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Almost α-regular spaces

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ABSTRACT

A new version of regularity is introduced as a simultaneous generalization of almost regularity and α-regulaity, and it is called *almost α-regularity*. We discuss many properties of this a new space and we give some properties that connect this a new space with some other topological spaces, also we present an examples and counter examples that show the relationships between almost α-regular and some other topological spaces.

1. Introduction

Murtinov a' in (Murtinová, 2001) introduced a weaker version of regularity called α-regularity. In this article we introduce a new notion of regularity as a simultaneous generalization of almost regularity (Singal and Arya, 1969) and α-regularity to get a decomposition of almost regularity in terms of α-regularity.

Let (Y, τ) be a topological space, and $B \subset Y$. In this work, the closure of a set B is denoted by $\bar{B}$ or $cl(B)$ and the interior by $int(B)$ or B°. A set $G \subset Y$ is called *regularly open* if $G = int(\bar{G})$, and *regularly closed* if $G = cl\left(G^{\circ}\right)$. The family of all regularly open sets defines a base for a topology on Y, we call it *semiregular* topology on Y. Note that every regular space is semiregular. A Tychonoff space is T_1 and completely regular space.

2. Preliminaries

2.1. Definition (Singal and Arya, 1969)

A topological space (Y, τ) is called *almost regular* if for any regularly closed subset B of Y not containing $y \in Y$ can be separated by two disjoint open sets in Y.

2.2. Definition (Murtinová, 2001)

A topological space (Y, τ) is called α-regular if for each $y \in Y$ and a closed subset $B \subset Y$ such that $y \notin B$ there are two disjoint open sets $H_1, H_2 \subset Y$ such that $y \in H_1$ and $\overline{B \cap H_2} = B$.

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2.3. Definition (Arhangel'skii and ludwig, 2001)

A topological space (Y, τ) is called α -normal if for every disjoint closed subsets E_1 and E_2 in Y there are two disjoint open subsets G_1 and G_2 of Y such that $\overline{E_1 \cap G_1} = E_1$ and $\overline{E_2 \cap G_2} = E_2$.

2.4. Definition (Gheith and Mohamed, 2018)

A topological space (Y, τ) is called *almost* α *-normal* if for every disjoint closed subsets A and B of Y one of which A is regulary closed, there are disjoint open subsets G and H such that $\overline{A \cap G} = A$ and $\overline{B \cap H} = B$.

2.5. Definition (Kalantan and Allahabi, 2008)

A topological space T_1 is called extremally disconnected if it is the closur of any open set is open.

2.6. Definition (Das et al., 2017)

A topological space (Y, τ) is called *almost* β *-normal* if for every disjoint closed subsets A and B of Y one of which A is regulary closed, there are disjoint open subsets G and H such that $\overline{A \cap G} = A$, $\overline{B \cap H} = B$, and $\bar{G} \cap \bar{H} = \varnothing$.

2.7. Proposition (Kalantan and Almontashery, 2016)

If X is a topological space. Then the following statements are equivalent:

1. Y is the only non-empty regularly closed set in Y.
2. For every non-empty open subset U of Y, $\bar{U} = Y$.
3. each non-empty proper subset of Y is nowhere dense in Y.

2.8. Definition (Engelking, 1977)

A topological space (Y, τ) is called *k -space* if Y is a housrorff and it is a quotient image of a locally compact space.

2.9. Definition (AlZahrani, 2018)

A topological space (Y, τ) is called *Epiregular* if there is a coarser topology τ' on Y such that (Y, τ') is T_3.

2.10. Lemma (Dugundji, 1966)

Let U and V be disjoint open sets in (Y, τ) , then $int(cl(U))$ and $int(cl(V))$ are disjoint open sets in (Y, τ_{semi}) containing U and V respectively.

2.11. Definition (Singal and Arya, 1969)

A topological space (Y, τ) is *ultra regular* if for any closed subset A in Y, and $y \in Y$ such that $y \notin A$ there are two disjoint clopen subsets in Y separates y and A.

2.12. Definition (Engelking, 1977)

A topological space T_1 is called *Zero-dimensional* if it has a base consisting of open-and-closed sets.

2.13. Definition (ALZahrani, 2018)

A topological space (Y, τ) is called C-regular if there exists a bijective function g from Y in to a regular space Z such that the restriction $g_{|_K} : K \to g(K)$ is a homeomorphism for each compact subspace $K \subseteq Y$.

2.14. Definition (Singal and Singal, 1973)

A topological space (Y, τ) is called *k-normal* space if for every two disjoint regularly closed sets of Y can be separated by disjoint open sets.

3. Almost α-Regular

Here we define a decomposition of almost regularity in terms of α-regularity as follows,

3.1. Definition

A topological space (Y, τ) is called almost α-regular if for any $y \in Y$ and a regularly closed subset $B \subset Y$ such that $y \notin B$ there are two disjoint open sets H_1 , $H_2 \subset Y$ such that $y \in H_1$ and $\overline{B \cap H_2} = B$.

3.2. Theorem

Any α-regular space is almost α-regular

Proof.

