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Bitopological rough approximations with medical applications

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Abstract In this paper, we introduce new generalizations concepts of lower and upper approximations of Pawlak rough sets by using two topological structures (bitopologies). Also, we study the concept of the generalized topological rough set and some of their basic properties. Applications for data reduction are done on medical data.

1. Introduction

Rough set theory, proposed by Pawlak in the early 1980s (Pawlak, 1981), is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Moreover, the theory may serve as a new mathematical tool to soft computing besides fuzzy set theory (Pawlak and Peters, 2007; Pawlak and Skowron, 2007a,b,c; Pawlak, 1981a, 1991, 2004; Peters et al., 2006a,b, 2007a,b; Peters and Henry, 2009; Peters and Ramanna, 2007, 2009; Peters, 2007a,b,c, 2008a,b, 2009), and has been successfully applied in machine learning, pattern recognition, expert systems, data analysis, and so on. Recently, lots of researchers are interested in the theory (Polkowski and Skowron, 1997; Polkowski, 2002; Puzio and Walczak, 2008; Randen and Husoy, 1999; Slowinski and Vanderpooten, 2000; Wasilewska, 1997; Yao, 1998a,b; Zadeh, 1965; Zakowski, 1983).

In Pawlak's original rough set theory, partition or equivalence (indiscernibility) relation is an important and primitive concept. But, partition or equivalence relation is still restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalence relation have been proposed in the past, such as tolerance relations (Orowska, 1985, 1998), similarity relations (Orowska, 1998), and others (Abd El-Monsef et al., 2007; Gupta and Patnaik, 2008; Hassanien et al., 2009; Henry and Peters, 2008, 2009; Hurtut et al., 2008; Meghdadi et al., 2009). Particularly, Peters has used coverings of an universe for establishing the generalized rough set (Peters and Ramanna, 2007). And an extensive body of research works has been developed (Peters et al., 2007b, 2008; Peters, 2007a,b; Salama and Abu-Donia, 2006; Salama, 2008a,b,c, 2010). In 1997, Wasilewska defined the topological rough algebras. Furthermore, Pawlak (2004) in his long paper

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have characterized a measure of roughness making use of the concept of rough fuzzy sets in 1995. He also suggested some possible applications of the measure in pattern recognition and image analysis problems. Some results about rough sets and fuzzy sets are obtained by Pawlak and Skowron (2007b).

In this paper, we investigate some important and basic issues of generalized rough sets induced by topological structures. The plan of this paper is as follows.

In Section 2, we recall the basic concepts and properties of the Pawlak's rough set theory. In Section 3, some new concepts and main results are considered in generalized rough sets induced by two topological structures. In Section 4, we define a measure of roughness based on generalized rough sets with the new approximations, and prove some properties of the measure. Finally, we give an example in order to indicate the use of the measure in Section 5.

2. Basic concepts and properties of the Pawlak's rough set theory

This section presents a review of some fundamental notions of rough sets. We refer to Hassanien et al. (2009), Orowska (1998), Pawlak and Skowron (2007a,b,c), Pawlak (1981a,b, 1991, 2004) for details.

Motivation for rough set theory has come from the need to represent subsets of an universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $A = (U, R)$, where U is a set called the universe and R is an equivalence relation (Orowska, 1985; Pawlak and Skowron, 2007c). The equivalence classes of R are also known as the granules, elementary sets or blocks; we will use $[x]_R \subseteq U$ to denote the equivalence class containing $x \in U$. In the approximation space, we consider two operators $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$ and $\underline{R}(X) = \{x \in U\text{: } [x]_R \subseteq X\}$, called the lower approximation and upper approximation of $X \subseteq U$, respectively. Also let $POS_R(X) = \underline{R}(X)$ denote the positive region of X, $NEG_R(X) = U - \overline{R}(X)$ denote the negative region of X and $BN_R(X) = \overline{R}(X) - \underline{R}(X)$ denote the borderline region of X.

The degree of completeness can also be characterized by the accuracy measure, in which $|X|$ represents the cardinality of set X as follows:

$$\alpha_R(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|}, \quad \text{where } X \neq \phi$$

Accuracy measures try to express the degree of completeness of knowledge. $\alpha_R(X)$ is able to capture how large the boundary region of the data sets is; however, we cannot easily capture the structure of the knowledge. A fundamental advantage of rough set theory is its ability to handle a category that cannot be sharply defined given a knowledge base. Characteristics of the potential data sets can be measured through the rough sets framework. We can measure inexactness and express topological characterization of imprecision with:

(1) If $\underline{R}(X) \neq \phi$ and $\overline{R}(X) \neq U$, then X is roughly R-definable.
(2) If $\underline{R}(X) = \phi$ and $\overline{R}(X) \neq U$, then X is internally R-undefinable.
(3) If $\underline{R}(X) \neq \phi$ and $\overline{R}(X) = U$, then X is externally R-undefinable.
(4) If $\underline{R}(X) = \phi$ and $\overline{R}(X) = U$, then X is totally R-undefinable.

