Journal of King Saud University – Science 31 (2019) 556–566

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

On soft topological ordered spaces

T.M. Al-shami [a,b,*], M.E. El-Shafei [b], M. Abo-Elhamayel [b]

[a] Department of Mathematics, Sana'a University, Sana'a, Yemen
[b] Department of Mathematics, Mansoura University, Mansoura, Egypt

ARTICLE INFO

Article history:
Received 9 December 2017
Accepted 25 June 2018
Available online 2 July 2018

2010 Mathematics Subject Classification:
54C99
54D10
54D15
54D30
54F05

Keywords:
Increasing (Decreasing) soft operator
Soft compactness and p-soft T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$

ABSTRACT

In this paper, the authors initiate a soft topological ordered space by adding a partial order relation to the structure of a soft topological space. Some concepts such as monotone soft sets and increasing (decreasing) soft operators are presented and their main properties are studied in detail. The notions of ordered soft separation axioms, namely p-soft T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ are introduced and the relationships among them are illustrated with the help of examples. In particular, the equivalent conditions for p-soft regularly ordered spaces and soft normally ordered spaces are given. Moreover, we define the soft topological ordered properties and then verify that the property of being p-soft T_i-ordered spaces is a soft topological ordered property, for $i = 0, 1, 2, 3, 4$. Finally, we investigate the relationships between soft compactness and some ordered soft separation axioms and point out that the condition of soft compactness is sufficient for the equivalent between p-soft T_2-ordered spaces and p-soft T_3-ordered spaces.
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1. Introduction

In 1965, Nachbin (1965) defined a topological ordered space by adding a partial order relation to the structure of a topological space. So it can be considered that the topological ordered spaces are one of the generalizations of the topological spaces. McCartan (1968) utilized monotone neighborhoods to introduce and study ordered separation axioms. Also, McCartan (1971) presented the notions of continuous and anti-continuous topological ordered spaces. Later on, many studies are done on ordered spaces (see, for example, Abo-Elhamayel and Al-shami, 2016; Arya and Gupta, 1991; Das, 2004; El-Shafei et al., 2017; Farajzadeh et al., 2012; Kumar, 2012; McCartan, 1971; Zangenehmehr et al., 2015).

In 1999, the notion of soft set theory was initiated by Molodtsov (1999) to approach problems associated with uncertainties. He demonstrated the advantages of soft set theory compared to probability theory and fuzzy theory. The applications of soft sets in many disciplines such as economics, medicine, engineering and game theory give rise to rapidly increase researches on it. Maji et al. (2002, 2003) presented the first application of soft sets in decision making problems and established several fundamental operators on soft sets. Aktas and Çağman (2007) studied soft groups and derived that every fuzzy set (rough set) may be considered soft set. Ali et al. (2009) pointed out that some results obtained in Maji et al. (2003) are not true and improved some operations on soft sets. Çağman and Enginoğlu (2010) defined soft matrices and then they constructed a soft max–min decision method which can be used in handling problems that contain vagueness without utilizing fuzzy sets and rough sets.

In 2011, the idea of soft topological spaces was formulated by Shabir and Naz (2011). They studied the main concepts regarding soft topologies such as soft closure operators, soft subspaces and soft separation axioms. Min (2011) studied further properties of these soft separation axioms and corrected some mistakes in Shabir and Naz (2011). As a continuation of the study of elementary concepts regarding soft topologies, Hussain and Ahmad (2011) studied the properties of soft interior and soft boundary operators, and investigated some findings that connected between them. Aygünoğlu and Aygün (2012) started to investigate soft compactness and soft product spaces. To study soft interior points and soft

* Corresponding author.
E-mail addresses: tareqalshami83@gmail.com (T.M. Al-shami), meshafei@hotmail.com (M.E. El-Shafei), maboelhamayle@mans.edu.eg (M. Abo-Elhamayel).
Peer review under responsibility of King Saud University.

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https://doi.org/10.1016/j.jksus.2018.06.005
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neighborhood systems, Zorlutuna et al. (2012) introduced an idea of soft points. Then the authors (Das and Samanta, 2013; Nazmul and Samanta, 2013) simultaneously modified a notion of soft points, which play the same role of the element in the crisp set, in order to study soft metric spaces and soft neighborhood systems. By the soft points, many results in soft sets and soft topologies are handled easily. The soft filter and soft ideal (Sahin and Kuçuk, 2013; Yüksel et al., 2014) notions were formulated and the main features were discussed. Kandil et al. (2014) generated a soft topological space stronger than the original soft topological space by utilizing a notion of soft ideal. Hida (2014) gave two formulations of soft compact spaces namely, SCPT1 and SCPT2, and compared these two formulations in relation with some important soft topological properties. Recently, we (El-Shafei et al., 2018) defined partial belong and total non belong relations which are more effective to theoretical and application studies in soft topological spaces and then utilized them to study partial soft separation axioms.

The idea of this study is to establish a soft topological ordered space which consists of a soft topological spaces endowed with a partial order relation. From this point of view, it can be consider that a generating soft topological ordered space and an original soft topological space are equivalent if a partial order relation is an equality relation. This paper starts by presenting the definitions and results of soft set theory and soft topological spaces which will be needed to probe results obtained herein. Then we define the concepts of monotone soft sets and increasing (decreasing) soft operators and illuminate their fundamental properties. One of the significant findings obtained in Section 3 is Theorem (3.8) which will be used to verify some results concerning soft product spaces. In the last section of this paper, we introduce the notions of ordered soft separation axioms, namely p-soft T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ and illustrate the relationships among them with the help of examples. Also, we investigate the characterizations of p-soft regularly ordered and soft normally ordered spaces, and point out that p-soft T_i-ordered spaces $(i = 0, 1, 2)$ are equivalent if these soft spaces are p-soft regularly ordered. Moreover, we use ordered embedding soft homeomorphism maps to define soft topological ordered properties and then verify that the property of being p-soft T_i-ordered spaces is a soft topological ordered property, for $(i = 0, 1, 2, 3, 4)$. Finally, we investigate soft compact spaces in connection with some ordered soft separation axioms and obtain interesting results.

2. Preliminaries

Let us recall some basic definitions and properties on soft sets, soft topological spaces and partial order relations which we shall need it to prove the sequels.

Definition 2.1. Molodtsov (1999) A pair (G, E) is said to be a soft set over X provided that G is a mapping of a set of parameters E into 2^X.