Let Y be an α-regular space. Let $y \in Y$ and $B \subset Y$ a regularly closed subset not containing y, then B is closed subset and $y \notin B$. Since Y is α-regular then there are two disjoint open sets G, $H \subset Y$ such that $y \in G$ and $\overline{B \cap H} = B$.

3.3. Theorem

Any almost regular space is almost α-regular.

Proof.

Let Y be an almost regular. Let $y \in Y$ and $B \subset Y$ a regularly closed subset not containing y. Since Y is almost regular then there are two disjoint open sets $H_1, H_2 \subset Y$ such that $y \in H_1$ and $B \subseteq H_2$. It follows that $\overline{B \cap H_2} = B$. Hence the space Y is almost α-regular

This guides automatically to ask the following questions:

Question 1: Is there an almost α-regular non almost regular space?

Question 2: Is there an almost α-regular non α-regular space?

Question 3: Is there an almost α-regular non regular? non Tychonoff space?

Question 4: Is there an almost α-regular non normal space?

Question 5: Which almost α-regular space is almost regular space? and which is α-regular?

Note that every regular space is almost α-regular.

Here will be an example of an almost α-regular Hausdorff non almost regular space, which has a large cardinal. To introduce this example, we need the definition of α-normal, which appears in Definition 2.3. Note that any α -normal space satisfying T_1 axiom is α-regular (Murtinová, 2001).

Our example is constructed by Murtinov a' in (Murtinová, 2002), and it is first countable space at each of his points except at one point. It is well known that the cardinality of local base of the point $\omega_1 \in Y$ is the same as the character of the Club filter on ω_1. For revision, see (Balcar & Simon, 1989).

3.4. Example (Murtinová, 2002)

Let $Y = \omega_1 + 1$ and define a topology τ such that: ω_1 with the ordinal topology is an open subspace and a base in the point ω_1 will be the collection:

$$U_F = \{\omega_1\} \cup \{\alpha + 1 : \alpha \in F\}$$

where F is a closed unbounded subset of ω_1 (Club).

The topology τ is Hausdorff since it is stronger than the order topology on $\omega_1 + 1$. The space is not almost regular, consider the set $F \subset \{\alpha < \omega_1 : \alpha \text{ is an ordinal element }\}$ such that F is regularly closed and unbounded, then F is closed (Club) not containing ω_1. However, For every Club D we have,

$$\overline{U_D} \cap F \supset D' \cap F \neq \varnothing$$

where D' is the set of the accumulation points of the club D. Hence ω_1 and F can not be separated.

This space is almost α-regular since it is α-normal satisfying T_1 axiom, see example (2) in (Murtinová, 2002), which implies that Y is α-regular, see (Murtinová, 2001). Hence Y is almost α-regular by Theorem 3.2.

The above example answers question (1). Observe that almost regularity and α-regularity do not imply each other, For example; the set $\mathbb{R}$ with the particlar point topology τ_p, see (Steen & Seebach, 1995), where $p \in \mathbb{R}$ is the paticular point, is not α-regular nor Hausdorff, but it is almost regular since the only regularly closed sets in the space are $\mathbb{R}$ and $\varnothing$. Hence by Theorem 3.3 the space is almost α -regular, and this answers question (2).

It is obvious that every regular space is almost α -regular which means that *Tychonoff Corkscrew* is an example of a regular, almost α -regular, non Tychonoff space, (for details, see (Steen & Seebach, 1995), Ex. 90). Also *Deleted Tychonoff Plank* (for details, see (Steen & Seebach, 1995), Ex. 87) is an example of a Tychonoff almost α -regular non normal space. Another example of an almost α -regular non normal space is *Tychonoff Corkscrew*, and we have answered questions (3) and (4).

3.5. Theorem

For any topological space (Y, τ), the following statements are equivelent:

(1) the space (Y, τ) is almost α -regular;
(2) For each $y \in Y$ and a regularly closed set $A \subseteq Y$ not containing y there is an open set $H \subseteq Y$ such that $\overline{A \cap H} = A \subseteq \bar{H} \subseteq Y - \{y\}$

Proof. (1)

$\Rightarrow$ (2): Let (Y, τ) be an almost α -regular space. Let $y \in Y$ and $A \subseteq Y$ a regularly closed set not containing y, then there are two disjoint open sets H and G such that $y \in G$ and $\overline{A \cap H} = A$, that is; $A \subseteq \bar{H}$, and $\bar{H} \subseteq Y - G \subseteq Y - \{y\}$. Hence $H \subseteq Y$ where $\overline{A \cap H} = A \subseteq \bar{H} \subseteq Y - \{y\}$.

(2) $\Rightarrow$ (1): Let $y \in Y$ and $A \subseteq Y$ a regularly closed set not containing y. By (2) there is an open set $H \subseteq Y$ such that $\overline{A \cap H} = A \subseteq \bar{H} \subseteq Y - \{y\}$. Let $G = Y - \bar{H}$, then $H \cap G = \varnothing$, $\overline{A \cap H} = A$, and $y \in G$. Hence (Y, τ) is almost α -regular.