We denote the set of all roughly R-definable (resp. internally R-undefinable, externally R-undefinable and totally R-undefinable) sets by $RD(U)$ (resp. $REUD(U)$, $RUD(U)$ and $RTUD(U)$).

With $\alpha_R(X)$ and classifications above we can characterize rough sets by the size of the boundary region and structure. Rough sets are treated as a special case of relative sets and integrated with the notion of Belnap's logic (Orowska, 1998).

A topological space (Hurtut et al., 2008) is a pair (U, τ) consisting of a set U and family τ of subset of U satisfying the following conditions:

(1) ϕ, $U \in \tau$.
(2) τ is closed under arbitrary union.
(3) τ is closed under finite intersection.

The pair (U, τ) is called a topological space, the elements of U are called points of the space, the subsets of U belonging to τ are called open sets in the space, and the complement of the subsets of U belonging to τ are called closed sets in the space; the family τ of open subsets of U is also called a topology for U.

$\overline{X}_\tau = \bigcap\{F \subseteq U : X \subseteq F, F \in \tau^c\}$ is called τ-closure of a subset $X \subset U$. Evidently, $\overline{X}_\tau$ is the smallest closed subset of U which contains X. Note that X is closed iff $X = \overline{X}_\tau$.

$\underline{X}_\tau = \bigcup\{G \subseteq U : G \subseteq X, G \in \tau\}$ is called the τ-interior of a subset $X \subseteq U$. Evidently, $\underline{X}_\tau$ is the union of all open subsets of U which containing in X. Note that X is open iff $X = \underline{X}_\tau$. And $X_b = \overline{X}_\tau - \underline{X}_\tau$ is called the τ-boundary of a subset $X \subseteq U$.

Let X be a subset of a topological spaces (U, τ). Let $\overline{X}_\tau$, $\underline{X}_\tau$ and X_b be closure, interior, and boundary of X, respectively. X is exact if $X_b = \phi$, otherwise X is rough. It is clear X is exact iff $\overline{X}_\tau = \underline{X}_\tau$. In Pawlak space a subset $X \subseteq U$ has two possibilities rough or exact.

Definition 2.1. Let (U, τ) be a topological space defined by a general relation R, then R-lower (resp. R-upper) approximation of any non-empty subset X of U is defined as:

$$\underline{R}_\tau(X) = \bigcup\{G \in \tau : G \subseteq X\} \quad \text{and}$$
$$\overline{R}_\tau(X) = \bigcap\{F \in \tau^c : F \supseteq X\}.$$

Definition 2.2. A subset X of a topological space (U, τ) is called upper lower upper set (shortly *ulu*-set) if $X \subseteq cl_\tau\,(int_\tau(cl_\tau(X)))$. The complement of *ulu*-set is ulu^c-set. We denote the set of all *ulu*-sets and ulu^c-sets by $ulu(U)$ and $ulu^C(U)$, respectively. For any topological space (U, τ). We have $\tau \subseteq ulu(U)$.

3. Generalized rough sets induced by two topological structures

In this section, we introduce and investigate the concept of ulu_{12} (ulu_{21})-approximation space. Also, we introduce the concepts of ulu_{12} (ulu_{21})-lower approximation and ulu_{12} (ulu_{21} and 12-21-*ulu*)-upper approximation and study their properties.

Definition 3.1. Let R be any binary general relation defined on the universe U. Then we can define two topologies, the subbase of the first topology τ_1 (right topology) is the right neighborhood xR. Also, the subbase of the second topology τ_2 (left topology) is the left neighborhood Rx, where $xR = \{y \in U: xRy\}$ and $Rx = \{y \in U: yRx\}$.

Definition 3.2. Let (U, τ_1, τ_2) be a generalized topological approximation space. Then the subset $X \subseteq U$ is called: 12-*ulu*-set (briefly ulu_{12}-set) if $X \subseteq cl_\tau(int_\tau(cl_\tau(X)))$ and it is called 21-*ulu*-set (briefly ulu_{21}-set) if $X \subseteq cl_{\tau_1}(int_{\tau_2}(cl_{\tau_1}(X)))$. The complement of $ulu_{12}(ulu_{21}$-set) is $ulu^c_{12}(ulu^c_{21}$-set), respectively.

Remark 3.1. The family of all ulu_{12}-sets (resp. ulu_{21}, ulu^c_{12} and ulu^c_{21}) sets in (U, τ_1, τ_2) is denoted by $Fulu_{12}(U)$ (resp. $Fulu_{21}(U)$, $Fulu^c_{12}(U)$ and $Fulu^c_{21}(U)$).