Remark 2.2.

(i) For short, we use the notation G_E instead of (G, E).
(ii) A soft set G_E can be defined as a set of ordered pairs $G_E = \{(e, G(e)) : e \in E \text{ and } G(e) \in 2^X\}$.

Definition 2.3. Molodtsov (1999) For a soft set G_E over X and $x \in X$, we say that:

(i) $x \in G_E$ if $x \in G(e)$, for each $e \in E$.
(ii) $x \notin G_E$ if $x \notin G(e)$, for some $e \in E$.

Definition 2.4. Maji et al. (2003) A soft set G_E over X is called:

(i) A null soft set, denoting by $\widetilde{\emptyset}$, if $G(e) = \emptyset$, for each $e \in E$.
(ii) An absolute soft set, denoting by $\widetilde{X}$, if $G(e) = X$, for each $e \in E$.

Definition 2.5. Maji et al. (2003) The union of soft sets G_A and F_B over X is the soft set V_D, where $D = A \bigcup B$ and a map $V : D \to 2^X$ is defined as follows

$$V(d) = \begin{cases} G(d) & : d \in A - B \\ F(d) & : d \in B - A \\ G(d) \bigcup F(d) & : d \in A \bigcap B \end{cases}$$

It is written briefly, $G_A \widetilde{\bigcup} F_B = V_D$.

Definition 2.6. Pei and Miao (2005) The intersection of soft sets G_A and F_B over X is the soft set V_D, where $D = A \bigcap B$, and a map $V : D \to 2^X$ is defined by $V(d) = G(d) \bigcap F(d)$, for all $d \in D$. It is written briefly, $G_A \widetilde{\bigcap} F_B = V_D$.

In this connection, we draw the attention of the readers to that there are other kinds of soft union and soft intersection of soft sets were originated and investigated in Ali et al. (2009).

Definition 2.7. Pei and Miao (2005) A soft set G_A is a soft subset of a soft set F_B if

(i) $A \subseteq B$.
(ii) For all $a \in A, G(a) \subseteq F(a)$.

The soft sets G_A and F_B are soft equal if each of them is a soft subset of the other. The set of all soft sets, over X under a parameter set A, is denoted by $S(X_A)$.

It should be noted that there are other kinds of soft subset and soft equal relations were introduced and discussed in Qin and Hong (2010).

Definition 2.8. Ali et al. (2009) The relative complement of a soft set G_E, denoted by G_E^c, where $G^c : E \to 2^X$ is the mapping defined by $G^c(e) = X \setminus G(e)$, for each $e \in E$.

Definition 2.9. Shabir and Naz (2011) A collection τ of soft sets over X under a fixed parameters set E is said to be a soft topology on X if it satisfies the following three axioms:

(i) $\widetilde{X}$ and $\widetilde{\emptyset}$ belong to τ.
(ii) The intersection of a finite family of soft sets in τ belongs to τ.
(iii) The union of an arbitrary family of soft sets in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space (briefly, STS). Every member of τ is called soft open and its relative complement is called soft closed.

Definition 2.10. Shabir and Naz (2011) A soft set x_E over X is defined by $x(e) = \{x\}$, for each $e \in E$.

Proposition 2.11. Shabir and Naz (2011) *If (X, τ, E) is an STS, then for each $e \in E$, a family $\tau_e = \{G(e) : G_E \in \tau\}$ forms a topology on X.*

Definition 2.12. Shabir and Naz (2011) Let Y be a non-empty subset of an STS (X, τ, E). Then $\tau_Y = \{\widetilde{Y} \widetilde{\bigcap} G_E : G_E \in \tau\}$ is said to be a soft relative topology on Y and the triple (Y, τ_Y, E) is said to be a soft subspace of (X, τ, E).

Definition 2.13. Shabir and Naz (2011) For a soft subset H_E of an STS $(X, \tau, E), Int(H_E)$ is the largest soft open set contained in H_E and $Cl(H_E)$ is the smallest soft closed set containing H_E.

Definition 2.14. Zorlutuna et al. (2012) A soft subset W_E of an STS (X, τ, E) is called soft neighborhood of $x \in X$, if there exists a soft open set G_E such that $x \in G_E \widetilde{\subseteq} W_E$.

Definition 2.15. Zorlutuna et al. (2012) A soft mapping between $S(X_A)$ and $S(Y_B)$ is a pair (f, ϕ), denoted also by f_ϕ, of mappings such that $f : X \to Y, \phi : A \to B$. Let G_K and H_L be soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then the image of G_K and pre-image of H_L are defined by:

(i) $f_\phi(G_K) = (f_\phi(G))_B$ is a soft subset of $S(Y_B)$ such that

$$f_\phi(G)(b) = \begin{cases} \widetilde{\bigcup}_{a \in \phi^{-1}(b) \bigcap K} f(G(a)) & : \quad \phi^{-1}(b) \bigcap K \neq \varnothing \\ \varnothing & : \quad \phi^{-1}(b) \bigcap K = \varnothing \end{cases}$$

for each $b \in B$.

(ii) $f_\phi^{-1}(H_L) = (f_\phi^{-1}(H))_A$ is a soft subset of $S(X_A)$ such that

$$f_\phi^{-1}(H)(a) = \begin{cases} f^{-1}(H(\phi(a))) & : \quad \phi(a) \in L \\ \varnothing & : \quad \phi(a) \notin L \end{cases}$$

for each $a \in A$.

Definition 2.16. Zorlutuna et al. (2012) A soft map $f_\phi : S(X_A) \to S(Y_B)$ is said to be:

(i) Injective if f and ϕ are injective.
(ii) Surjective if f and ϕ are surjective.
(iii) Bijective if f and ϕ are bijective.

Proposition 2.17. Nazmul and Samanta (2013) *Let $f_\phi : S(X_A) \to S(Y_B)$ be a soft map. Then for each soft subsets G_A and H_B of $S(X_A)$ and $S(Y_B)$, respectively, we have the following results*:

(i) $G_A \widetilde{\subseteq} f_\phi^{-1} f_\phi(G_A)$ *and* $G_A = f_\phi^{-1} f_\phi(G_A)$ *if* f_ϕ *is injective.*
(ii) $f_\phi f_\phi^{-1}(H_B) \widetilde{\subseteq} H_B$ *and* $f_\phi f_\phi^{-1}(H_B) = H_B$ *if* f_ϕ *is surjective.*

Definition 2.18. (Nazmul and Samanta, 2013; Zorlutuna et al., 2012) A soft map $f_\phi : (X, \tau, A) \to (Y, \theta, B)$ is said to be:

(i) Soft continuous if the inverse image of each soft open subset of (Y, θ, B) is a soft open subset of (X, τ, A).
(ii) Soft open (resp. soft closed) if the image of each soft open (resp. soft closed) subset of (X, τ, A) is a soft open (resp. soft closed) subset of (Y, θ, B).
(iii) Soft homeomorphism if it is bijective, soft continuous and soft open.