3.6. Theorem

A topological space (Y, τ) is almost α -regular if and only if for every regularly closed subset $B \subseteq Y$ and $y \in Y$ such that $y \notin B$ there exists an open subset $G \subseteq Y$ such that $y \in G$ and $\bar{G} \cap B$ is nowhere dense in B.

Proof.

Let (Y, τ) be an almost α -regular space, $B \subseteq Y$ be a regularly closed subset, and $y \in Y$ such that $y \notin B$, then there are two disjoint open subsets G and H such that $y \in G$ and $\overline{H \cap B} = B$. It follows that $B \subseteq G^c$ which means that $B \cap G = \varnothing$. Now let $int\left(\overline{\bar{G} \cap B}\right)_B \neq \varnothing$ then there is at least an element $y \in int\left(\overline{\bar{G} \cap B}\right)_B$ so $y \in int\left(\overline{\bar{G} \cap B}\right) \cap B$ which means that $y \in int(\bar{G} \cap B)$ and $y \in B$ so there is an open set W containing y and contained in B such that $W \cap G \neq \varnothing$. This contradicts $B \subseteq G^c$. Therefore $int\left(\overline{\bar{G} \cap B}\right)_B = \varnothing$. Hence $\bar{G} \cap B$ is nowhere dense in B.

On the other direction, Let $B \subseteq Y$ be a regularly closed subset and $y \in Y$ such that $y \notin B$, then there exists an open subset $G \subseteq Y$ such that $y \in G$ and $\bar{G} \cap B$ is nowhere dense in B. Then $int\left(\overline{\bar{G} \cap B}\right)_B = \varnothing$ which implies that $int\left(\overline{\bar{G} \cap B}\right) \cap B = \varnothing$. Now let $y \in B$ then $y \notin int\left(\overline{\bar{G} \cap B}\right)$. It follows that for any open subset W in B containing y then W is not contained in $\overline{\bar{G} \cap B}$. and then $W \cap (\bar{G}^c \cap B) \neq \varnothing$. Let $H = \bar{G}^c$, then H is an open set disjoint from G and $y \in \overline{H \cap B}$, and so $B = \overline{H \cap B}$. Hence (Y, τ) be an almost α-regular space.

3.7. Theorem

Let (Y, τ) be a T_1 space such that for some $y \in Y$, Y is almost α -regular in y , (that is, for every regularly closed E not containing y there are two disjoint open sets G_1 , $G_2 \subseteq Y$ such that $y \in G_1$ and $E = \overline{G_2 \cap E}$) and $Y \backslash \{y\}$ is almost α -normal, then Y is almost α-normal.

Proof.

Let (Y, τ) be a T_1 space such that for some $y \in Y$, Y is almost α -regular in y. Then $\{y\}$ is a closed subset of Y and disjoint from regular closed E and there are two disjoint open sets G_1, $G_2 \subseteq Y$ such that $y \in G_1$ and $E = \overline{G_2 \cap E}$, so consequently Y is almost α -regular in $\{y\}$. Let A, B be two disjoint closed subsets, one of which A is regularly closed, then we have three cases:

1. $y \notin A$, $y \notin B$, then A, B be two disjoint closed subsets, one of which A is regularly closed in $Y \backslash \{y\}$. Since $Y \backslash \{y\}$ is almost α -normal, then there are two disjoint open set G_1 and G_2 such that $A \cap G_1$ is dense in A and $B \cap G_2$ is dense in B.

2. $y \in A$, $y \notin B$. Since Y is T_1 then $Y \backslash \{y\}$ is a dense subset of Y which easily implies that $A \backslash \{y\}$ is regularly closed in $Y \backslash \{y\}$ and $A \backslash \{y\}$ is disjoint from B in $Y \backslash \{y\}$, and since $Y \backslash \{y\}$ is almost α -normal, then there are two disjoint open set G_1 and G_2 in $Y \backslash \{y\}$ such that $(A \backslash \{y\}) \cap G_1$ is dense in $A \backslash \{y\}$ and $B \cap G_2$ is dense in B. Since $Y \backslash \{y\}$ is open, then G_2 is open in Y and for convenience we choose an open set H in Y containing y and $H = G_1 \cap Y \backslash \{y\}$, then $A \cap H$ is dense in A and $B \cap G_2$ is dense in B.

3. $y \notin A$, $y \in B$. Since Y is almost α-regular in some y then there are two disjoint open sets $G, H \subseteq Y$ such that $y \in G$ and $A = \overline{H \cap A}$, it follows $B \subseteq G$ and then $B = \overline{G \cap B}$.

Hence Y is almost α-normal.

The following two theorems give an answer for the first part of question (5).

Recall the definition

$$\mathfrak{p} = \min\{|\mathscr{P}|; \mathscr{P} \subset [\omega]^{\omega}, \mathscr{P} isclosedunderfiniteintersection\}$$

$[\omega]^{\omega} = \{B \subset \omega; |B| = \omega\}$ and $\mathfrak{p}$ be any regular cardinal between $\aleph_1$ and $\mathfrak{c}$.