Definition 3.3. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R. Then the ulu_{12} -lower and ulu_{12} -upper approximations of any subset $X \subseteq U$ are defined as: $\underline{R}_{ulu_{12}}(X) = ulu_{12}(\underline{R}_\tau(X))$ and $\overline{R}_{ulu_{12}}(X) = ulu_{12}(\overline{R}_\tau(X))$.

Definition 3.4. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R. Then the 12-21-*ulu* -lower and 12-21-*ulu*-upper approximations of any subset $X \subseteq U$ are defined as: $\underline{R}_{12\text{-}21\text{-}ulu}(X) = \underline{R}_{ulu_{12}}(X) \cup \underline{R}_{ulu_{21}}(X)$, $\overline{R}_{12\text{-}21\text{-}ulu}(X) = \overline{R}_{ulu_{12}}(X) \cap \overline{R}_{ulu_{21}}(X)$.

Definition 3.5. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R. Then we can characterize the degree of completeness by a new tool named 12-21-*ulu*-accuracy measure defined as follows:

$$\alpha_{12\text{-}21\text{-}ulu}(X) = \frac{|\underline{R}_{12\text{-}21\text{-}ulu}(X)|}{|\overline{R}_{12\text{-}21\text{-}ulu}(X)|}, \quad \text{where } X \neq \phi.$$

Example 3.1. Let $U = \{a, b, c, d\}$ be an universe and the relation R defined on U by $R = \{(a,a), (a,c), (a,d), (b,b), (b,d), (c,a), (c,b), (c,d), (d,a)\}$. Table 1 shows the degree of accuracy measure $\alpha_R(X)$, *ulu*-accuracy measure $\alpha_{ulu_{12}}(X)$ and 12-21-*ulu*-accuracy measure $\alpha_{12\text{-}21\text{-}ulu}(X)$ for some subsets $X \subseteq U$.

We see from Table 1 that the degree of exactness of these subsets by using 12-21-*ulu*-accuracy measure is equal to 100% of the chosen subsets. Consequently 12-21-*ulu*-accuracy measure is refinement of the last measures.

The universe U can be divided into 24 regions with respect to any $X \subseteq U$ as follows:

Table 1 Comparison of the three accuracy measures.

$X \subseteq U$	$\alpha_R(X)$	$\alpha_{ulu_{12}}(X)$	$\alpha_{12\text{-}21\text{-}ulu}(X)$
{c}	0	0	1
{d}	1/3	1/2	1
{a, b}	1/3	1/2	1
{a, d}	1/2	1/2	1
{b, c}	0	0	1
{c, d}	1/3	2/3	1
{a, b, c}	1/3	2/3	1
{a, b, d}	3/4	3/4	1

(1) The internal edge of X, $\underline{Edg}(X) = X - \underline{R}(X)$.
(2) The τ-internal edge of X, $\underline{Edg}_\tau(X) = X - \underline{R}_\tau(X)$.
(3) The 12-21-*ulu*-internal edge of X, $\underline{Edg}_{12\text{-}21\text{-}ulu}(X) = X - \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(4) The external edge of X, $\overline{Edg}(X) = \overline{R}(X) - X$.
(5) The τ-external edge of X, $\overline{Edg}_\tau(X) = \overline{R}_\tau(X) - X$.
(6) The 12-21-*ulu*-external edge of X, $\overline{Edg}_{12\text{-}21\text{-}ulu}(X) = \overline{R}_{12\text{-}21\text{-}ulu}(X) - X$.
(7) The boundary of X, $b(X) = \overline{R}(X) - \underline{R}(X)$.
(8) The τ-boundary of X, $X_b = \overline{R}_\tau(X) - \underline{R}_\tau(X)$.
(9) The 12-21-*ulu*-boundary of X, $X_{12\text{-}21\text{-}ulu} = \overline{R}_{12\text{-}21\text{-}ulu}(X) - \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(10) The exterior of X, $X^{ex} = X - \overline{R}(X)$.
(11) The τ-exterior of X, $X^{ex}_\tau = X - \overline{R}_\tau(X)$.
(12) The 12-21-*ulu*-exterior of X, $X^{ex}_{12\text{-}21\text{-}ulu} = X - \overline{R}_{12\text{-}21\text{-}ulu}(X)$.
(13) $\overline{R}(X) - \underline{R}_\tau(X)$.
(14) $\overline{R}(X) - \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(15) $\overline{R}_\tau(X) - \underline{R}(X)$.
(16) $\overline{R}_\tau(X) - \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(17) $\overline{R}_{12\text{-}21\text{-}ulu}(X) - \underline{R}(X)$.
(18) $\overline{R}_{12\text{-}21\text{-}ulu}(X) - \underline{R}_\tau(X)$.
(19) $\underline{R}_{12\text{-}21\text{-}ulu}(X) - \underline{R}_\tau(X)$.
(20) $\underline{R}_{12\text{-}21\text{-}ulu}(X) - \underline{R}(X)$.
(21) $\underline{R}_\tau(X) - \underline{R}(X)$.
(22) $\overline{R}(X) - \overline{R}_\tau(X)$.
(23) $\overline{R}(X) - \overline{R}_{12\text{-}21\text{-}ulu}(X)$.
(24) $\overline{R}_\tau(X) - \overline{R}_{12\text{-}21\text{-}ulu}(X)$.