Definition 2.19. Aygünoğlu and Aygün (2012)

(i) A collection $\{G_{i_E} : i \in I\}$ of soft open sets is called soft open cover of an STS (X, τ, E) if $\widetilde{X} = \widetilde{\bigcup_{i \in I}} G_{i_E}$.
(ii) An STS (X, τ, E) is called soft compact (resp. soft Lindelöf) provided that every soft open cover of $\widetilde{X}$ has a finite (resp. countable) subcover.

Proposition 2.20. Aygünoğlu and Aygün (2012) *Every soft closed subset H_E of a soft compact (resp. soft Lindelöf) space is soft compact (resp. soft Lindelöf).*

Definition 2.21. Aygünoğlu and Aygün (2012) Let G_A and H_B be soft sets over X and Y, respectively. Then the cartesian product of G_A and H_B is denoted by $(G \times H)_{A \times B}$ and is defined as $(G \times H)(a, b) = G(a) \times H(b)$, for each $(a, b) \in A \times B$.

Theorem 2.22. Aygünoğlu and Aygün (2012) *Let (X, τ, A) and (Y, θ, B) be two STSs. Let $\Omega = \{G_A \times F_B : G_A \in \tau$ and $F_B \in \theta\}$. Then the family of all arbitrary union of elements of Ω is a soft topology on $X \times Y$.*

Definition 2.23. Das and Samanta (2013) A soft set H_E over X is called countable (resp. finite) if $H(e)$ is countable (resp. finite), for each $e \in E$.

Definition 2.24. (Das and Samanta, 2013; Nazmul and Samanta, 2013) A soft subset P_E of $\widetilde{X}$ is called soft point if there exists $e \in E$ and there exists $x \in X$ such that $P(e) = \{x\}$ and $P(\alpha) = \varnothing$, for each $\alpha \in E \setminus \{e\}$. A soft point will be shortly denoted by P_e^x and we say that $P_e^x \in G_E$, if $x \in G(e)$.

It is noteworthy that the above definition of soft point is a special case of the definition of soft point which introduced in Zorlutuna et al. (2012).

Definition 2.25. El-Shafei et al. (2018) For a soft set G_E over X and $x \in X$, we say that

(i) $x \Subset G_E$ if $x \in G(e)$, for some $e \in E$.
(ii) $x \not\Subset G_E$ if $x \notin G(e)$, for each $e \in E$.

Definition 2.26. El-Shafei et al. (2018) A soft set G_E in $S(X_E)$ is said to be stable if there exists a subset S of X such that $G(e) = S$, for each $e \in E$.

Definition 2.27. El-Shafei et al. (2018) An STS (X, τ, E) is said to be:

(i) p-soft T_0-space if for every pair of distinct points $x, y \in X$, there is a soft open set G_E such that $x \in G_E$, $y \not\Subset G_E$ or $y \in G_E$, $x \not\Subset G_E$.
(ii) p-soft T_1-space if for every pair of distinct points $x, y \in X$, there are soft open sets G_E and F_E such that $x \in G_E$, $y \not\Subset G_E$ and $y \in F_E$, $x \not\Subset F_E$.
(iii) p-soft T_2-space if for every pair of distinct points $x, y \in X$, there are disjoint soft open sets G_E and F_E containing x and y, respectively.
(iv) p-soft regular if for every soft closed set H_E and $x \in X$ such that $x \not\Subset H_E$, there are disjoint soft open sets G_E and F_E such that $H_E \widetilde{\subseteq} G_E$ and $x \in F_E$.

(vi) (Shabir and Naz, 2011) Soft normal if for every two disjoint soft closed sets H_{1_E} and H_{2_E}, there are two disjoint soft open sets G_E and F_E such that $H_{1_E} \widetilde{\subseteq} G_E$ and $H_{2_E} \widetilde{\subseteq} F_E$.
(vii) p-soft T_3-space if it is both p-soft regular and p-soft T_1-space.
(viii) p-soft T_4-space if it is both soft normal and p-soft T_1-space.

Lemma 2.28. El-Shafei et al. (2018) *If $H_{E_1\times E_2}$ is a soft closed subset of a soft product space $(X\times Y, \tau_1\times\tau_2, E_1\times E_2)$, then $H_{E_1\times E_2} = \left[(G_{E_1})^c \times \widetilde{Y}\right] \widetilde{\bigcup} \left[\widetilde{X} \times (F_{E_2})^c\right]$, for some $G_{E_1}\in\tau_1$ and $F_{E_2}\in\tau_2$.*

Definition 2.29. Kelley (1975) a binary relation $\preceq$ on a non-empty set X is called a partial order relation if it is reflexive, anti-symmetric and transitive.

$\{(x,x)$: for each $x\in X\}$ is the equality relation on X and it is indicated by $\triangle$.

Definition 2.30. Kelley (1975) Let $(X,\preceq)$ be a partially ordered set. An element $a\in X$ is called:

(i) A smallest element of X provided that $a\preceq x$, for all $x\in X$.
(ii) A largest element of X provided that $x\preceq a$, for all $x\in X$.

Definition 2.31. Nachbin (1965) A triple $(X,\tau,\preceq)$ is said to be a topological ordered space, where $(X,\preceq)$ is a partially ordered set and (X,τ) is a topological space.

Definition 2.32. McCartan (1968) A topological ordered space $(X,\tau,\preceq)$ is called:

(i) Lower (Upper) T_1-ordered if for each $x\not\preceq y$ in X, there is an increasing (resp. a decreasing) neighborhood W of a(resp. b) such that b(resp. a) belongs to W^c.
(ii) T_0-ordered if it is lower T_1-ordered or upper T_1-ordered.
(iii) T_1-ordered if it is lower T_1-ordered and upper T_1-ordered.
(iv) T_2-ordered if for each $x\not\preceq y$ in X, there are disjoint neighborhoods W_1 and W_2 of x and y, respectively, such that W_1 is increasing and W_2 is decreasing.