3.8. Theorem

Let (Y, τ) be a countable almost α -regular, T_2 space, and $w(Y) < \mathfrak{p}$, then Y is almost regular.

Proof.

Let (Y, τ) be a countable almost α -regular T_2 space, $w(Y) < \mathfrak{p}$, $|Y| = \aleph_0$. Suppose that the space Y is not almost regular, and consider $y \in Y$ and a regularly closed set $A \subset Y$ attesting it. Fix an open base $\mathscr{G}$ in y where $|\mathscr{G}| < \mathfrak{p}$.

Therefore we have a centered family of infinite sets $\mathscr{F} = \left\{\overline{B \cap A^{\circ}}; B \in \mathscr{G}\right\}$, $|\mathscr{F}| < \mathfrak{p}$. Then there is an infinite set $F \subset A$ such for any $B \in \mathscr{G}$, $F\ \bar{B}$ is finite. Since Y is T_2, then it is not hard to see that F is closed and discrete not containing y, then F is open so that it is regularly closed and not containing y.

Now by almost α -regularity there is an open set G such that $B \cap G = \varnothing$, $y \in B$, and $F = \overline{G \cap F}$ then $F \subset \bar{G}$, and resulting that $F \backslash (\bar{B} \cap G = \varnothing)$ is finite, which is a contradiction.

The following theorem gives an answer to the first part of question (5).

3.9. Theorem

In extremally disconnected spaces, every almost α-regular is almost regular.

Proof.

Let (Y, τ) be an extremally disconnected spaces and almost α-regular space. Let $y \in Y$ and $A \subseteq Y$ a regularly closed set not containing y. Since Y is almost α-regular, then there are two disjoint open sets G and H such that $y \in H$ and $\overline{A \cap G} = A$, which implies that $A \subseteq \bar{G}$ and $H \cap \bar{G} = \varnothing$. Now since Y is extremally disconnected, then $\bar{G}$ is open. Hence (Y, τ) is almost regular.

3.10. Theorem

Every semiregular, almost β-normal space satisfying T_1 axiom is α -regular.

Proof.

Let Y be a T_1, semiregular, and almost β-normal space. Let $y \in Y$ and A a closed subset of Y such that $y \notin A$, then $Y - A$ is an open subset containing y. By semi-regularity, there is a regularly open set G where $y \in G \subset Y - A$. Here $F = Y - G$ is regularly closed not containing y. The set $\{y\}$ is closed because Y is a T_1 and disjoint from F. But since Y is almost β -normal, there exist two disjoint open sets H_1 and H_2 such that $y \in H_1$, $\overline{F \cap H_2} = F$, and $\bar{H_1} \cap \bar{H_2} = \varnothing$, that is; $F \subseteq \overline{H_2}$. Let $O = Y - \overline{H_1}$ then H_1 and O are disjoint open sets in Y where $y \in H_1$ and $A \subseteq O$. Therefore $\overline{A \cap O} = A$. Hence Y is α -regular.

The following obvious theorem gives an answer to the second part of question (5).

3.11. Theorem

In extremaly disconnected spaces. Every semiregular, almost α -regular space is α -regular.

Proof..

Let (Y, τ) be an extremally disconnected spaces and almost α -regular space. By Theorem 3.9, (Y, τ) is almost regular, and by theorem 3.1 in (Singal and Arya, 1969), every semiregular, almost α-regular space is α-regular.

The following characterization of almost regular spaces induced from (2.5 in (xxxx), and has an important role on many properties of our space which will not be mentioned in this article.

3.12. Theorem

If (Y, τ) is an extremally disconnected space, then (Y, τ) is almost α-regular if and only if for every open set G in (Y, τ) , int$\bar{G}$ is θ -open.

Proof.

Since by Theorem 3.9 every extremally disconnected almost α-regular space is almost regular and by theorem (2.5 in (xxxx) proves the statement.

4. Topological properties of an almost α-regular space

4.1. Theorem

A regularly closed subspace of an almost α-regular space is almost α-regular.

Proof.

Assume Y is almost α-regular space and Z be a regularly closed subspace of Y. Let A be a regularly closed subset of Z and $z \in Z$ such that $z \notin A$ then since Z is regularly closed and by (exercise 2.1.B in (Engelking, 1977) the set A is regularly closed in Y and $z \in Y$ such that $z \notin A$. Now by almost α-regularity of Y, there are two disjoint open sets G_1 and G_2 in Y such that $z \in G_1$ and $\overline{G_2 \cap A} = A$. Thus $G_1 \cap Z$ and $G_2 \cap Z$ are open sets in Z, $(G_1 \cap Z) \cap (G_2 \cap Z) = \varnothing$, $z \in G_1 \cap Z$, and $\overline{G_2 \cap Z \cap A} = A$. Hence Z is almost α-regular.

4.2. Theorem

A regularly open subspace of an almost α-regular space is almost α-regular.

Proof.