4. Properties of 12-21-*ulu*-approximations

In this section, we introduce a generalization for some of the concepts of rough set theory by using the 12-21-*ulu-lower* and the 12-21-*ulu-upper* approximations.

Definition 4.1. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, for any subset $X \subseteq U$. Then we define two membership relations $\underline{\in}_{12\text{-}21\text{-}ulu}$ and $\overline{\in}_{12-21-ulu}$, say, 12-21-*ulu*-strong and 12-21-*ulu*-weak memberships, respectively, and defined by:

$$x \underline{\in}_{12\text{-}21\text{-}ulu} X \text{ iff } x \in \underline{R}_{12\text{-}21\text{-}ulu}(X) \quad \text{and}$$
$$x \overline{\in}_{12\text{-}21\text{-}ulu} X \text{ iff } x \in \overline{R}_{12\text{-}21\text{-}ulu}(X).$$

Remark 4.1. According to Definition 4.1, 12-21-*ulu*-lower and 12-21-*ulu*-upper approximations of a set $X \subseteq U$ can be rewritten as: $\underline{R}_{12\text{-}21\text{-}ulu}(A) = \{x \in X : x \underline{\in}_{12\text{-}21\text{-}ulu} X\}$, $\overline{R}_{12\text{-}21\text{-}ulu}(X) = \{x \in X : x \overline{\in}_{12\text{-}21\text{-}ulu} X\}$.

Remark 4.2. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, for any subset $X \subseteq U$. Then $x \underline{\in} X \wedge x \underline{\in}_\tau X \Rightarrow x \underline{\in}_{12\text{-}21\text{-}ulu} X$ and $x \overline{\in} X \vee x \overline{\in}_\tau X \Rightarrow x \overline{\in}_{12\text{-}21\text{-}ulu} X$.

The converse of Remark 4.2 may not be true in general as seen in the following examples.

Example 4.1. In Example 3.1, if $X = \{a, b, c\}$, then $\underline{R}(X) = \{a\}$, $\underline{R}_\tau(X) = \{a, c\}$ and $\underline{R}_{12\text{-}21\text{-}ulu}(X) = \{a, b, c\}$, hence $b \underline{\in}_{12\text{-}21\text{-}ulu} X$, $b \underline{\notin}_\tau X$ and $b \underline{\notin} X$. Also $c \underline{\in}_\tau X$, but $c \underline{\in} X$.

Example 4.2. In Example 3.1, if $X = \{d\}$, then $\overline{R}(X) = \{b,c,d\}, \overline{R}_\tau(X) = \{b,d\}$ and $\overline{R}_{12\text{-}21\text{-}ulu}(X) = \{d\}$. So $b \mathrel{\overline{\in}} X$, $b \mathrel{\overline{\in}}_\tau X$, but $b \mathrel{\overline{\in}}_{12\text{-}21\text{-}ulu} X$ and $c \mathrel{\overline{\in}} X$, but $c \mathrel{\overline{\in}}_\tau X$.

We investigate 12-21-*ulu*-rough equality and 12-21-*ulu*-rough inclusion based on rough equality and inclusion.

Definition 4.2. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, and let $X, Y \subseteq U$ be two subsets of U. Then we say that X and Y are:

(i) 12-21-*ulu*-roughly bottom equal $(X \eqsim_{12\text{-}21\text{-}ulu} Y)$ if $\underline{R}_{12\text{-}21\text{-}ulu}(X) = \underline{R}_{12\text{-}21\text{-}ulu}(Y)$.
(ii) 12-21-*ulu*-roughly top equal $(X \simeq_{12\text{-}21\text{-}ulu} Y)$ if $\overline{R}_{12\text{-}21\text{-}ulu}(X) = \overline{R}_{12\text{-}21\text{-}ulu}(Y)$.
(iii) 12-21-*ulu*-roughly equal $(X \approx_{12\text{-}21\text{-}ulu} Y)$ if $(X \eqsim_{12\text{-}21\text{-}ulu} Y)$ and $(X \simeq_{12\text{-}21\text{-}ulu} Y)$.

Example 4.3. In Example 3.1, we have the sets $\{b\}$, ϕ are 12-21-*ulu*-roughly bottom equal and $\{a,c,d\}$, U are 12-21-*ulu*-roughly top equal.

