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Estimation type results associated to k-fractional integral inequalities with applications

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ABSTRACT

This paper aims to establish k-fractional integral inequalities with multiple parameters involving the first differentiable mappings. Applications of our results to random variables are also given.

1. Introduction

If $h:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping and $e_1,e_2\in\mathcal{I}$ with $e_1<e_2$, then one has

$$h\left(\frac{e_1+e_2}{2}\right)\leqslant\frac{1}{e_2-e_1}\int_{e_1}^{e_2}h(t)\mathrm{d}t\leqslant\frac{h(e_1)+h(e_2)}{2}. \tag{1.1}$$

The subsequent inequality is an improvement of the inequality (1.1).

$$h\left(\frac{e_1+e_2}{2}\right)\leqslant\frac{1}{2}\left[h\left(\frac{3e_1+e_2}{4}\right)+h\left(\frac{e_1+3e_2}{4}\right)\right]\leqslant\frac{1}{e_2-e_1}\int_{e_1}^{e_2}h(t)\mathrm{d}t \leqslant\frac{1}{2}\left[h\left(\frac{e_1+e_2}{2}\right)+\frac{h(e_1)+h(e_2)}{2}\right]\leqslant\frac{h(e_1)+h(e_2)}{2}. \tag{1.2}$$

Many different generalizations, new extensions and improvements related to the inequality (1.1) can be found in Awan et al. (2016), Chu et al. (2016, 2017), Du et al. (2017), Iqbal et al. (2018), İşcan et al. (2016), Khan et al. (2017a,b,c), Khan et al. (2018a,b,c), Sarikaya and Karaca (2014), Set et al. (2016), Xi et al. (2015).

Let us consider an invex set $\mathcal{K}$. A set $\mathcal{K}\subseteq\mathbb{R}^n$ is named invex set with respect to the mapping $\delta:\mathcal{K}\times\mathcal{K}\to\mathbb{R}^n$, if $v+\lambda\delta(\mu,v)\in\mathcal{K}$ holds, for all $\mu,v\in\mathcal{K}$ and $\lambda\in[0,1]$. A mapping $h:\mathcal{K}\to\mathbb{R}$ is called preinvex respecting δ, if the following inequality:

$$h(v+\lambda\delta(\mu,v))\leqslant(1-\lambda)h(v)+\lambda h(\mu) \tag{1.3}$$

grips, for every $\mu,v\in\mathcal{K}$ and $\lambda\in[0,1]$.

The preinvex function is an important substantive generalization of the convex function.

Example 1.1. The function $h(x)=\frac{1}{2}-\left|x-\frac{1}{2}\right|$, $(x\in\mathbb{R})$ with respect to the following

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$$\delta(y,x)=\begin{cases}2(y-x), & x\geqslant\frac{1}{2},y\geqslant\frac{1}{2},y\geqslant x,\\ 0, & x>\frac{1}{2},y\geqslant\frac{1}{2},y<x,\\ 0, & x<\frac{1}{2},y\leqslant\frac{1}{2},y>x,\\ y-x, & x\leqslant\frac{1}{2},y\leqslant\frac{1}{2},y\leqslant x,\\ 1-x-y, & x<\frac{1}{2},y>\frac{1}{2},x+y\geqslant 1,\\ 0, & x<\frac{1}{2},y>\frac{1}{2},x+y\leq 1,\\ 0, & x>\frac{1}{2},y<\frac{1}{2},x+y\geqslant 1,\\ 1-x-y, & x>\frac{1}{2},y<\frac{1}{2},x+y\leqslant 1\end{cases}\tag{1.4}$$

is preinvex on $\mathbb{R}$ while it is not convex on $\mathbb{R}$.

This paper aims to obtain estimation type results related to k-fractional integral operators. For this, we suppose that the absolute value of the derivative of the considered mapping is preinvex. Next, we substitute this hypothesis with the boundedness of the derivative and with a Lipschitzian condition for the derivative of the considered mapping to derive integral inequalities with new bounds. Applications of our results to random variables are also provided.

We end this section by reciting a well-known k-fractional integral operators in the literature.

Definition 1.1 Mubeen and Habibullah (2012). Let $h\in L^1[\mu,\nu]$, the k-fractional integrals ${}_k\mathcal{J}^{\tau}_{\mu^+}h(x)$ and ${}_k\mathcal{J}^{\tau}_{\nu^-}h(x)$ of order $\tau>0$ are defined by

$${}_k\mathcal{J}^{\tau}_{\mu^+}h(x)=\frac{1}{k\Gamma_k(\tau)}\int_{\mu}^{x}(x-\lambda)^{\frac{\tau}{k}-1}h(\lambda)\mathrm{d}\lambda,\quad(0\leqslant\mu<x<\nu)$$

and

$${}_k\mathcal{J}^{\tau}_{\nu^-}h(x)=\frac{1}{k\Gamma_k(\tau)}\int_{x}^{\nu}(\lambda-x)^{\frac{\tau}{k}-1}h(\lambda)\mathrm{d}\lambda,\quad(0\leqslant\mu<x<\nu),$$

respectively, where $k>0$, and Γ_k is the k-gamma function defined as $\Gamma_k(x):=\int_0^{\infty}\lambda^{x-1}e^{-\frac{\lambda^k}{k}}\mathrm{d}\lambda$, $\mathrm{Re}(x)>0$, with the properties $\Gamma_k(x+k)=x\Gamma_k(x)$ and $\Gamma_k(k)=1$.