Let Z be a regularly open subspace of an almost α-regular space Y. Let A be a regularly closed subset of Z and $z \in Z$ where $z \notin A$, then A^c is regularly open in Z containing z. Since Z is a regularly open subspace of Y then by exercise 2.1.B in (Engelking, 1977) A^c is regularly open in Y containing z, and so A is a regularly closed subset of Y and $z \in Y$ such that $z \notin A$. By almost α-regularity of Y there are two disjoint open sets G_1 and G_2 in Y such that $z \in G_1$ and $\overline{G_2 \cap A} = A$. Since Z is open, then G_1 an G_2 are disjoint open sets in Z such that $z \in G_1$ and $\overline{G_2 \cap A} = A$. Therefore Z is almost α-regular.

4.3. Corollary

Every subspace of an almost α-regular space which is both closed and open is almost α-regular.

Proof.

Any subspace which is both open and closed is obviously regularly closed and therefore the result follows by Theorem 4.1.

It is unknown if every open or closed subspace of an almost α-regular space is almost α-regular space. It is easy to prove the following result,

4.4. Lemma (Engelking, 1977)

(Exercise 1.4.D) *The inverse image of a regularly closed subset under a continuous open map between topological spaces is regularly closed.*

Proof.

Let $h: Y \to Z$ be a continuous open map between topological spaces Y and Z, B be a regularly closed subset of Z, then clearly $\overline{int\left(h^{-1}(B)\right)} \subseteq h^{-1}(B)$, since h is continuous then $h^{-1}(B)$ is closed. Now let $x \in h^{-1}(B)$, since $h^{-1}(B)$ is closed then $\left(h^{-1}(B)\right)' \subseteq h^{-1}(B)$, so for any open set W of x, we have $h^{-1}(B) \cap W \neq \emptyset$. Since h is open then $h(W)$ is open subset of Z and $B \cap h(W) = \overline{int(B)} \cap h(W) \neq \emptyset$. It follows that $int(B) \cap h(W) \neq \emptyset$ and so $h^{-1}(int(B)) \cap W \neq \emptyset$ which implies that $x \in \overline{h^{-1}(int(B))}$. Again by continuity $x \in \overline{int\left(h^{-1}(B)\right)}$, and therefore $h^{-1}(B) \subseteq \overline{int(h^{-1}(B))}$. Thus $h^{-1}(B) = \overline{int\left(h^{-1}(B)\right)}$. Hence $h^{-1}(B)$ is regularly closed.

4.5. Theorem

Let Y be an almost α-regular space, $h: Y \to Z$ is an onto, continuous, open, and closed function. Then Z is almost α-regular.

Proof.

Let Y be an almost α-regular space, B be a regularly closed subset of Z and $z \in Z$ such that $z \notin B$. Then $h^{-1}(B)$ is a closed subset of Y and there exists $y \in Y$ such that $h(y) = z$ and $y \notin h^{-1}(B)$. By Lemma 4.4 the set $h^{-1}(B)$ is regularly closed, and since Y is an almost α-regular space, there are disjoint open subsets G and H of Y such that $y \in H$ and $\overline{h^{-1}(B) \cap G} = h^{-1}(B)$, since $H \cap G = \emptyset$, so $y \notin \bar{G}$. Then $z \notin h(\bar{G})$. It is clear that $h(\bar{G})$ is a closed set containing the open set $h(G)$, $\overline{h(G)} \subseteq h(\bar{G})$. Thus $z \notin \overline{h(G)}$ which implies $z \in h(H)$ and $h(G) \cap h(H) = \emptyset$. Now we show that $\overline{B \cap h(G)} = B$, it is sufficient to show that $B \subseteq \overline{B \cap h(G)}$. Let $b \in B$ and W be any open set containing b, then $h^{-1}(b) \subseteq h^{-1}(B) \cap h^{-1}(W)$. Since $\overline{h^{-1}(B) \cap G} = h^{-1}(B)$, $h^{-1}(B) \cap G \cap h^{-1}(W) \neq \emptyset$. Hence by surjectivity of h, $B \cap h(G) \cap W \neq \emptyset$ which implies $b \in \overline{B \cap h(G)}$.

The following theorem shows a result on a quotient space defined on an almost α-regular space, and it is induced from (Singal and Arya, 1969)

4.6. Theorem

Let (Y, τ) be an almost α-regular space and define an equivalence relation R in Y by setting xRy iff $\{\bar{x}\} = \{\bar{y}\}$. If $p: Y \to Y/R$ be the projection map of Y onto the quotient space Y/R . Then Y/R is almost α -regular.

Proof.

Similar argument of Theorem 4.5.

4.7. Corollary

Almost α-regularity is a topological property.

4.8. Theorem

Let Y be an extremally disconnected, Hausdorff, almost α-regular space, $B \subseteq Y$ is a regularly closed set. Then Y/B is a Hausdorff space.

Proof.

