R(t)$  for different choices of  $n$  and  $t = 3.5$ .

n	$\hat{R}(t)$	$\tilde{R}(t)_{BS}$	$\tilde{R}(t)_{BL}$		$\tilde{R}(t)_{BE}$			
			m=-0.1	m = 0.5	m = 1.0	w=-0.5	w = 0.5	w = 1.0
20	0.514616	0.527684	0.530718	0.529619	0.529991	0.53047	0.528675	0.528817
	0.029253	0.009876	0.013444	0.014415	0.016898	0.018317	0.026075	0.027501
40	0.512092	0.517769	0.519411	0.517908	0.517019	0.517133	0.514269	0.510783
	0.006862	0.003961	0.005891	0.005915	0.006187	0.006018	0.006802	0.007134
60	0.512211	0.514399	0.514769	0.514043	0.512922	0.513329	0.509912	0.508635
	0.003199	0.002557	0.002861	0.002998	0.003056	0.002699	0.002713	0.003395
80	0.513245	0.514885	0.514998	0.514296	0.513731	0.513776	0.511432	0.510449
	0.002345	0.001809	0.001901	0.001903	0.001908	0.001914	0.001985	0.001991
100	0.511976	0.513229	0.51332	0.512771	0.512313	0.512326	0.510527	0.509638
	0.001876	0.001561	0.001601	0.001602	0.001605	0.00161	0.001642	0.001668

**Table 54**  
MSEs and average values of all estimates of  $h(t)$  for different choices of  $n$  and  $t = 2$ .

$n$	$\hat{h}(t)$	$\tilde{h}(t)_{BS}$	$\tilde{h}(t)_{BL}$			$\tilde{h}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
20	0.160252	0.165234	0.164647	0.163015	0.161781	0.161103	0.164748	0.172922
	0.03286	0.014645	0.010189	0.00983	0.009682	0.012783	0.012448	0.011214
40	0.15967	0.161263	0.161327	0.161002	0.160741	0.159442	0.155677	0.154505
	0.001633	0.000896	0.000901	0.000896	0.000889	0.000962	0.000872	0.000901
60	0.15924	0.160761	0.16081	0.160519	0.160278	0.160036	0.156198	0.154836
	0.001097	0.000670	0.000672	0.000665	0.000661	0.000688	0.000647	0.000658
80	0.158287	0.159732	0.159771	0.15954	0.159348	0.15854	0.156166	0.15501
	0.000802	0.000539	0.000546	0.000535	0.000532	0.000527	0.000519	0.000522
100	0.158326	0.159593	0.159625	0.159433	0.159274	0.158547	0.156525	0.155562
	0.000637	0.000460	0.000463	0.00046	0.000458	0.000459	0.000454	0.000454

**Table 55**  
MSEs and average values of all estimates of  $h(t)$  for different choices of  $n$  and  $t = 3.5$ .

$n$	$\hat{h}(t)$	$\tilde{h}(t)_{BS}$	$\tilde{h}(t)_{BL}$			$\tilde{h}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
20	0.536141	0.398157	0.395956	0.397877	0.3847	0.387033	0.417482	0.400376
	0.123038	0.094173	0.085498	0.083405	0.07687	0.009644	0.009787	0.104902
40	0.514487	0.412248	0.412021	0.413491	0.41444	0.413399	0.422225	0.424833
	0.028533	0.016722	0.016784	0.016345	0.016308	0.016363	0.016912	0.017433
60	0.506869	0.440939	0.440891	0.44092	0.441035	0.441005	0.442087	0.442777
	0.009296	0.008135	0.008176	0.008033	0.007906	0.006934	0.007148	0.007268
80	0.502095	0.454709	0.454731	0.454505	0.454313	0.454415	0.453816	0.453645
	0.006797	0.005074	0.005079	0.005039	0.005009	0.004896	0.004941	0.005004
100	0.499315	0.462004	0.462063	0.461715	0.461439	0.461484	0.460589	0.460214
	0.004931	0.003688	0.003689	0.003683	0.003679	0.003580	0.003671	0.00367