2. Main results

Throughout this work, let $\mathcal{K}\subseteq\mathbb{R}$ be an open invex subset respecting $\delta:\mathcal{K}\times\mathcal{K}\to\mathbb{R}\setminus\{0\}$ and $r_1,r_2\in\mathcal{K}$ with $r_1<r_2$, and let $h:\mathcal{K}\to\mathbb{R}$ be a differentiable mapping such that h' is integrable on the δ-path $P_{x\nu}:\ \nu=x+\delta(y,x)$ with $x,y\in[r_1,r_2]$. Before stating the results, we use notations below.

$$\begin{aligned}&\mathcal{L}_{\delta}(\tau,k,\lambda;x)\\&:=(1-\lambda)\frac{\delta^{\frac{\tau}{k}}(x,r_1)(h(r_1+\delta(x,r_1))+h(r_1))+\delta^{\frac{\tau}{k}}(r_2,x)(h(x)+h(x+\delta(r_2,x)))}{\delta(r_2,r_1)}\\&\quad+\frac{2\lambda\left(\delta^{\frac{\tau}{k}}(x,r_1)h\left(r_1+\frac{1}{2}\delta(x,r_1)\right)+\delta^{\frac{\tau}{k}}(r_2,x)h\left(x+\frac{1}{2}\delta(r_2,x)\right)\right)}{\delta(r_2,r_1)}\\&\quad-\frac{2^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{\delta(r_2,r_1)}\left[{}_k\mathcal{J}^{\tau}_{[r_1+\delta(x,r_1)]^-}h\left(r_1+\frac{1}{2}\delta(x,r_1)\right)+{}_k\mathcal{J}^{\tau}_{r_1^+}h\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right.\\&\quad\left.+{}_k\mathcal{J}^{\tau}_{x^+}h\left(x+\frac{1}{2}\delta(r_2,x)\right)+{}_k\mathcal{J}^{\tau}_{[x+\delta(r_2,x)]^-}h\left(x+\frac{1}{2}\delta(r_2,x)\right)\right].\end{aligned}\tag{2.1}$$

If $\delta(\mu,\nu)=\mu-\nu$ with $\mu,\nu\in[r_1,r_2]$, then we have

$$\begin{aligned}&\mathcal{L}(\tau,k,\lambda;x)\\&:=(1-\lambda)\frac{(x-r_1)^{\frac{\tau}{k}}[h(r_1)+h(x)]+(r_2-x)^{\frac{\tau}{k}}[h(x)+h(r_2)]}{r_2-r_1}\\&\quad+\frac{2\lambda\left[(x-r_1)^{\frac{\tau}{k}}h\left(\frac{r_1+x}{2}\right)+(r_2-x)^{\frac{\tau}{k}}h\left(\frac{x+r_2}{2}\right)\right]}{r_2-r_1}-\frac{2^{\frac{\tau}{k}}\Gamma_k(\tau+k)}{r_2-r_1}\\&\quad\times\left[{}_k\mathcal{J}^{\tau}_{r_1^+}h\left(\frac{r_1+x}{2}\right)+{}_k\mathcal{J}^{\tau}_{x^-}h\left(\frac{r_1+x}{2}\right)+{}_k\mathcal{J}^{\tau}_{x^+}h\left(\frac{x+r_2}{2}\right)+{}_k\mathcal{J}^{\tau}_{r_2^-}h\left(\frac{x+r_2}{2}\right)\right].\end{aligned}\tag{2.2}$$

(a) Choosing $\lambda=0$ and $\tau=k=1$, the Eq. (2.2) reduces to

$$\mathcal{L}(1,1,0;x):=h(x)+\frac{(r_2-x)h(r_2)+(x-r_1)h(r_1)}{r_2-r_1}-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u.$$

Specially,

$$\mathcal{L}\left(1,1,0;\frac{r_1+r_2}{2}\right):=\frac{h(r_1)+h(r_2)}{2}+h\left(\frac{r_1+r_2}{2}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u.$$

(b) Choosing $\lambda=1$ and $\tau=k=1$, the Eq. (2.2) degenerates into

$$\mathcal{L}(1,1,1;x):=\frac{2\left[(x-r_1)h\left(\frac{r_1+x}{2}\right)+(r_2-x)h\left(\frac{x+r_2}{2}\right)\right]}{r_2-r_1}-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u.$$

Specially,

$$\mathcal{L}\left(1,1,1;\frac{r_1+r_2}{2}\right):=h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u.$$

We need the following lemma for our results.