3. Monotone soft sets

In this section, we first formulate the definitions of partially ordered soft sets, increasing (decreasing) soft sets and increasing (decreasing, ordered embedding) soft maps. Then we present and investigate the main properties of these new concepts.

Definition 3.1. Let $\preceq$ be a partial order relation on a non-empty set X and let E be a set of parameters. A triple $(X,E,\preceq)$ is said to be a partially ordered soft set.

Definition 3.2. Let $(X,E,\preceq)$ be a partially ordered soft set. We define an increasing soft operator $i:(S(X_E),\preceq)\to(S(X_E),\preceq)$ and a decreasing soft operator $d:(S(X_E),\preceq)\to(S(X_E),\preceq)$ as follows, for each soft subset G_E of $S(X_E)$

(i) $i(G_E)=(iG)_E$, where iG is a mapping of E into X given by $iG(e)=i(G(e))=\{x\in X: y\preceq x$, for some $y\in G(e)\}$.
(ii) $d(G_E)=(dG)_E$, where dG is a mapping of E into X given by $dG(e)=d(G(e))=\{x\in X: x\preceq y$, for some $y\in G(e)\}$.

Definition 3.3. A soft subset G_E of a partially ordered soft set $(X,E,\preceq)$ is said to be:

(i) Increasing if $G_E=i(G_E)$.
(ii) Decreasing if $G_E=d(G_E)$.

Proposition 3.4. *We have the following results for a soft subset G_E of a partially ordered soft set $(X,E,\preceq)$.*

(i) G_E is increasing if and only if for each $P_e^x\in i(G_E)$, then $P_e^x\in G_E$.
(ii) G_E is decreasing if and only if for each $P_e^x\in d(G_E)$, then $P_e^x\in G_E$.
(iii) If G_E is increasing, then for each $x\in i(G_E)$, we have $x\in G_E$.
(iv) If G_E is decreasing, then for each $x\in d(G_E)$, we have $x\in G_E$.

Proof. We only prove case **(i)**, and the other follow similar lines.

Necessity: It comes immediately from Definition (3.3).

Sufficiency: By hypothesis, $P_e^x\in i(G_E)$ implies that $P_e^x\in G_E$. Then $x\in G(e)$. Since $\preceq$ is reflexive, then $x\in i(G_E)$. So $P_e^x\in i(G_E)$. This means that $i(G_E)\widetilde{\subseteq} G_E$. Thus $G_E=i(G_E)$. Hence a soft set G_E is increasing. □

Proposition 3.5. *Let $\{G_{j_E}: j\in J\}$ be a collection of increasing (resp. decreasing) soft subsets of a partially ordered soft set $(X,E,\preceq)$. Then:*

(i) $\widetilde{\bigcup}_{j\in J} G_{j_E}$ *is increasing (resp. decreasing).*
(ii) $\widetilde{\bigcap}_{j\in J} G_{j_E}$ *is increasing (resp. decreasing).*

Proof. (**i**): We prove this case when a collection consists of increasing soft sets. Let $P_e^x\in\widetilde{\bigcup}_{j\in J}G_{j_E}$. Then there exists $j_0\in J$ such that $P_e^x\in G_{j_{0E}}$. Therefore $i(P_e^x)\widetilde{\subseteq} i(G_{j_{0E}})=G_{j_{0E}}\widetilde{\subseteq}\widetilde{\bigcup}_{j\in J}G_{j_E}$. Thus a soft set $\widetilde{\bigcup}_{j\in J}G_{j_E}$ is increasing.

A similar proof is given for the case between parentheses.

By analogy with (**i**), one can prove (**ii**). □

Corollary 3.6. *A collection of all increasing (resp. decreasing) soft subsets of a partially ordered soft set $(X,E,\preceq)$ forms a soft topology on X.*

Proposition 3.7. *A soft subset G_E of a partially ordered soft set $(X,E,\preceq)$ is increasing (resp. decreasing) if and only if G_E^c is decreasing (resp. increasing).*

Proof. Let G_E be an increasing soft set. Suppose, to the contrary, that G_E^c is not decreasing. Then there exists $P_e^x\in d(G_E^c)$ and $P_e^x\notin G_E^c$. So $x\in d(G^c(e))$ and $x\notin G^c(e)$. This means that there exists $y\in G^c(e)$ such that $x\preceq y$. Since $x\in G(e)$ and the soft set G_E is increasing, then $y\in G(e)$. But this contradicts that $G(e)\bigcap G^c(e)=\varnothing$. Hence G_E^c is decreasing. Similarly, one can prove the proposition in case of G_E is decreasing. □

Theorem 3.8. *The finite product of increasing (resp. decreasing) soft sets is increasing (resp. decreasing).*

Proof. We only prove the theorem for two soft sets in case of increasing soft sets and one can prove it similarly for finite soft sets.

Let G_A and F_B be two increasing soft subsets of $(X, A, \preceq_1)$ and $(Y, B, \preceq_2)$, respectively. Setting $H_{A\times B} = G_A \times F_B$ such that $H(a,b) = G(a) \times F(b)$, for each $(a,b) \in A \times B$. Suppose, to the contrary, that $H_{A\times B}$ is not increasing. Then there exists a soft point $P^{(x,y)}_{(\alpha,\beta)}$ such that $P^{(x,y)}_{(\alpha,\beta)} \in i(H_{A\times B})$ and $P^{(x,y)}_{(\alpha,\beta)} \notin H_{A\times B}$. This means that $(x,y) \in i(H(\alpha,\beta))$ and $(x,y) \notin H(\alpha,\beta)$. So $(x,y) \in i(G(\alpha) \times F(\beta))$ implies that

$$x \in i\,(G(\alpha)) = G(\alpha) \text{ and } y \in i(F(\beta)) = F(\beta) \tag{1}$$

and $(x,y) \notin G(\alpha) \times F(\beta)$ implies that

$$x \notin G(\alpha) \text{ or } y \notin F(\beta) \tag{2}$$

From (1) and (2), we obtain a contradiction. Since the contradiction arises by assuming that the soft set $H_{A\times B}$ is not increasing, then $H_{A\times B}$ is increasing.

A similar proof is given for the case between parentheses. □

In the following two results, we present the main properties of the increasing and decreasing soft operators.