Definition 4.3. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, and let $X, Y \subseteq U$ be two subsets of U. Then we say that:

(i) X is 12-21-*ulu*-roughly bottom included in Y $(X \subset_{\sim 12\text{-}21\text{-}ulu} Y)$ if $\underline{R}_{12\text{-}21\text{-}ulu}(X) \subseteq \underline{R}_{12\text{-}21\text{-}ulu}(Y)$.
(ii) X is 12-21-*ulu*-roughly top included in $Y (X \mathrel{\tilde{\subset}}_{12\text{-}21\text{-}ulu} Y)$ if $\overline{R}_{12\text{-}21\text{-}ulu}(X) \subseteq \overline{R}_{12\text{-}21\text{-}ulu}(Y)$.
(iii) X is 12-21-*ulu*-roughly included in $Y (X \mathrel{\tilde{\subset}}_{12\text{-}21\text{-}ulu} Y)$ if $(X \subset_{\sim 12\text{-}21\text{-}ulu} Y)$ and $(X \mathrel{\tilde{\subset}}_{12\text{-}21\text{-}ulu} Y)$.

Example 4.4. In Example 3.1, we have $X_1 = \{b\}$, $X_2 = \{c\}$, $Y_1 = \{a,b,d\}$ and $Y_2 = \{a,c,d\}$, then X_1 is 12-21-*ulu*-roughly bottom included in X_2 and Y_1 is 12-21-*ulu*-roughly top included in Y_2.

Definition 4.4. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, and let $X \subseteq U$ Then X is called:

(i) 12-21-*ulu*-definable (12-21-*ulu*-exact), if $\overline{R}_{12\text{-}21\text{-}ulu}(X) = \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(ii) 12-21-*ulu*-rough, if $\overline{R}_{12\text{-}21\text{-}ulu}(X) \neq \underline{R}_{12\text{-}21\text{-}ulu}(X)$.
(iii) Roughly 12-21-*ulu*-definable, if $\underline{R}_{12\text{-}21\text{-}ulu}(X) \neq \phi$ and $\overline{R}_{12\text{-}21\text{-}ulu}(X) \neq U$.
(iv) Internally 12-21-*ulu*-undefinable, if $\underline{R}_{12\text{-}21\text{-}ulu}(X) = \phi$ and $\overline{R}_{12\text{-}21\text{-}ulu}(X) \neq U$.
(v) Externally 12-21-*ulu*-undefinable, if $\underline{R}_{12\text{-}21\text{-}ulu}(X) \neq \phi$ and $\overline{R}_{12\text{-}21\text{-}ulu}(X) = U$.
(vi) Totally 12-21-*ulu*-undefinable, if $\underline{R}_{12\text{-}21\text{-}ulu}(X) = \phi$ and $\overline{R}_{12\text{-}21\text{-}ulu}(X) = U$.

Proposition 4.1. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, then:

(i) Every exact set in U is 12-21-*ulu*-exact.
(ii) Every τ-exact set in U is 12-*ulu*-exact.
(iii) Every 12-21-*ulu*-rough set in U is rough.
(iv) Every 12-21-*ulu* -rough set in U is τ-rough.

Proof. Obvious. □

The converse of all parts of Proposition 4.1, may not be true in general as seen in the following example.

Example 4.5. In Example 3.1, the sets $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,d\}$, $\{b,c\}$, $\{c,d\}$, $\{a,b,c\}$ and $\{a,b,d\}$ are 12-21-*ulu*-exact but neither τ-exact nor exact.

Remark 4.3. Let (U, τ_1, τ_2) be a generalized topological approximation space generated by any binary relation R, then:

(i) The intersection of two 12-21-*ulu*-exact sets need not be 12-21-*ulu*-exact set.
(ii) The union of two 12-21-*ulu*-exact sets need not be 12-21-*ulu*-exact set.

The following example shows the above remark.

Example 4.6. In Example 3.1, let $X_1 = \{a\}$, $X_2 = \{c,d\}$, $Y_1 = \{b,c\}$ and $Y_2 = \{b,d\}$, are 12-21-*ulu*-exact. Then $Y_1 \cap Y_2$ and $X_1 \cup X_2$ are not 12-21-*ulu*-exact.

5. Medical applications

In this section, we briefly describe the rheumatic fever data sets used in this study as a topological application of data reduction (Abd El-Monsef et al., 2007; Salama and Abu-Donia, 2006, 2008; Salama, 2008a,b,c, 2010). No doubt that rheumatic fever is a very common disease and it has many symptoms that differ from one patient to another though the diagnosis is the same. So, we obtained the following example on four rheumatic fever patients. All patients were between 9 and 12 years old with a history of Arthurian which began from age 3 to 5 years. This disease has many symptoms and it usually starts at young age and persists with the patient all through his life. Table 3 contains information on seven patients characterized by eight symptoms (attributes) which were used to decide the diagnosis for each patient (decision attribute), where the attributes are shown in Table 2.

Here we will give the main conventions that we will apply in this section. These conventions will be indicated by examples.