Let $y, z \in Y/B$ such that $y \neq z$. If neither one is an element of $[B]$, the existence of disjoint open sets follows by nothing that $Y - B$ is Hausdorff as a subspace of Y. If $z = [B]$, then $p^{-1}(y)$ is a single point and $p^{-1}(y) \notin B$. By almost α -regularity of Y and Theorem 3.5 there exists an open set G such that $\overline{B \cap G} = B \subseteq \bar{G} \subseteq Y - \{p^{-1}(y)\}$. Let $K = \bar{G}^c$ then $p^{-1}(y) \in K$, and $B \subseteq \bar{G}$. Since Y is extremally disconnected then $\bar{G}$ is an open set containing B and disjoint from K. Now clearly the images $p(\bar{G})$ and $p(K)$ are disjoint open sets in Y/B containing B and y respectively. Hence Y/B is a Hausdorff space.

Note that any almost β-normal T_1 is almost regular and hence almost α -regular. However; Example 3.4 is an example of an almost α -regular non almost β -normal space since it is not almost regular.

It is well known that regularity and almost regularity are invariant under products, however, this is not the case for α -regularity as Murtinov a' in (Murtinová, 2001) proved that α -regularity is not preserved under products. Regarding Murtinov a' result in (Murtinová, 2001), the following theorem proves that almost α -regularity does not be preserved by products and at the same time we construct a non almost α -regular space from a non almost regular space.

4.9. Theorem

Let $A(\kappa)$ is the one-point compactification of a discrete set of cardinality κ . Then for every non-almost regular T_1 space Y there is $\kappa \leq \chi(Y)$ such that $Y \times A(\kappa)$ is not almost α -regular.

Proof.

Suppose that $y \in Y$ and L is a regularly closed subset of Y such that $y \notin L$ and there are no disjoint open sets in Y that separate y and L. Choose an open base $\mathscr{B}$ for y such that $|\mathscr{B}| \leq \chi(Y)$.

Now let $Z \simeq A(|\mathscr{B}|)$ be the one-point compactification of the set $\mathscr{B}$ with discrete topology, and ∞ be the compactifying point. The aim of this theorem is to show that $Y \times Z$ is not an almost α -regular space.

Define the set

$$F = \left\{(z, B); z \in Y, B \in \mathscr{B}, z \in \overline{L^{\circ} \cap B}\right\} \cup \left(L \times \bar{B}\right)\}$$

Claim:

The set F is regularly closed.

Note that the set F is closed, and clearly $\overline{F^{\circ}} \subseteq F$. Now let $(z, B) \notin \overline{F^{\circ}}$, then $(Y \backslash (\overline{L^{\circ} \cap B}\})) \times \{B\}$ is an open neighborhood of (z, B) which does not meet $\overline{F^{\circ}}$ since $\overline{L^{\circ} \cap B}$ is regularly closed.

Therefore $\left(Y \backslash \left(\overline{L^{\circ} \cap B}\right\}\right)) \times \{B\}$ is an open neighborhood of (z, B) which does not meet F. Therefore F is regularly closed. Moreover, $((Y \backslash L) \times Y)$ does not meet F which implies that $((Y \backslash L) \times \{\infty\})$. So clearly $(z, \infty) \notin F$.

Suppose that $Y \times Z$ is an almost α -regular space in the point (z, ∞). Then in paretical, there is $B \in \mathscr{B}$ and an open neighborhood $H = Z \backslash \{B_1, B_2, \ldots, B_k\}$ of ∞ in Z where

$$\overline{F \backslash (\overline{B} \times H)} = \overline{F \backslash (\bar{B} \times H)} = F$$

Since z is not isolated we can use T_1 axiom in Y to find $B_0 \in \mathscr{B}$ such that $B_0 \subset B$, $B_0 \notin \{B_1, B_2, \ldots, B_k\}$. Pick $z_0 \in \overline{L^{\circ} \cap B_0}$, then $(z_0, B_0) \in F$ and $W = Y \times \{B_0\}$ is an open neighborhood of (z_0, B_0) in $Y \times Z$. Observe that $B_0 \in H$ and $(z, B_0) \in F$ implies that $z \in \bar{F}$. Now it is not hard to see that $W \cap F \subset \bar{B} \times H$, therefore $W \cap (F \backslash (\bar{B} \times H)) = \varnothing$ which is a contradiction. Hence $Y \times Z$ is not an almost α -regular space

It is not hard to show that any T_1 space meets the statements in proposition 2.7 is almost regular and hence it is almost α -regular. By a similar argument of corollary (0.8 in (Kalantan and Almontashery, 2016), we prove the following.

4.10. Corollary

If Y is a T_1 space meets the conditions of proposition 2.7, then the Alexandroff Duplicate $A(Y)$ of it is almost α -regular.

The Topological space of Alexandroff Duplicate $A(Y)$ in (Engelking, 1968), we have the following theorem.

4.11. Theorem (Gheith, 2019)

If (Y, τ) is α -regular satisfying T_1 axiom, then the Alexandroff Duplicate $A(Y)$. is α -regular.

Proof.