Proposition 3.9. *Let G_E and F_E be two soft subsets of $(X, E, \preceq)$ and let $i : (S(X_E), \preceq) \to (S(X_E), \preceq)$ be an increasing soft operator. Then:*

(i) $i(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(ii) $G_E \widetilde{\subseteq} i(G_E)$.
(iii) $i(i(G_E)) = i(G_E)$.
(iv) $i\left[G_E \widetilde{\bigcup} F_E\right] = i(G_E) \widetilde{\bigcup} i(F_E)$.

Proof. The proof of items (**i**) and (**ii**) are obvious.

(**iii**): From (**ii**), we get that $i(G_E) \widetilde{\subseteq} i(i(G_E))$. On the other hand, let $x \widetilde{\in} i(i(G_E))$. Then there exists $y \widetilde{\in} i(G_E)$ such that $y \preceq x$. Also, there exists $z \widetilde{\in} G_E$ such that $z \preceq y$. Since $\preceq$ is transitive, then $z \preceq x$. So $x \widetilde{\in} i(G_E)$. Thus $i(i(G_E)) \widetilde{\subseteq} i(G_E)$. This completes the proof of this property.

(**iv**): Obviously, $i(G_E) \widetilde{\bigcup} i(F_E) \widetilde{\subseteq} i\left[G_E \widetilde{\bigcup} F_E\right]$. On the other hand, $G_E \widetilde{\bigcup} F_E \widetilde{\subseteq} i(G_E) \widetilde{\bigcup} i(F_E)$. From (**iii**) and Definition (3.3), we infer that $i(G_E)$ and $i(F_E)$ are increasing. From Proposition (3.5), we infer that $i(G_E) \widetilde{\bigcup} i(F_E)$ is increasing. So $i\left[G_E \widetilde{\bigcup} F_E\right] \widetilde{\subseteq} i(G_E) \widetilde{\bigcup} i(F_E)$. Hence this part of the proposition holds. □

Proposition 3.10. *Let G_E and F_E be two soft subsets of $(X, E, \preceq)$ and let $d : (S(X_E), \preceq) \to (S(X_E), \preceq)$ be a decreasing soft operator. Then:*

(i) $d(\widetilde{\emptyset}) = \widetilde{\emptyset}$.
(ii) $G_E \widetilde{\subseteq} d(G_E)$.
(iii) $d(d(G_E)) = d(G_E)$.
(iv) $d\left[G_E \widetilde{\bigcup} F_E\right] = d(G_E) \widetilde{\bigcup} d(F_E)$.

Proof. The proof is similar to that of Proposition (3.9). □

Proposition 3.11. *The following two results hold for a soft map $f_\phi : S(X_A) \to S(Y_B)$.*

(i) *The image of each soft point is soft point.*
(ii) *If f_ϕ is bijective, then the inverse image of each soft point is soft point.*

Proof.

(i) Consider P^x_α is a soft point in the domain. Then $f_\phi(P^x_\alpha) = (f_\phi(P))_B$ such that $f_\phi(P)(b) = \bigcup_{a\in\phi^{-1}(b)} \phi(P(a))$. Since

$$P(a) = \begin{cases} \textit{singleton element of } X & : \; a = \alpha \\ \emptyset & : \; a \neq \alpha \end{cases},$$

then this part of proposition holds.

(ii) Consider P^y_β is a soft point in the codomain. Then $f^{-1}_\phi(P^y_\beta) = (f^{-1}_\phi(P))_A$ such that $f^{-1}_\phi(P)(a) = \bigcup f^{-1}(P(\phi(a)))$. Since $\phi(a)$ is a singleton element in B and P^y_β is a soft point, then

$$P(\phi(a)) = \begin{cases} y & : \; \phi(a) = \beta \\ \emptyset & : \; \phi(a) \neq \beta \end{cases}$$

Since ϕ and f are bijective, then

$$f^{-1}(P(\phi(a))) = \begin{cases} \textit{singleton element in } X & : \; \phi(a) = \beta \\ \emptyset & : \; \phi(a) \neq \beta \end{cases}$$

This completes the proof of this part of proposition. □

Definition 3.12. Let P^x_α and P^y_α be two soft points in a partially ordered soft set $(X, E, \preceq)$. We say that $P^x_\alpha \preceq P^y_\alpha$ if $x \preceq y$.

Definition 3.13. A soft map $f_\phi : (S(X_A), \preceq_1) \to (S(Y_B), \preceq_2)$ is said to be:

(i) Increasing if $P^x_\alpha \preceq_1 P^y_\alpha$, then $f_\phi(P^x_\alpha) \preceq_2 f_\phi(P^y_\alpha)$.
(ii) Decreasing if $P^x_\alpha \preceq_1 P^y_\alpha$, then $f_\phi(P^y_\alpha) \preceq_2 f_\phi(P^x_\alpha)$.
(iii) Ordered embedding if $P^x_\alpha \preceq_1 P^y_\alpha$ if and only if $f_\phi(P^x_\alpha) \preceq_2 f_\phi(P^y_\alpha)$.

Theorem 3.14. *The following two results hold for a soft map $f_\phi : (S(X_A), \preceq_1) \to (S(Y_B), \preceq_2)$.*

(i) *If f_ϕ is increasing, then the inverse image of each increasing (resp. decreasing) soft subset of $\widetilde{Y}$ is an increasing (resp. a decreasing) soft subset of $\widetilde{X}$.*
(ii) *If f_ϕ is decreasing, then the inverse image of each increasing (resp. decreasing) soft subset of $\widetilde{Y}$ is a decreasing (resp. an increasing) soft subset of $\widetilde{X}$.*

Proof. (**i**): Let G_K be an increasing soft subset of $\widetilde{Y}$. Suppose that $f^{-1}_\phi(G_K)$ is not increasing. Then there exists $x \in X$ and there exists $\alpha \in A$ such that $P^x_\alpha \in i(f^{-1}_\phi(G_K))$ and $P^x_\alpha \notin f^{-1}_\phi(G_K)$. So we infer that there exists $P^y_\alpha \in f^{-1}_\phi(G_K)$ such that $P^y_\alpha \preceq_1 P^x_\alpha$. Since f_ϕ is increasing, then $f_\phi(P^y_\alpha) \preceq_2 f_\phi(P^x_\alpha)$ and since $f_\phi(P^y_\alpha) \in f_\phi(f^{-1}_\phi(G_K)) \widetilde{\subseteq} G_K$, then $f_\phi(P^x_\alpha) \in G_K$. This implies that $P^x_\alpha \in f^{-1}_\phi(G_K)$. But this contradicts that $P^x_\alpha \notin f^{-1}_\phi(G_K)$. Hence the soft set $f^{-1}_\phi(G_K)$ is increasing.