Lemma 2.1. *One has the subsequent identity*

$$\begin{aligned}\mathcal{L}_{\delta}(\tau,k,\lambda;x)=&\frac{\delta^{\frac{\tau}{k}+1}(x,r_1)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)\mathrm{d}t\right.\\&\left.-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)\mathrm{d}t\right\}\\&-\frac{\delta^{\frac{\tau}{k}+1}(r_2,x)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)\mathrm{d}t\right.\\&\left.-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)\mathrm{d}t\right\}\end{aligned}\tag{2.3}$$

for k-fractional integrals with $x\in(r_1,r_2)$, $\tau>0$, $k>0$ and $\lambda\in[0,1]$.

Proof. Using integration by parts and appropriate substitutions, such as $\mu=r_1+\frac{1+t}{2}\delta(x,r_1)$, $\nu=r_1+\frac{1-t}{2}\delta(x,r_1)\ldots$, we can obtain the identity (2.3). □

Example 2.1. If we take $r_1=\frac{1}{2}$, $r_2=1$, $\tau=1=k$, $h(x)=\frac{1}{2}-\left|x-\frac{1}{2}\right|$, and

$$\delta(\mu,\nu)=\begin{cases}2(\mu-\nu), & \mu\geqslant\nu,\\ 0, & \mu<\nu,\end{cases}\tag{2.4}$$

then all the assumptions in Lemma 2.1 are satisfied. Clearly, the left of the Eq. (2.3) is

$$\begin{aligned}\mathcal{L}_{\delta}(1,1,\lambda;x)=&(1-\lambda)\frac{2\left(x-\frac{1}{2}\right)\left[\left(\frac{3}{2}-2x\right)-\frac{1}{2}\right]+2(1-x)[(1-x)+(x-1)]}{2\left(1-\frac{1}{2}\right)}\\&+2\lambda\frac{2\left(x-\frac{1}{2}\right)[1-x]+2(1-x)[1-1]}{2\left(1-\frac{1}{2}\right)}\\&-2\left[\int_{\frac{1}{2}}^{x}(1-t)\mathrm{d}t+\int_x^{2x-\frac{1}{2}}(1-t)\mathrm{d}t+\int_x^1(1-t)\mathrm{d}t\right.\\&\left.+\int_1^{2-x}(1-t)\mathrm{d}t\right]=0\end{aligned}$$

and the right of the Eq. (2.3) is

$$4\left(x-\frac{1}{2}\right)^2\left\{\int_0^1 -(t-\lambda)\mathrm{d}t+\int_0^1 (t-\lambda)\mathrm{d}t\right\}$$
$$-4(1-x)^2\left\{\int_0^1 -(t-\lambda)\mathrm{d}t+\int_0^1 (t-\lambda)\mathrm{d}t\right\}=0.$$

Corollary 2.1. *If $\delta(\mu,v)=\mu-v$ with $\mu,v\in[r_1,r_2]$ in Lemma 2.1, then one has*

$$\begin{aligned}\mathcal{L}(\tau,k,\lambda;x)=&\frac{(x-r_1)^{\frac{\tau}{k}+1}}{2(r_2-r_1)}\left\{\int_0^1 (t^{\frac{\tau}{k}}-\lambda)h'\left(\frac{1+t}{2}x+\frac{1-t}{2}r_1\right)\mathrm{d}t\right.\\&\left.-\int_0^1 (t^{\frac{\tau}{k}}-\lambda)h'\left(\frac{1-t}{2}x+\frac{1+t}{2}r_1\right)\mathrm{d}t\right\}\\&-\frac{(r_2-x)^{\frac{\tau}{k}+1}}{2(r_2-r_1)}\left\{\int_0^1 (t^{\frac{\tau}{k}}-\lambda)h'\left(\frac{1+t}{2}x+\frac{1-t}{2}r_2\right)\mathrm{d}t\right.\\&\left.-\int_0^1 (t^{\frac{\tau}{k}}-\lambda)h'\left(\frac{1-t}{2}x+\frac{1+t}{2}r_2\right)\mathrm{d}t\right\},\end{aligned}\tag{2.5}$$

which is a new form of Lemma 2.1.

Remark 2.1. (a) In Lemma 2.1, putting $\lambda=0$ and $k=1$, one has Lemma 2.8 in Noor et al. (2016). (b) In Corollary 2.1, taking $\lambda=0$ and $k=1$, one has Lemma 1 in Mihai and Mitroi (2014). Further, choosing $\tau=1$, one has Lemma 1 in Latif (2015).

Theorem 2.1. *If $|h'|^q$ for $q>1$ is preinvex on $\mathcal{K}$, then the coming inequality for k-fractional integrals with $x\in(r_1,r_2)$, $\tau>0$, $k>0$, $p^{-1}+q^{-1}=1$ and $\lambda\in[0,1]$ grips:*