The structure $GMIS = (U, At, \{V_a : a \in At\}, f_a, \{\eta_B : B \subseteq At\})$ is called generalized multi-valued information system, where U is a non-empty finite set of objects (persons, planets, cars, digits, etc.) called the universe. Any set $X \subseteq U$ is called a category in U. V_a is a collection of value sets corresponding to the attribute $a \in At$. f_a: $U \to V_a$ is a total information function such that $f_a(x) \in V_a$. η_B is a binary relation defined on U, which is not necessary to be an equivalence relation. Here, we consider η_a as an example of non-equivalence relation on U which is defined by: for $B \in At$, $\eta_B = \{(x,y) : [f_a(x)]^c \subseteq f_a(y)\} \forall a \in B$, $B \subseteq At$. Clearly, η_B is not reflexive, not transitive, but it is symmetric. For $a \in At$, the class A_{η_a} is defined by: $A_{\eta_a} = \{\eta_{ax} : x \in U\}$, where $\eta_{ax} = \{y : x\eta_a y\}$.

If D is the decision attribute, then the generalized multi-valued information system will take the form $GMIS = (U, At \cup D, \{V_a : a \in At\}, f_a, \{\eta_B : B \subseteq At\})$. In this case, we suggest the following non-equivalence relation for the decision attribute:

Table 2 Single-valued medical information system.

Attribute symbol	Refers to ?	Attribute values	Refers to ?
S	Sex	s_1	Male
		s_2	Female
F	Pharyngitis	f_1	Yes
		f_2	No
A	Arthritis	a_0	No arthritis
		a_1	Began in the knee
		a_2	Began in the ankle
R	Carditis	r_1	Affected
		r_2	Not affected
K	Chorea	k_1	Yes
		k_2	No
E	ESR	e_1	Normal
		e_2	High
P	Abdominal Pain	p_1	Absent
		p_2	Present
H	Headache	h_1	Yes
		h_2	No
D	Diagnosis	d_1	Rheumatic arthritis
		d_2	Rheumatic carditis
		d_3	Rheumatic arthritis and carditis

$\eta_D = \{(x,y) : f_D(x) \subseteq f_D(y)\}$.

The concept of this relation is defined as $\eta_{Dx} = \{y: x\eta_D y\}$. The set of all concepts is defined by $A_{\eta_a} = \{\eta_{ax} : x \in U\}$. Also, if D is the decision attribute and for $a \in At$, we have $POS_a(D) = \cup_{Y \in A_{\eta_D}} a - \underline{R}_{12\text{-}21\text{-}ulu}(Y)$, where

$$a - \underline{R}_{12\text{-}21\text{-}ulu}(Y) = a - \underline{R}_{ulu_{12}}(Y) \cup a - \underline{R}_{ulu_{21}}(Y)$$

where $a - \underline{R}_{ulu_{12}}(Y)$ and $a - \underline{R}_{ulu_{21}}(Y)$ are the lower approximations defined in Definition 3.3, by using the attribute $a \in At$.

Let us take $\{A_{\eta_a} : a \in At\}$ as a subbase of a topological space τ_a (the set of all finite intersections and arbitrary unions of members of A_{η_a}) and $\{A_{\eta_B} : B \subseteq At\}$ as a subbase of a topological space τ_B. The decision makes the topology τ_D which has $\{A_{\eta_D}\}$ as a subbase. Hence, we can say that the set of attributes $B \subseteq At$ is a reduct if $\tau_B < \tau_D$ and B is a minimal, where $\tau_B < \tau_D$ iff $\forall G \in \tau_B$, $\exists G' \in \tau_D$ s.t. $G \subseteq G'$, $G, G' \neq U$.

A set of attribute B depends totally on a set of attributes A denoted by $A \Rightarrow B$, if all values sets of attributes from B are uniquely determined by values sets of attributes from A. Let A and B be subsets of At, we say that B depends on A in a degree K $(0 \leqslant K \leqslant 1)$, denoted by: $A \Rightarrow_K B$ if $K = \gamma(A,B) = \frac{|POS_A(B)|}{|U|}$.

If $K = 1$, B depends totally on A. If $K < 1$, B depends partially (in a degree K) on A.

If we take $A = At$ and $B = D$ in the above two issues, where At is the set of condition attributes and D is the decision attribute, then we say that, D depends totally on At, denoted by $At \Rightarrow D$, if all values of attributes from D are uniquely determined by values sets of attributes from At. Otherwise, we say that D depends on At in a degree K, denoted by $At \Rightarrow_K D$.

Table 4 shows the coding of the data, which is described as follows: Sex $(S) = \{M,F\} = \{0,1\}$, Pharyngitis $(F) = \{yes, no\} = \{1,0\}$, Arthritis $A = \{a_0, a_1, a_2\} = \{0,1,2\}$, Carditis $R = \{affected, not\ affected\} = \{1,0\}$, Chorea $K = \{yes, no\} = \{1,0\}$, ESR $E = normal, high = \{0,1\}$, Abdominal Pain $P = \{absent, present\} = \{0,1\}$ and Headache $H = \{yes, no\} = \{1,0\}$. The decision attribute is Diagnosis $D = \{rheumatic\ arthritis, rheumatic\ carditis, rheumatic\ arthritis\ and\ carditis\} = \{d1, d2, d3\}$.