A similar proof is given for the case between parentheses.

By analogy with (**i**), one can prove (**ii**). □

Theorem 3.15. *Let $f_\phi : (S(X_A), \preceq_1) \to (S(Y_B), \preceq_2)$ be a bijective ordered embedding soft map. Then the image of each increasing (resp. decreasing) soft subset of $\widetilde{X}$ is an increasing (resp. a decreasing) soft subset of $\widetilde{Y}$.*

Proof. Let G_L be an increasing soft subset of $\widetilde{X}$. Suppose that $f_\phi(G_L)$ is not increasing. Then there exists $y \in Y$ and there exists $\beta \in B$ such that $P^y_\beta \in i(f_\phi(G_L))$ and $P^y_\beta \notin f_\phi(G_L)$. So we infer that there exists $P^z_\beta \in f_\phi(G_L)$ such that $P^z_\beta \preceq_2 P^y_\beta$. Since f_ϕ is ordered embedding, then $f^{-1}_\phi(P^z_\beta) \preceq_1 f^{-1}_\phi(P^y_\beta)$ and since $f^{-1}_\phi(P^z_\beta) \in f^{-1}_\phi(f_\phi(G_L)) = G_L$, then $f^{-1}_\phi(P^y_\beta) \in G_L$. This implies that $P^y_\beta \in f_\phi(G_L)$. But this contradicts that $P^y_\beta \notin f_\phi(G_L)$. Hence the soft set $f_\phi(G_L)$ is increasing.

A similar proof is given for the case between parentheses. □

4. Ordered soft separation axioms

We devote this section to introducing soft ordered separation axioms namely, p-soft T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ and to studying their main properties. Various examples are considered to show the relationships among them and to illustrate some results obtained herein.

Definition 4.1. A quadrable system $(X, \tau, E, \preceq)$ is said to be a soft topological ordered space, where (X, τ, E) is a soft topological space and $(X, E, \preceq)$ is a partially ordered soft set.

Henceforth, we use the abbreviation STOS in a place of soft topological ordered space.

Definition 4.2. A soft subset W_E of an STOS $(X, \tau, E, \preceq)$ is said to be:

(i) Increasing soft neighborhood of $x \in X$ if W_E is soft neighborhood of $x \in X$ and increasing.
(ii) Decreasing soft neighborhood of $x \in X$ if W_E is soft neighborhood of $x \in X$ and decreasing.

Definition 4.3. For two soft subsets G_E and H_E of an STOS $(X, \tau, E, \preceq)$ and $x \in X$, we say that:

(i) G_E containing x provided that $x \in G_E$.
(ii) G_E containing H_E provided that $H_E \widetilde{\subseteq} G_E$.
(iii) G_E is a soft neighborhood of H_E provided that there exists a soft open set F_E such that $H_E \widetilde{\subseteq} F_E \widetilde{\subseteq} G_E$.

Definition 4.4. An STOS $(X, \tau, E, \preceq)$ is said to be:

(i) Lower p-soft T_1-ordered if for every distinct points $x \npreceq y$ in X, there exists an increasing soft neighborhood W_E of x such that $y \notin W_E$.
(ii) Upper p-soft T_1-ordered if for every distinct points $x \npreceq y$ in X, there exists a decreasing soft neighborhood W_E of y such that $x \notin W_E$.
(iii) p-soft T_0-ordered if it is lower soft T_1-ordered or upper soft T_1-ordered.
(iv) p-soft T_1-ordered if it is lower soft T_1-ordered and upper soft T_1-ordered.
(v) p-soft T_2-ordered if for every distinct points $x \npreceq y$ in X, there exist disjoint soft neighborhoods W_E and V_E of x and y, respectively, such that W_E is increasing and V_E is decreasing.

Proposition 4.5. *Every p-soft T_i-ordered space $(X, \tau, \preceq, E)$ is p-soft T_{i-1}-ordered, for $i = 1, 2$.*

Proof. It is obtained immediately from the above definition. □

In what follows, we present two examples to illustrate that the converse of the above proposition fails.

Example 4.6. Let $E = \{e_1, e_2\}$ be a set of parameters, $\preceq = \triangle \bigcup \{(x, y), (x, z)\}$ be a partial order relation on $X = \{x, y, z\}$ and $\tau = \{\widetilde{\emptyset}, \widetilde{X}, G_{1_E}, G_{2_E}, G_{3_E}\}$ be a soft topology on X. The soft sets G_{1_E}, G_{2_E} and G_{3_E} are defined as follows:

$$G_{1_E} = \{(e_1, \{y\}), (e_2, \{y\})\},$$

$$G_{2_E} = \{(e_1, \{z\}), (e_2, \{z\})\},$$

$$G_{3_E} = \{(e_1, \{y, z\}), (e_2, \{y, z\})\}.$$

Then $(X, \tau, \preceq, E)$ is a lower p-soft T_1-ordered space. So it is p-soft T_0-ordered. On the other hand, there does not exist a soft open set containing x and does not contain y or z. Thus $(X, \tau, \preceq, E)$ is not p-soft T_1-ordered.

Example 4.7. Let $E = \{e_1, e_2\}$ be a set of parameters, $\preceq = \triangle \bigcup \{(1, x) : x \in \mathcal{R}\}$ be a partial order relation on the set of real numbers $\mathcal{R}$ and $\tau = \{\widetilde{\emptyset}, G_E \widetilde{\subseteq} \widetilde{\mathcal{R}} : G^c_E$ is finite $\}$ be a soft topology on $\mathcal{R}$. Obviously, $(\mathcal{R}, \tau, \preceq, E)$ is p-soft T_1-ordered, but is not p-soft T_2-ordered.