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k,\lambda;x)|\leqslant&\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\rho^{\frac{1}{p}}(\tau,k,\lambda,p)\left\{\left[\frac{|h'(r_1)|^q+3|h'(x)|^q}{4}\right]^{\frac{1}{q}}\right.\\&\left.+\left[\frac{3|h'(r_1)|^q+|h'(x)|^q}{4}\right]^{\frac{1}{q}}\right\}\\&+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\rho^{\frac{1}{p}}(\tau,k,\lambda,p)\left\{\left[\frac{3|h'(x)|^q+|h'(r_2)|^q}{4}\right]^{\frac{1}{q}}\right.\\&\left.+\left[\frac{|h'(x)|^q+3|h'(r_2)|^q}{4}\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{2.6}$$

where

$$\rho(\tau,k,\lambda,p)=\begin{cases}\frac{k}{\tau p+k}, & \lambda=0,\\ \frac{k}{\tau p+k}-\frac{2k}{\tau p+k}\lambda^{\frac{\tau p+k}{\tau}}-\lambda^p+2\lambda^{\frac{\tau p+k}{\tau}}, & 0<\lambda<1,\\ \frac{\Gamma(p+1)\Gamma\left(\frac{k}{\tau}+1\right)}{\Gamma\left(\frac{k}{\tau}+1+p\right)}, & \lambda=1.\end{cases}\tag{2.7}$$

Proof. According to Lemma 2.1, the Holder inequality and preinvexity of $|h'|^q$, one gets

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k,\lambda;x)|\leqslant&\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\left[\int_0^1 |t^{\frac{\tau}{k}}-\lambda|^p\mathrm{d}t\right]^{\frac{1}{p}}\left((\chi_1)^q+(\chi_2)^q\right)\\&+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\left[\int_0^1 |t^{\frac{\tau}{k}}-\lambda|^p\mathrm{d}t\right]^{\frac{1}{p}}\left((\chi_3)^q+(\chi_4)^q\right),\end{aligned}$$

where

$$\begin{aligned}\chi_1=&\int_0^1\left|h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)\right|^q\mathrm{d}t\\ \leqslant&\int_0^1\left(\frac{1-t}{2}|h'(r_1)|^q+\frac{1+t}{2}|h'(x)|^q\right)\mathrm{d}t=\frac{|h'(r_1)|^q+3|h'(x)|^q}{4},\end{aligned}$$

$$\chi_2=\int_0^1\left|h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)\right|^q\mathrm{d}t\leqslant\frac{3|h'(r_1)|^q+|h'(x)|^q}{4},$$

$$\chi_3=\int_0^1\left|h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)\right|^q\mathrm{d}t\leqslant\frac{3|h'(x)|^q+|h'(r_2)|^q}{4},$$

$$\chi_4=\int_0^1\left|h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)\right|^q\mathrm{d}t\leqslant\frac{|h'(x)|^q+3|h'(r_2)|^q}{4}.$$

When $\lambda=0$, we have

$$\int_0^1 |t^{\frac{\tau}{k}}-\lambda|^p\mathrm{d}t=\frac{k}{\tau p+k},$$

when $\lambda=1$, we have

$$\int_0^1 |t^{\frac{\tau}{k}}-\lambda|^p\mathrm{d}t=\frac{\Gamma(p+1)\Gamma\left(\frac{k}{\tau}+1\right)}{\Gamma\left(\frac{k}{\tau}+1+p\right)},$$

when $0<\lambda<1$, we have

$$\begin{aligned}\int_0^1 |t^{\frac{\tau}{k}}-\lambda|^p\mathrm{d}t=&\int_0^{\lambda^{\frac{k}{\tau}}}\left(\lambda-t^{\frac{\tau}{k}}\right)^p\mathrm{d}t+\int_{\lambda^{\frac{k}{\tau}}}^1\left(t^{\frac{\tau}{k}}-\lambda\right)^p\mathrm{d}t\\ \leqslant&\int_{\lambda^{\frac{k}{\tau}}}^1\left(t^{\frac{\tau p}{k}}-\lambda^p\right)\mathrm{d}t+\int_0^{\lambda^{\frac{k}{\tau}}}\left(\lambda^p-t^{\frac{\tau p}{k}}\right)\mathrm{d}t\\ =&\frac{k}{\tau p+k}-\frac{2k}{\tau p+k}\lambda^{\frac{\tau p+k}{\tau}}-\lambda^p+2\lambda^{\frac{\tau p+k}{\tau}}.\end{aligned}$$

The above inequality is obtained by using the fact that

$(\alpha-\beta)^\kappa\leqslant\alpha^\kappa-\beta^\kappa,\alpha\geqslant\beta\geqslant0,\kappa\geqslant1.$

The proof of Theorem 2.1 is completed. □

Corollary 2.2. *Choosing $\lambda=1$, $k=1$ and $\delta(\mu,v)=\mu-v$ for $\mu,v\in[r_1,r_2]$ in Theorem 2.1, one obtains*