Let F be a closed subset in $A(Y)$ and $y' \in A(Y)$ such that $y' \notin E$. Write $F = F_1 \cup F_2$ where $F_1 = F \cap Y$ and $F_2 = F \cap Y'$. Since $y' = (y, 1)$ where $y \in Y$, then $y \notin F_1$ in Y. By α -regularity of Y there are two disjoint open sets G and H in Y where $F_1 \cap G$ is dense in F_1 and $y \in H$. since Y is T_1 then we can choose $W_1 = (G \cup G' \cup F_2) \backslash \{y'\}$ and $W_2 = (H \cup H' \cup \{y'\})$. Then W_1 and W_2 are disjoint open sets in $A(Y)$ and $y' \in W_2$. It is enough to show that $W_1 \cap F$ is dense in F. Observe that $W_1 \cap F = (W_1 \cap F_1) \cup (W_1 \cap F_2) = (G \cap F_1) \cap F_2$. Therefore $\overline{W_1 \cap F} = \overline{(G \cap F_1) \cap F_2} = \left(\overline{G \cap F_1}\right) \cap \overline{F_2} \supset F_1 \cup \overline{F_2} \supset F$, and so $W_1 \cap F$ is dense in F. Hence $A(Y)$ is α -regular.

4.12. Theorem

Let (Y, τ) be an almost α -regular space satisfying T_1 axiom. If (Y, τ) is α -regular, then the Alexandroff Duplicate $A(Y)$ is almost α -regular.

Proof.

Let (Y, τ) be an almost α -regular space. If (Y, τ) is α -regular, then by Theorem 4.11 the Alexandroff Duplicate $A(Y)$ is α -regular, and since every α -regular space is almost α -regular by Theorem 3.2, then the Alexandroff Duplicate $A(Y)$ is almost α -regular.

5. Relations with some other separation axioms

5.1. Theorem

Every almost α -normal space satisfying T_1 axiom is almost α -regular.

Proof.

Let Y be an almost α -normal space satisfying T_1. Let $y \in Y$ and A a regularly closed subset of Y such that $y \notin A$. Since Y is T_1, then $\{y\}$ is closed set disjoint from A. Now by almost α -normality, there are two disjoint open sets $G_1, G_2 \subset Y$ such that $y \in G_1$ and $\overline{A \cap G_2} = A$ as required.

The Michael product space $\mathbb{M} \times \mathbb{P}$, (Steen & Seebach, 1995) is an almost α -regular T_1 non almost α -normal space. It is almost α -regular because it is regular, and it is not an almost α -normal space by similar argument used in (Kalantan and Allahabi, 2008). Hausdorff spaces and almost α -regular spaces do not imply each others, For example, the finite complement topology defined on an infinite set Y see (Steen & Seebach, 1995) is an example of an almost α -regular non Huasdorrf space. And Alexandroff plank (Steen & Seebach, 1995) is an example of a Hausdorff non almost α -regular space.

The definition of k -space appears in Definition 2.8. Note that a closed subset of a k -space is a k -space [Arhangel'skii]. we get the following result.

5.2. Theorem

In the class of extremally disconnected spaces, every regularly open subspace of almost α -regular space, k -space is k -space.

Proof.

Straightforward by Theorem 4.2 and theorem [7.1] in (Singal and Arya, 1969).

Murtinová in (Murtinová, 2001) proved that Every first countable, Hausdorff, α -regular space is regular. Regarding her result, we have the following,

5.3. Theorem

Each first countable, Hausdorff, almost α -regular space is almost regular.

Proof.

Using a contradiction, we suppose that Y is a first countable, Hausdorff and non almost regular. Then there is an $y \in Y$ and a regularly closed subset F of Y such that $y \notin F$ where there are no disjoint open sets that separate them. Let $\{U_n : n \in \omega\}$ be an open base in y such that $U_{n+1} \subset U_n$ for all $n \in \omega$. Let $H = \left\{y_n : y_n \in \overline{U_n \cap F^{\circ}}, n \in \omega\right\}$.

Note that y_n was chosen inductively and because the space Y is Hausdorff, we can also suppose at each step of the induction that $y_n \notin \overline{U_{n+1}}$, it follows that $y_n \in \overline{U_m}$ if and only if $m \leq n$.

The set H is regularly closed. Indeed, if $y \notin \overline{H^{\circ}}$, then $Y \Big\backslash \left(\overline{U_n \cap F^{\circ}}\right)$ is a regularly open set containing y and not intersecting $\overline{H^{\circ}}$ which implies that $Y \backslash (\overline{U_n} \cap F)$ is a neighborhood open set containing y and not intersecting H. Therefore $y \notin H$. Since y and H can not be separated, so Y is not an almost α -regular space.

An example of almost α -regular non regular is an example (113 in (Steen & Seebach, 1995), this answer for the first part of question (3), and it is as follows:

Let $\mathbb{N}$ be the set of nutural numbers and $\mathscr{F}$ be the collection of all ultra filters defined on $\mathbb{N}$. Let $Y = \mathbb{N} \cup \mathscr{F}$. Let τ the topology defined on Y generated by the points $y \in \mathbb{N}$, that is; the point $y \in \mathbb{N}$ are isolated, and the collection $\{A \cup \{F\} : A \in F \in \mathscr{F}\}$. Kalantan in (Kalantan and Allahabi, 2008) showed that this is an extremally disconnected almost normal Hausdorff non regular space. Therefore this is another example of an almost regular, almost α -regular non regular space.