Then we constrain the MIS as shown in Table 5.

From the relation $R_a = \{(x,y): f_a(x) \subseteq f_a(y)\}$, where a is an element of the power set of the set of condition attributes $\{\alpha, \beta, \delta\}$. The two subbases of two topologies for each element of the power set of $\{\alpha, \beta, \delta\}$ are defined as: $\xi_1^a = \{xR_a :$

Table 4 Coding medical data.

Attribute symbol	Refers to ?	Attribute values	Refers to ?
α	$\{S,K\}$	α_1	S takes s_1
		α_2	K takes k_1
		α_3	Each of $\{S,K\}$ takes $\{s_2,k_2\}$
β	$\{F,A,E\}$	β_1	F takes f_1
		β_2	A takes a_1
		β_3	A takes a_2
		β_4	E takes e_1
		β_5	Each of $\{F,A,E\}$ takes $\{f_2,a_0,e_2\}$
δ	$\{R,P,H\}$	δ_1	R takes r_1
		δ_2	P takes p_1
		δ_3	H takes h_1
		δ_4	Each of $\{R,P,H\}$ takes $\{r_2,p_2,h_2\}$
D	Diagnosis	d_1	Rheumatic arthritis
		d_2	Rheumatic carditis
		d_3	Rheumatic arthritis and carditis

Table 3 Rheumatic fever data.

Patients	History							
	S	F	A	R	K	E	P	H	D
p_1	s_2	f_1	a_1	r_1	k_1	e_1	p_1	h_2	d_3
p_2	s_1	f_1	a_1	r_1	k_1	e_2	p_1	h_1	d_3
p_3	s_2	f_1	a_2	r_1	k_2	e_1	p_1	h_2	d_3
p_4	s_1	f_1	a_1	r_2	k_2	e_1	p_1	h_2	d_1
p_5	s_1	f_2	a_0	r_1	k_2	e_1	p_2	h_2	d_2
p_6	s_1	f_1	a_1	r_1	k_2	e_2	p_1	h_2	d_3
p_7	s_1	f_1	a_2	r_1	k_2	e_1	p_1	h_1	d_3

Table 5 Multi-valued information system.

	α	β	δ	D
p_1	$\{\alpha_2\}$	$\{\beta_1,\beta_2,\beta_4\}$	$\{\delta_1\}$	$\{d_3\}$
p_2	$\{\alpha_1,\alpha_2\}$	$\{\beta_1,\beta_2\}$	$\{\delta_1,\delta_3\}$	$\{d_3\}$
p_3	$\{\alpha_3\}$	$\{\beta_1,\beta_3\}$	$\{\delta_1\}$	$\{\underline{d}_3\}$
p_4	$\{\alpha_1\}$	$\{\beta_1,\beta_2,\beta_4\}$	$\{\delta_4\}$	$\{\underline{d}_1\}$
p_5	$\{\alpha_1\}$	$\{\beta_5\}$	$\{\delta_1,\delta_2\}$	$\{\underline{d}_2\}$
p_6	$\{\alpha_1\}$	$\{\beta_1,\beta_2\}$	$\{\delta_1\}$	$\{d_3\}$
p_7	$\{\alpha_1\}$	$\{\beta_1,\beta_3,\beta_4\}$	$\{\delta_1,\delta_3\}$	$\{d_3\}$

$x \in U\}$, where $xR_a = \{y\colon xR_a y\}$ and $\xi_2^a = \{R_a x : x \in U\}$, where $R_a x = \{y\colon yR_a x\}$. Then according to Table 5 we have the following couples of topologies:

$$\tau_1^\alpha = \{U, \phi, \{p_2\}, \{p_3\}, \{p_2,p_3\}, \{p_1,p_2\}, \{p_1,p_2,p_3\}, \{p_2,p_3,p_4,p_5,p_6,p_7\}, \{p_1,p_2,p_4,p_5,p_6,p_7\}, \{p_2,p_4,p_5,p_6,p_7\}\},$$

$$\tau_2^\alpha = \{U, \phi, \{p_1\}, \{p_3\}, \{p_1,p_3\}, \{p_4,p_5,p_6,p_7\}, \{p_3,p_4,p_5,p_6,p_7\}, \{p_1,p_4,p_5,p_6,p_7\}, \{p_1,p_2,p_4,p_5,p_6,p_7\}, \{p_2,p_3,p_4,p_5,p_6,p_7\}\},$$