$$\begin{aligned}|\mathcal{L}(\tau,1,1;x)|:=&\left|\frac{2\left[(x-r_1)^\tau h\left(\frac{r_1+x}{2}\right)+(r_2-x)^\tau h\left(\frac{x+r_2}{2}\right)\right]}{r_2-r_1}-\frac{2^\tau\Gamma(\tau+1)}{r_2-r_1}\right.\\&\times\left[\mathcal{J}^\tau_{r_1^+}h\left(\frac{r_1+x}{2}\right)+\mathcal{J}^\tau_{x^-}h\left(\frac{r_1+x}{2}\right)+\mathcal{J}^\tau_{x^+}h\left(\frac{x+r_2}{2}\right)\right.\\&\left.\left.+\mathcal{J}^\tau_{r_2^-}h\left(\frac{x+r_2}{2}\right)\right]\right|\leqslant\left(\frac{1}{2}\right)^{\frac{2}{q}+1}\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\tau}+1\right)}{\Gamma\left(\frac{1}{\tau}+1+p\right)}\\&\times\left\{\frac{(x-r_1)^{\tau+1}}{r_2-r_1}\left[\left(|h'(r_1)|^q+3|h'(x)|^q\right)^{\frac{1}{q}}+\left(3|h'(r_1)|^q+|h'(x)|^q\right)^{\frac{1}{q}}\right]\right.\\&\left.+\frac{(r_2-x)^{\tau+1}}{r_2-r_1}\left[\left(|h'(x)|^q+3|h'(r_2)|^q\right)^{\frac{1}{q}}+\left(3|h'(x)|^q+|h'(r_2)|^q\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}\tag{2.8}$$

Specially, putting $\tau=1$ and $x=\frac{r_1+r_2}{2}$, one has

$$\begin{aligned}&\left|h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u\right|\\ \leqslant&\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{2}{q}+1}\left(\frac{r_2-r_1}{4}\right)\left\{\left(|h'(r_1)|^q+3\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\&+\left(3|h'(r_1)|^q+\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q+3|h'(r_2)|^q\right)^{\frac{1}{q}}\\&\left.+\left(3\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q+|h'(r_2)|^q\right)^{\frac{1}{q}}\right\}\\ \leqslant&\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left(\frac{r_2-r_1}{4}\right)\left[1^{\frac{1}{q}}+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right]\left(|h'(r_1)|+|h'(r_2)|\right).\end{aligned}\tag{2.9}$$

The second inequality is obtained by applying the convexity of $|h'|^q$ and the succeeding fact that

$$\sum_{i=1}^{n}(\varrho_i+\sigma_i)^s \leqslant \sum_{i=1}^{n}(\varrho_i)^s+\sum_{i=1}^{n}(\sigma_i)^s, \quad \varrho_i, \quad \sigma_i>0, \quad 0\leqslant s<1.$$

Theorem 2.2. *If $|h'|^q$ for $q\geqslant 1$ is preinvex on $\mathcal{K}$, then the coming inequality for k-fractional integrals with $x\in(r_1,r_2)$, $\tau>0$, $k>0$ and $\lambda\in[0,1]$ grips:*

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k,\lambda;x)| \leqslant &\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\Delta_1^{1-\frac{1}{q}}\left\{\left[\frac{1}{2}\left((\Delta_1-\Delta_2)|h'(r_1)|^q+(\Delta_1+\Delta_2)|h'(x)|^q\right)\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{1}{2}\left((\Delta_1+\Delta_2)|h'(r_1)|^q+(\Delta_1-\Delta_2)|h'(x)|^q\right)\right]^{\frac{1}{q}}\right\}\\ &+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\Delta_1^{1-\frac{1}{q}}\left\{\left[\frac{1}{2}\left((\Delta_1+\Delta_2)|h'(x)|^q+(\Delta_1-\Delta_2)|h'(r_2)|^q\right)\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{1}{2}\left((\Delta_1-\Delta_2)|h'(x)|^q+(\Delta_1+\Delta_2)|h'(r_2)|^q\right)\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{2.10}$$

where

$$\Delta_1=\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt=\frac{k}{\tau+k}-\lambda+2\lambda^{\frac{\tau+k}{\tau}}\frac{\tau}{\tau+k} \tag{2.11}$$

and

$$\Delta_2=\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt=\frac{k}{\tau+2k}-\frac{\lambda}{2}+\lambda^{\frac{\tau+2k}{\tau}}\frac{\tau}{\tau+2k}. \tag{2.12}$$

Proof. Using Lemma 2.1 and the power mean inequality, one has

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k,\lambda;x)| \leqslant &\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\left[\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt\right]^{1-\frac{1}{q}}\left((\mathcal{I}_1)^{\frac{1}{q}}+(\mathcal{I}_2)^{\frac{1}{q}}\right)\\ &+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\left[\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt\right]^{1-\frac{1}{q}}\left((\mathcal{I}_3)^{\frac{1}{q}}+(\mathcal{I}_4)^{\frac{1}{q}}\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_1&=\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)\right|^q dt\\ &\leqslant\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left(\frac{1-t}{2}|h'(r_1)|^q+\frac{1+t}{2}|h'(x)|^q\right)dt\\ &=\frac{1}{2}\left[\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt-\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(r_1)|^q\right.\\ &\left.+\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt+\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(x)|\right],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2&=\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)\right|^q dt\\ &\leqslant\frac{1}{2}\left[\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt+\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(r_1)|^q\right.\\ &\left.+\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt-\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(x)|\right],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_3&=\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)\right|^q dt\\ &\leqslant\frac{1}{2}\left[\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt+\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(x)|^q\right.\\ &\left.+\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt-\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(r_2)|\right],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_4&=\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)\right|^q dt\\ &\leqslant\frac{1}{2}\left[\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt-\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(x)|^q\right.\\ &\left.+\left(\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt+\int_0^1 t|t^{\frac{\tau}{k}}-\lambda|dt\right)|h'(r_2)|\right].\end{aligned}$$

These last four inequalities clasp due to the preinvexity of $|h'|^q$. This ends the proof.