Theorem 4.8. *Let $(X, \tau, E, \preceq)$ be an STOS. Then the following three statements are equivalent:*

(i) *$(X, \tau, E, \preceq)$ is upper (resp. lower) p-soft T_1-ordered;*
(ii) *For all $x, y \in X$ such that $x \npreceq y$, there is a soft open set G_E containing y(resp.x) in which $x \npreceq a$(resp. $a \npreceq y$) for every $a \widetilde{\in} G_E$;*
(iii) *For all $x \in X$, $(i(x))_E$(resp.$(d(x))_E$) is soft closed.*

Proof. (**i**) → (**ii**): Consider $(X, \tau, E, \preceq)$ is an upper p-soft T_1-ordered space and let $x, y \in X$ such that $x \npreceq y$. Then there exists a decreasing soft neighbourhood U_E of y such that $x \notin U_E$. Putting $G_E = Int(U_E)$. Suppose that $G_E \widetilde{\nsubseteq} (i(x))^c_E$. Then there exists $a \widetilde{\in} G_E$ and $a \notin (i(x))^c_E$. Therefor $a \in (i(x))_E$ and this implies that $x \preceq a$. Now, $a \widetilde{\in} U_E$ implies that $x \widetilde{\in} U_E$. But this contradicts that $x \notin U_E$. Thus $G_E \widetilde{\subseteq} (i(x))^c_E$. Hence $x \npreceq a$, for every $a \widetilde{\in} G_E$.

(**ii**) → (**iii**): Consider $x \in X$ and let $a \widetilde{\in} (i(x))^c_E$. Then $x \npreceq a$. Therefore there exists a soft open set G_E containing a such that $G_E \widetilde{\subseteq} (i(x))^c_E$. Since a and x are chosen arbitrary, then a soft set $(i(x))^c_E$ is soft open, for all $x \in X$. Hence $(i(x))_E$ is soft closed, for all $x \in X$.

(**iii**) → (**i**): Let $x \npreceq y$ in X. Obviously, $(i(x))_E$ is increasing and by hypothesis, $(i(x))_E$ is soft closed. Then $(i(x))^c_E$ is a decreasing soft open set satisfies that $y \in (i(x))^c_E$ and $x \notin (i(x))^c_E$. Hence the proof is completed.

A similar proof can be given for the case between parentheses. □

Corollary 4.9. *If a is the smallest element of a lower p-soft T_1-ordered space $(X, \tau, E, \preceq)$, then a_E is decreasing soft closed.*

Corollary 4.10. *If a is the largest element of an upper p-soft T_1-ordered space $(X, \tau, E, \preceq)$, then a_E is increasing soft closed.*

Proposition 4.11. *If a is the smallest (resp. largest) element of a finite p-soft T_1-ordered space $(X, \tau, E, \preceq)$, then a_E is decreasing (resp. increasing) soft open.*

Proof. We will start with the proof for the smallest element, as the proof for the largest element is analogous. Since a is the smallest element of X, then $a \preceq x$, for all $x \in X$. By the anti-symmetric of $\preceq$, we have $x \npreceq a$, for all $x \in X$. By hypothesis, there is a decreasing neighborhood W_E of a such that $x \notin W_E$. It follows that $a_E = \widetilde{\bigcap} W_E$. Since X is finite, then a_E is a decreasing soft open set. □

Proposition 4.12. *A finite STOS $(X, \tau, E, \preceq)$ is p-soft T_1-ordered if and only if it is p-soft T_2-ordered.*

Proof. Necessity: For each $y \in X \setminus (i(x))_E$, we have $(d(y))_E$ is soft closed. Since X is finite, then $\widetilde{\bigcup}_{y \in X \setminus (i(x))_E} (d(y))_E$ is soft closed. Therefore $\left(\widetilde{\bigcup}_{y \in X \setminus (i(x))_E} (d(y))_E\right)^c = (i(x))_E$ is a soft open set. Thus $(X, \tau, E, \preceq)$ is a p-soft T_2-ordered space.

Sufficiency: It follows immediately from Proposition (4.5). □

Theorem 4.13. *An STOS $(X, \tau, E, \preceq)$ is p-soft T_2-ordered if and only if for all $x \npreceq y$ in X, there exist soft open sets G_E and H_E containing x and y, respectively, such that $a \npreceq b$ for every $a \in G(e)$ and $b \in H(e)$.*

Proof. Necessity: Consider $(X, \tau, E, \preceq)$ is p-soft T_2-ordered and let $x, y \in X$ such that $x \npreceq y$. Then there exist disjoint soft neighborhoods W_E and V_E of x and y, respectively, such that W_E is increasing and V_E is decreasing. Putting $U_{E \times E} = Int(W_E) \times Int(V_E)$. Let $a \widetilde{\in} Int(W_E) = G_E$ and $b \widetilde{\in} Int(V_E) = H_E$. Suppose that $a \in G(e)$ and $b \in H(e)$ such that $a \preceq b$. As W_E is increasing and V_E is decreasing, then it follows, by assumption, that $W_E \widetilde{\bigcap} V_E \neq \widetilde{\varnothing}$. But this contradicts the disjointness between W_E and V_E. Therefore $a \npreceq b$, for every $a \in G(e)$ and $b \in H(e)$.

Sufficiency: Let $x \npreceq y$ in X and assume that for any soft open sets G_E and H_E containing x and y, respectively, we have that $i(G_E) \widetilde{\bigcap} d(H_E) \neq \widetilde{\varnothing}$. Then there exists $e \in E$ such that $x \in i(G(e)) \widetilde{\bigcap} d(H(e))$. Therefore there exist $a \in G(e)$ and $b \in H(e)$ such that $a \preceq x$ and $x \preceq b$. This means that $a \preceq b$. But this contradicts, the given hypothesis, that $a \npreceq b$ for every $a \in G(e)$ and $b \in H(e)$. Thus $i(G_E) \widetilde{\bigcap} d(H_E) = \widetilde{\varnothing}$. This completes the proof. □

Proposition 4.14. *If $(X, \tau, E, \preceq)$ is an STOS, then for each $e \in E$, a family $\tau_e = \{G(e) : G_E \in \tau\}$ with a partial order relation $\preceq$, form an ordered topology on X.*

Proof. From Proposition (2.11), a family τ_e forms a topology on X. From Definition (2.31), the triple $(X, \tau_e, \preceq)$ forms a topological ordered space. □

Proposition 4.15. *If an STOS $(X, \tau, E, \preceq)$ is p-soft T_i-ordered, then a topological ordered space $(X, \tau_e, \preceq)$ is always T_i-ordered, for $i = 0, 1, 2$.*