$$\tau_1^\beta = \{U, \phi, \{p_5\}, \{p_7\}, \{p_3,p_7\}, \{p_1,p_4\}, \{p_5,p_7\}, \{p_3,p_5,p_7\}, \{p_1,p_4,p_5\}, \{p_1,p_4,p_5,p_7\}, \{p_1,p_2,p_4,p_6\}\},$$

$$\tau_2^\beta = \{U, \phi, \{p_3,p_5,p_7\}, \{p_2,p_3,p_6\}, \{p_2,p_3,p_6,p_7\}, \{p_1,p_2,p_4,p_6\}, \{p_1,p_2,p_3,p_4,p_6\}, \{p_2,p_3,p_5,p_6,p_7\}, \{p_1,p_2,p_4,p_5,p_6\}, \{p_1,p_2,p_3,p_4,p_5,p_6\}, \{p_1,p_2,p_3,p_4,p_6,p_7\}\},$$

$$\tau_1^\delta = \{U, \phi, \{p_4\}, \{p_5\}, \{p_2,p_7\}, \{p_4,p_5\}, \{p_2,p_4,p_7\}, \{p_2,p_5,p_7\}, \{p_2,p_4,p_5,p_7\}, \{p_1,p_2,p_3,p_5,p_6,p_7\}\},$$

$$\tau_2^\delta = \{U, \phi, \{p_4\}, \{p_1,p_3,p_6\}, \{p_1,p_3,p_4,p_6\}, \{p_1,p_3,p_5,p_6\}, \{p_1,p_2,p_3,p_6,p_7\}, \{p_1,p_2,p_3,p_5,p_6,p_7\}, \{p_1,p_3,p_4,p_5,p_6\}, \{p_1,p_2,p_3,p_4,p_6,p_7\}\}$$

$$\tau_1^{\alpha\beta} = \tau_1^\alpha \cap \tau_1^\beta = \{U, \phi\},$$

$$\tau_2^{\alpha\beta} = \tau_2^\alpha \cap \tau_2^\beta = \{U, \phi\},$$

$$\tau_1^{\alpha\delta} = \tau_1^\alpha \cap \tau_1^\delta = \{U, \phi\},$$

$$\tau_2^{\alpha\delta} = \tau_2^\alpha \cap \tau_2^\delta = \{U, \phi\},$$

$$\tau_1^{\beta\delta} = \tau_1^\beta \cap \tau_1^\delta = \{U, \phi, \{p_5\}\},$$

$$\tau_2^{\beta\delta} = \tau_2^\beta \cap \tau_2^\delta = \{U, \phi, \{p_1,p_2,p_3,p_4,p_6,p_7\}\},$$

$$\tau_1^{\alpha\beta\delta} = \tau_1^\alpha \cap \tau_1^\beta \cap \tau_1^\delta = \{U, \phi\},$$

$$\tau_2^{\alpha\beta\delta} = \tau_2^\alpha \cap \tau_2^\beta \cap \tau_2^\delta = \{U, \phi\}.$$

Now we will deal with the decision attribute D applying the relation: $\eta_D = \{(x,y)\colon D(x) \subseteq D(y)\}$, then the subbase of the decision topology is $\xi_1^D = \{xR_D : x \in U\} = \{\{p_1,p_2,p_3,p_6,p_7\}, \{p_4\}, \{p_5\}\}$. Then the decision topology is given by: $\tau_D = \{U, \phi, \{p_1,p_2,p_3,p_6,p_7\}, \{p_4\}, \{p_5\}, \{p_4,p_5\}, \{p_1,p_2,p_3,p_4,p_6,p_7\}, \{p_1,p_2,p_3,p_5,p_6,p_7\}\}$, the complement decision topology is

$$\tau_D^c = \{U, \phi, \{p_4,p_5\}, \{p_1,p_2,p_3,p_5,p_6,p_7\}, \{p_1,p_2,p_3,p_4,p_6,p_7\}, \{p_1,p_2,p_3,p_6,p_7\}, \{p_5\}, \{p_4\}\}.$$

We can observe that $\tau_1^{\beta\delta} \leqslant \tau_D$ and $\tau_2^{\beta\delta} \leqslant \tau_D^c$, which lead to $\{\beta,\delta\} = \{F,A,E,R,P,H\}$ which is the reduct and the core of our system. This means that we can remove the attributes $\{S,K\}$ without losing any information.

6. Conclusion

It is well known that rough set theory has been regarded as a generalization of classical set theory in one way. Furthermore, this is an important mathematical tool to deal with uncertainty. As a natural need, it is a fruitful way to extend classical rough sets to generalized rough sets induced by topological spaces. In this paper, new lower and upper approximations are proposed in generalized rough set induced by a topological structure, and some important properties are obtained. Also, we define the concept of a rough membership function in generalized topological approximation spaces. It is a generalization of classical rough membership function of Pawlak rough sets. The rough membership function can be used to analyze which decision should be made according to a conditional attribute in decision table.

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