Corollary 2.3. *Choosing $\lambda=1$, $k=1$ and $\delta(\mu,\nu)=\mu-\nu$ with $\mu,\nu\in[r_1,r_2]$ in Theorem 2.2, one obtains*

$$\begin{aligned}|\mathcal{L}(\tau,1,1;x)| \leqslant &\frac{\tau(x-r_1)^{\tau+1}}{2(\tau+1)(r_2-r_1)}\left\{\left[\frac{\tau+3}{4(\tau+2)}|h'(r_1)|^q+\frac{3\tau+5}{4(\tau+2)}|h'(x)|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{3\tau+5}{4(\tau+2)}|h'(r_1)|^q+\frac{\tau+3}{4(\tau+2)}|h'(x)|^q\right]^{\frac{1}{q}}\right\}\\ &+\frac{\tau(r_2-x)^{\tau+1}}{2(\tau+1)(r_2-r_1)}\left\{\left[\frac{\tau+3}{4(\tau+2)}|h'(x)|^q+\frac{3\tau+5}{4(\tau+2)}|h'(r_2)|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\frac{3\tau+5}{4(\tau+2)}|h'(x)|^q+\frac{\tau+3}{4(\tau+2)}|h'(r_2)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{2.13}$$

Specially, taking $\tau=1$, $x=\frac{r_1+r_2}{2}$ and utilizing similar arguments as in Corollary 2.2, one gets

$$\begin{aligned}&\left|h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)du\right|\\ &\leqslant\frac{(r_2-r_1)}{16}\left\{\left[\frac{1}{3}|h'(r_1)|^q+\frac{2}{3}\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.\\ &+\left[\frac{2}{3}|h'(r_1)|^q+\frac{1}{3}\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q\right]^{\frac{1}{q}}+\left[\frac{1}{3}\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q+\frac{2}{3}|h'(r_2)|^q\right]^{\frac{1}{q}}\\ &\left.+\left[\frac{2}{3}\left|h'\left(\frac{r_1+r_2}{2}\right)\right|^q+\frac{1}{3}|h'(r_2)|^q\right]^{\frac{1}{q}}\right\}\\ &\leqslant\frac{(r_2-r_1)}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[1^{\frac{1}{q}}+2^{\frac{1}{q}}+4^{\frac{1}{q}}+5^{\frac{1}{q}}\right]\left(|h'(r_1)|+|h'(r_2)|\right).\end{aligned} \tag{2.14}$$

Remark 2.2. Theorems 2.1 and 2.2 will be reduced to Theorem 2 and 3 in Latif (2015), respectively, if we choose $\lambda=0$, $k=1=\tau$ and $\delta(\mu,\nu)=\mu-\nu$ for $\mu,\nu\in[r_1,r_2]$.

3. New estimation results

For obtaining new estimation type results, we deal with the boundedness and the Lipschitzian condition of h', respectively.

Theorem 3.1. *Assume that there exists constants $m<M$ such that $-\infty<m\leqslant h'(y)\leqslant M<\infty$ for all $y\in\mathcal{K}$, then the inequality*

$$|\mathcal{L}_\delta(\tau,k,\lambda;x)|\leqslant\frac{(M-m)\left(|\delta^{\frac{\tau}{k}+1}(x,r_1)|+|\delta^{\frac{\tau}{k}+1}(r_2,x)|\right)}{2|\delta(r_2,r_1)|}\left[\frac{2\tau\lambda^{\frac{\tau+k}{\tau}}+k}{\tau+k}-\lambda\right] \tag{3.1}$$

holds with $\tau>0$, $k>0$, $\lambda\in[0,1]$ and $x\in(r_1,r_2)$.

Proof. From Lemma 2.1, we have

$$\mathcal{L}_\delta(\tau,k,\lambda;x)=\frac{\delta^{\frac{\tau}{k}+1}(x,r_1)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)-\frac{m+M}{2}\right]dt-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)-\frac{m+M}{2}\right]dt\right\}-\frac{\delta^{\frac{\tau}{k}+1}(r_2,x)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)-\frac{m+M}{2}\right]dt-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)-\frac{m+M}{2}\right]dt\right\}.$$

Using the fact that $m-\frac{m+M}{2}\leqslant h'(y)-\frac{m+M}{2}\leqslant M-\frac{m+M}{2}$, one obtains