**Table 6.1**  
Estimates of all estimates of  $\alpha, \beta$  and  $\theta$ .

$\hat{\alpha}$	$\tilde{\alpha}_{BS}$	$\tilde{\alpha}_{BL}$	$\tilde{\alpha}_{BE}$	$\hat{\beta}$	$\tilde{\beta}_{BS}$	$\tilde{\beta}_{BL}$	$\tilde{\beta}_{BE}$	$\hat{\theta}$	$\tilde{\theta}_{BS}$	$\tilde{\theta}_{BL}$			$\tilde{\theta}_{BE}$		
										$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
0.27537	0.299551	0.300119	0.296669	0.292778	0.293419	0.293791	0.291564	1.56221	1.6848	1.70183	1.59505	1.50635	1.62785	1.51311	1.46127

**Table 6.2**  
Estimates of all estimates of  $R(t)$  for different choices of  $t$ .

$t$	$\hat{R}(t)$	$\tilde{R}(t)_{BS}$	$\tilde{R}(t)_{BL}$			$\tilde{R}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
1	0.748772	0.744989	0.745232	0.74378	0.742577	0.743375	0.740193	0.738634
3	0.115773	0.127536	0.127645	0.126985	0.126428	0.122605	0.112519	0.107982

**Table 6.3**  
Estimates of all estimates of  $h(t)$  for different choices of  $t$ .

$t$	$\hat{h}(t)$	$\tilde{h}(t)_{BS}$	$\tilde{h}(t)_{BL}$			$\tilde{h}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
1	0.514616	0.527684	0.530718	0.529619	0.529991	0.53047	0.528675	0.528817
3	2.96113	2.01797	2.01962	2.04233	2.09011	2.03021	2.06521	2.08488

study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and failure caused by normal product wear. The times are: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65,

0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.

The maximum likelihood estimates and different Bayesian estimates of  $\alpha, \beta$  and  $\theta$  are presented in Table 6.1. The estimates of  $R(t)$  and  $h(t)$  are tabulated for different choices of  $t$  in Table 6.2 and 6.3

**Table 6.4**  
Estimates of all estimates of  $\alpha, \beta$  and  $\theta$ .

$\hat{\alpha}$ $\hat{\beta}$ $\hat{\theta}$	$\tilde{\alpha}_{BS}$ $\tilde{\beta}_{BS}$ $\tilde{\theta}_{BS}$	$\tilde{\alpha}_{BL}$ $\tilde{\beta}_{BL}$ $\tilde{\theta}_{BL}$			$\tilde{\alpha}_{BE}$ $\tilde{\beta}_{BE}$ $\tilde{\theta}_{BE}$		
		$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
0.075027	0.083163	0.083227	0.082841	0.082516	0.078713	0.069954	0.066329
0.207059	0.192319	0.192957	0.18916	0.186063	0.17779	0.156861	0.150765
0.303958	0.417462	0.418344	0.412668	0.407224	0.39854	0.345298	0.316704

**Table 6.5**  
Estimates of all estimates of  $R(t)$  for different choices of  $t$ .

$t$	$\hat{R}(t)$	$\tilde{R}(t)_{BS}$	$\tilde{R}(t)_{BL}$			$\tilde{R}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
4	0.885334	0.880524	0.880622	0.880035	0.879548	0.879971	0.878878	0.878338
8	0.569962	0.572353	0.572639	0.570922	0.569491	0.569842	0.564855	0.562411

**Table 6.6**  
Estimates of all estimates of  $h(t)$  for different choices of  $t$ .

$t$	$\hat{h}(t)$	$\tilde{h}(t)_{BS}$	$\tilde{h}(t)_{BL}$			$\tilde{h}(t)_{BE}$		
			$m=-0.1$	$m = 0.5$	$m = 1.0$	$w=-0.5$	$w = 0.5$	$w = 1.0$
4	0.033615	0.030971	0.030977	0.030942	0.030913	0.030146	0.028801	0.028295
6	0.088253	0.049886	0.049833	0.050151	0.05041	0.053167	0.057698	0.059372

respectively. Conclusions similar to the Section 5 can easily be drawn from these tables.

**Example 1 (simulated data):** We consider a simulated data presented in Merovci and Elbatal (2015). As in previous example, here again we have obtained various estimates of  $\alpha, \beta$  and  $\theta$ . These results are tabulated in Table 6.4. The corresponding estimates of  $R(t)$  and  $h(t)$  are presented in Table 6.5 and 6.6 respectively.

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