We introduced epiregular spaces in (AlZahrani, 2018) and it appears in Definition 2.9. Almost α -regular spaces do not imply epiregular spaces. An example of this, Each indiscrete space which has more than one point is an example of an almost α -regular non epiregular space. On the other direction, if (Y,τ) is epiregular and the witness of epiregularity (Y,τ') is semiregular, then (Y,τ) is almost regular, and hence it is almost α -regular. The family of all regularly open sets formes a base for a topology τ_{semi} on Y, this topology τ_{semi} is known as semi-regularization of (Y,τ). Note that $\tau_{semi} \subset \tau$.

Considering the semi-regularization τ_{semi} of a space Y we still have the following correct.

5.4. Theorem

If the space (Y,τ_{semi}) is α -regular, then (Y,τ) is almost α -regular.

Proof.

Suppose that (Y,τ_{semi}) is α -regular. Let A be a regularly closed subset in (Y,τ), and $y \in Y$ such that $y \notin A$. Now since (Y,τ_{semi}) is α -regular, then it is almost α -regular by Theorem 1.2. Therefore there are two disjoint open subsets G_1 and G_2 in (Y,τ_{semi}) such that $y \in G_2$ and $\overline{G_1 \cap A} = A$. Since (Y,τ_{semi}) is coarser than (Y,τ), then G_1 and G_2 are in (Y,τ). Hence (Y,τ) is almost α -regular.

5.5. Definition

A topological space (Y,τ) is called weakly α -regular if for any $y \in Y$ and a closed subset $B \subset Y$ not containing y there are two disjoint open sets G_1 , $G_2 \subset Y$ such that $y \in G_1$ and $B \subseteq \bar{G}_2$.

Note that every α -regular space is weakly α -regular. Howevere; In extremallly disconnected space, the concept of an α -regular space and a weakly α -regular space are the same.

We still have the following true.

5.6. Theorem

If the space (Y,τ) is almost α -regular, then (Y,τ_{semi}) is weakly α -regular.

Proof.

Assume (Y,τ) is almost α -regular. Let A be a closed subset in (Y,τ_{semi}), and $y \in Y$ such that $y \notin A$. Consider $A = \cap_{i\in I} A_i$ where A_i are regularly closed in τ and $y \notin A_i$ for every $i \in I$. By almost α -regularity of (Y,τ), there are two disjoint open subsets G and H in (Y,τ) where $y \in G$ and $\overline{H \cap A_i} = A_i$. Therefore $A_i \subseteq \bar{H}$ for all $i \in I$, then $A \subseteq \bar{H}$. By lemma 2.10 there are two disjoint open sets W_1 and W_2 in (Y,τ_{semi}) where $G \subseteq W_1$ and $H \subseteq W_2$. Therefore $y \in W_1$ and $A \subseteq \overline{W_2}$, and $W_1 \cap W_2 = \emptyset$. Hence (Y,τ_{semi}) is weakly α -regular.

Observe that in extremally disconnected spaces, Theorem 5.6 and Theorem 5.4 will be as follows:

5.7. Corollary

If (Y,τ) is an extremally disconnected space, then the space (Y,τ) is almost α -regular if and only if (Y,τ_{semi}) is almost α -regular.

5.8. Corollary

Every semiregular, almost α -regular space is weakly α -regular.

5.9. Corollary

Every subspace of a semiregular almost α -regular space is weakly α -regular space.

We referred to ultra regular in Definition 2.11. Therefore the following is clear.

5.10. Theorem

Every ultra regular is almost α -regular.

The other side of the above theorem is not always correct. Example 3.4 is an example of almost α -regular non ultra regular space. In zero-dimensional spaces, we have the following result,

5.11. Corollary

If (Y,τ) is a zero-dimensional semiregular space, then the space (Y,τ) is almost α -regular if and only if (Y,τ) is ultra regular.

Note that any compact C -regular space is regular, see (AlZahrani, 2018), and hence is almost α -regular. Considering that, the finite particular point topology is an example of a compact non regular and hence non C -regular space. However this is an example of an almost α -regular space. On the other side, C -regularity does not imply almost α -regular. For example; Theorem 4.9 shows that the product of any almost α -regular, non almost regular T_1 space Y and a compact zero-dimensional space fail to be almost α -regular, and if we assume that Y is C -regular, then the product space would be an example of C -regular, non almost α -regular space since C -regularity is multiplicative property, see theorem [2.16] in (AlZahrani, 2018).

We referred to k-normal in Definition 2.14. Since every almost regular lindel ö f space is k -normal (Singal and Singal, 1973), the proof of the following theorem is immediate by applying Theorem 3.8.

5.12. Theorem

Every countable almost α -regular, lindel ö f, T_2 space, and $w(Y) < \mathfrak{p}$, then Y is k -normal.

6. Conclusion

The aim of this paper is to introduce a new weaker version of regularity called almost α -regular. This difenition leads naturally to ask five questions, we have answered this questions in pages 5, 7, 8, and 15. We discuss some topological properties of almost α-regular and we show that some relationships between this a new space and some other topological spaces.

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