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k,\lambda;x)|&\leqslant\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\left\{\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)-\frac{m+M}{2}\right|dt+\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)-\frac{m+M}{2}\right|dt\right\}\\&\quad+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\left\{\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)-\frac{m+M}{2}\right|dt+\int_0^1|t^{\frac{\tau}{k}}-\lambda|\left|h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)-\frac{m+M}{2}\right|dt\right\}\\&\leqslant\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{|\delta(r_2,r_1)|}\frac{M-m}{2}\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{|\delta(r_2,r_1)|}\frac{M-m}{2}\int_0^1|t^{\frac{\tau}{k}}-\lambda|dt\\&=\frac{(M-m)\left(|\delta^{\frac{\tau}{k}+1}(x,r_1)|+|\delta^{\frac{\tau}{k}+1}(r_2,x)|\right)}{2|\delta(r_2,r_1)|}\left[\frac{2\tau\lambda^{\frac{\tau+k}{\tau}}+k}{\tau+k}-\lambda\right].\end{aligned}\tag{3.2}$$

The proof is completed. □

Corollary 3.1. *In Theorem 3.1, choosing $\delta(\mu,v)=\mu-v$ for $\mu,v\in[r_1,r_2]$, and $\tau=k=1$, one gets*

$$\begin{aligned}|\mathcal{L}(1,1,\lambda;x)|&=\left|(1-\lambda)\frac{(x-r_1)[h(r_1)+h(x)]+(r_2-x)[h(x)+h(r_2)]}{r_2-r_1}+2\lambda\frac{(x-r_1)h\left(\frac{r_1+x}{2}\right)+(r_2-x)h\left(\frac{x+r_2}{2}\right)}{r_2-r_1}-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)du\right|\\&\leqslant\frac{(M-m)\left[(x-r_1)^2+(r_2-x)^2\right]}{2(r_2-r_1)}\left[\frac{2\lambda^2-2\lambda+1}{2}\right].\end{aligned}\tag{3.3}$$

Remark 3.1. Taking $x=\frac{r_1+r_2}{2}$ in Corollary 3.1, one can see the following.

(a) For $\lambda=0$, we have

$$\left|\frac{h(r_1)+h(r_2)}{2}+h\left(\frac{r_1+r_2}{2}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)du\right|\leqslant\frac{(M-m)(r_2-r_1)}{8}.\tag{3.4}$$

(b) For $\lambda=\frac{1}{2}$, we have

$$\left|\frac{h(r_1)+h(r_2)}{4}+\frac{1}{2}h\left(\frac{r_1+r_2}{2}\right)+\frac{h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)}{2}-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\right|\leqslant\frac{(M-m)(r_2-r_1)}{16}.\tag{3.5}$$

(c) For $\lambda=1$, we have

$$\left|h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)du\right|\leqslant\frac{(M-m)(r_2-r_1)}{8}.\tag{3.6}$$

Theorem 3.2. *Assume that h' satisfies Lipschitz condition on $\mathcal{K}$ for some $L>0$, then the inequality*

$$|\mathcal{L}_\delta(\tau,k,\lambda;x)|\leqslant\frac{L\left(|\delta^{\frac{\tau}{k}+2}(x,r_1)|+|\delta^{\frac{\tau}{k}+2}(r_2,x)|\right)}{4|\delta(r_2,r_1)|}\left[\frac{2\tau\lambda^{\frac{2k}{\tau}+1}+2k}{\tau+2k}-\lambda\right]\tag{3.7}$$

grips for all $x\in(r_1,r_2)$, $\mu>0$, $k>0$ and $\lambda\in[0,1]$.

Proof. From Lemma 2.1, we have

$$\begin{aligned}\mathcal{L}_\delta(\tau,k,\lambda;x)&=\frac{\delta^{\frac{\tau}{k}+1}(x,r_1)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right]dt-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right]dt\right\}\\&\quad-\frac{\delta^{\frac{\tau}{k}+1}(r_2,x)}{2\delta(r_2,r_1)}\left\{\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)-h'\left(x+\frac{1}{2}\delta(r_2,x)\right)\right]dt-\int_0^1\left(t^{\frac{\tau}{k}}-\lambda\right)\left[h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)-h'\left(x+\frac{1}{2}\delta(r_2,x)\right)\right]dt\right.\end{aligned}$$

Utilizing the fact that h' satisfies Lipschitz condition on $\mathcal{K}$ for $L>0$, we have

$$\begin{aligned}|\mathcal{L}_\delta(\tau,k;\lambda,x)|&\leqslant\frac{|\delta^{\frac{\tau}{k}+1}(x,r_1)|}{2|\delta(r_2,r_1)|}\left\{\int_0^1|(t^{\frac{\tau}{k}}-\lambda)|\left|h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right|dt+\int_0^1|(t^{\frac{\tau}{k}}-\lambda)|\left|h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right|dt\right\}+\frac{|\delta^{\frac{\tau}{k}+1}(r_2,x)|}{2|\delta(r_2,r_1)|}\\&\quad\times\left\{\int_0^1|(t^{\frac{\tau}{k}}-\lambda)|\left|h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)-h'\left(x+\frac{1}{2}\delta(r_2,x)\right)\right|dt+\int_0^1|(t^{\frac{\tau}{k}}-\lambda)|\left|h'\left(x+\frac{1+t}{2}\delta(r_2,x)\right)-h'\left(x+\frac{1}{2}\delta(r_2,x)\right)\right|dt\right\},\end{aligned}$$

where

$$\left|h'\left(r_1+\frac{1+t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right|\leqslant L\left|r_1+\frac{1+t}{2}\delta(x,r_1)-\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right|=\frac{tL}{2}|\delta(x,r_1)|,$$

$$\left|h'\left(r_1+\frac{1-t}{2}\delta(x,r_1)\right)-h'\left(r_1+\frac{1}{2}\delta(x,r_1)\right)\right|\leqslant\frac{tL}{2}|\delta(x,r_1)|,$$

$$\left|h'\left(x+\frac{1-t}{2}\delta(r_2,x)\right)-h'\left(a+\frac{1}{2}\delta(r_2,x)\right)\right|\leqslant\frac{tL}{2}|\delta(r_2,x)|,$$

$$\left|h'\left(x+\frac{n+t}{n+1}\delta(r_2,x)\right)-h'\left(a+\frac{1}{2}\delta(r_2,x)\right)\right|\leqslant\frac{tL}{2}|\delta(r_2,x)|.$$

After a simple calculation, we obtain the desired result. □

Corollary 3.2. *In Theorem 3.2, choosing $\delta(\mu,v)=\mu-v$ for $\mu,v\in[a,b]$, and $\tau=k=1$, one has*

$$|\mathcal{L}(1,1,\lambda;x)|\leqslant\frac{L\left[(x-r_1)^3+(r_2-x)^3\right]}{4(r_2-r_1)}\left[\frac{2\lambda^3+2}{3}-\lambda\right].\tag{3.8}$$

Remark 3.2. Putting $x=\frac{r_1+r_2}{2}$ in Corollary 3.2, one can see the following.

(a) For $\lambda=0$, we have

$$\left|\frac{h(r_1)+h(r_2)}{2}+h\left(\frac{r_1+r_2}{2}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)du\right|\leqslant\frac{L(r_2-r_1)^2}{24}.\tag{3.9}$$

(b) For $\lambda = \frac{1}{2}$, we have

$$\left|\frac{h(r_1)+h(r_2)}{4}+\frac{1}{2}h\left(\frac{r_1+r_2}{2}\right)+\frac{h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)}{2}-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\right| \leqslant \frac{L(r_2-r_1)^2}{64}. \tag{3.10}$$

(c) For $\lambda = 1$, we have

$$\left|h\left(\frac{3r_1+r_2}{4}\right)+h\left(\frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}\int_{r_1}^{r_2}h(u)\mathrm{d}u\right| \leqslant \frac{L(r_2-r_1)^2}{48}. \tag{3.11}$$

4. Applications for random variables

Let X be a random variable in $[r_1, r_2]$, with the probability density mapping $p : [r_1, r_2] \to [0, 1]$, and with the cumulative distribution mapping

$$F(x) = \mathrm{P}(X \leqslant x) = \int_a^x p(\lambda)\mathrm{d}\lambda.$$

Using the fact that $E(X) = \int_{r_1}^{r_2} \lambda \mathrm{d}F(\lambda) = r_2 - \int_{r_1}^{r_2} F(\lambda)\mathrm{d}\lambda$, one has the following results.

Proposition 4.1. *In Theorem 2.2, taking $\lambda = 1$, $k = 1 = \tau$, $x = \frac{r_1+r_2}{2}$ and $\delta(u, v) = u - v$ with $u, v \in [r_1, r_2]$, one has*

$$\left|\mathrm{P}\left(X \leqslant \frac{3r_1+r_2}{4}\right)+\mathrm{P}\left(X \leqslant \frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}(r_2-E(X))\right| \leqslant \frac{(r_2-r_1)}{16}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[1^{\frac{1}{q}}+2^{\frac{1}{q}}+4^{\frac{1}{q}}+5^{\frac{1}{q}}\right](|p(r_1)|+|p(r_2)|). \tag{4.1}$$

Proposition 4.2. *In Theorem 2.1, putting $\lambda = 1$, $k = 1 = \tau$, $x = \frac{r_1+r_2}{2}$ and $\delta(u, v) = u - v$ with $u, v \in [r_1, r_2]$, one gets*

$$\left|\mathrm{P}\left(X \leqslant \frac{3r_1+r_2}{4}\right)+\mathrm{P}\left(X \leqslant \frac{r_1+3r_2}{4}\right)-\frac{2}{r_2-r_1}(r_2-E(X))\right| \leqslant \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{3}{q}+1}\left(\frac{r_2-r_1}{4}\right)\left[1^{\frac{1}{q}}+3^{\frac{1}{q}}+5^{\frac{1}{q}}+7^{\frac{1}{q}}\right](|p(r_1)|+|p(r_2)|). \tag{4.2}$$

Remark 4.1. Applications can be given based on the obtained results to special means, and we omit the details.

5. Conclusion

Four main results of the trapezium-like inequalities involving the mappings δ are hereby obtained. More new results can be derived by choosing different mappings δ and the special parameter values for λ, k and τ.

Competing interests

The authors declare that there are no competing interests.

Author's contributions

All authors read and approved the final manuscript.

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