Proof. We prove the proposition when $i = 2$ and the other two cases are proven similarly. Let x, y be two distinct points in $(X, \tau_e, \preceq)$ such that $x \npreceq y$. As $(X, \tau, \preceq, E)$ is p-soft T_2-ordered, then there exist disjoint an increasing soft neighborhood W_E of a and a decreasing soft neighborhood V_E of b such that $b \notin W_E$ and $a \notin V_E$. Therefore $W(e)$ is an increasing neighborhood of a and $V(e)$ is a decreasing neighborhood of b in $(X, \tau_e, \preceq)$ such that $W(e) \bigcap V(e) = \varnothing$. Thus a topological ordered space $(X, \tau_e, \preceq)$ is T_2-ordered. □

Corollary 4.16. *A p-soft T_1-ordered space $(X, \tau, E, \preceq)$ contains at least $2^{|X|}$ soft open sets.*

Definition 4.17. Let $Y \subseteq X$ and $(X, \tau, E, \preceq)$ be an STOS. Then $(Y, \tau_Y, E, \preceq_Y)$ is called soft ordered subspace of $(X, \tau, E, \preceq)$ provided that (Y, τ_Y, E) is soft subspace of (X, τ, E) and $\preceq_Y = \preceq \bigcap Y \times Y$.

Lemma 4.18. *If U_E is an increasing (resp. a decreasing) soft subset of an STOS $(X, \tau, E, \preceq)$, then $U_E \widetilde{\bigcap} \widetilde{Y}$ is an increasing (resp. a decreasing) soft subset of a soft ordered subspace $(Y, \tau_Y, E, \preceq_Y)$.*

Proof. Let U_E be an increasing soft subset of an STOS $(X, \tau, E, \preceq)$. In a soft ordered subspace $(Y, \tau_Y, E, \preceq_Y)$, let $a \in i_{\preceq_Y}(U_E \widetilde{\bigcap} \widetilde{Y})$. Since $i_{\preceq_Y}(U_E \widetilde{\bigcap} \widetilde{Y}) \widetilde{\subseteq} i_{\preceq_Y}(U_E)\ \widetilde{\bigcap} i_{\preceq_Y}(\widetilde{Y}) \widetilde{\subseteq} U_E \widetilde{\bigcap} \widetilde{Y}$, then $a \in U_E \widetilde{\bigcap} \widetilde{Y}$. Therefore $i_{\preceq_Y}(U_E \widetilde{\bigcap} \widetilde{Y}) = U_E \widetilde{\bigcap} \widetilde{Y}$. Thus $U_E \widetilde{\bigcap} \widetilde{Y}$ is an increasing soft subset of a soft ordered subspace $(Y, \tau_Y, E, \preceq_Y)$.

The proof is similar in case of U_E is decreasing. □

Theorem 4.19. *The property of being a p-soft T_i-ordered space is hereditary, for $i = 0, 1, 2$.*

Proof. Let $(Y, \tau_Y, E, \preceq_Y)$ be a soft ordered subspace of a p-soft T_2-ordered space $(X, \tau, E, \preceq)$. If $a, b \in Y$ such that $a \npreceq_Y b$, then $a \npreceq b$. So by hypothesis, there exist disjoint soft neighborhoods W_E and V_E of a and b, respectively, such that W_E is increasing and V_E is decreasing. Setting $U_E = \widetilde{Y} \widetilde{\bigcap} W_E$ and $G_E = \widetilde{Y} \widetilde{\bigcap} V_E$, then from the above lemma, we obtain that U_E is an increasing soft neighborhood of a and G_E is a decreasing soft neighborhood of b. Since the soft neighborhoods U_E and G_E are disjoint, it follows that $(Y, \tau_Y, E, \preceq_Y)$ is p-soft T_2-ordered.

The theorem can be proven similarly in case of $i = 0, 1$. □

Proposition 4.20. *Every p-soft T_i-ordered space $(X, \tau, E, \preceq)$ is p-soft T_i-space, for $i = 0, 1, 2$.*

Proof. The proof comes immediately from the definition of p-soft T_i-ordered spaces and the definition of p-soft T_i-spaces, for $i = 0, 1, 2$. □

It can be given some examples to illustrate that the converse of the above theorem fails. However, for the sake of economy, we consider a set of parameters E is singleton and suffice with Example 1 and Example 6 in McCartan (1968).

Definition 4.21. An STOS $(X, \tau, E, \preceq)$ is said to be:

(i) Lower (resp. Upper) p-soft regularly ordered if for each decreasing (resp. increasing) soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint soft neighbourhoods W_E of H_E and V_E of x such that W_E is decreasing (resp. increasing) and V_E is increasing (resp. decreasing).
(ii) p-soft regularly ordered if it is both lower p-soft regularly ordered and upper p-soft regularly ordered.
(iii) Lower (resp. Upper) p-soft T_3-ordered if it is both lower (resp. upper) p-soft T_1-ordered and lower (resp. upper) p-soft regularly ordered.
(iv) p-soft T_3-ordered if it is both lower p-soft T_3-ordered and upper p-soft T_3-ordered.

Theorem 4.22. *An STOS $(X,\tau,E,\preceq)$ is lower (resp. upper) p-soft regularly ordered if and only if for all $x\in X$ and every increasing (resp. decreasing) soft open set U_E containing x, there is an increasing (resp. a decreasing) soft neighbourhood V_E of x satisfies that $Cl(V_E)\widetilde{\subseteq}U_E$.*

Proof. Necessity: Let $x\in X$ and U_E be an increasing soft open set containing x. Then U_E^c is decreasing soft closed such that $x\notin U_E^c$. By hypothesis, there exist disjoint soft neighbourhoods V_E of x and W_E of U_E^c such that V_E is increasing and W_E is decreasing. So there is a soft open set G_E such that $U_E^c\widetilde{\subseteq}G_E\widetilde{\subseteq}W_E$. Since $V_E\widetilde{\subseteq}W_E^c$, then $V_E\widetilde{\subseteq}W_E^c\widetilde{\subseteq}G_E^c\widetilde{\subseteq}U_E$ and since G_E^c is soft closed, then $Cl(V_E)\widetilde{\subseteq}G_E^c\widetilde{\subseteq}U_E$.

Sufficiency: Let $x\in X$ and H_E be a decreasing soft closed set such that $x\notin H_E$. Then H_E^c is an increasing soft open set containing x. So that, by hypothesis, there is an increasing soft neighbourhood V_E of x such that $Cl(V_E)\widetilde{\subseteq}H_E^c$. Consequently, $(Cl(V_E))^c$ is a soft open set containing H_E. Thus $d((Cl(V_E))^c)$ is a decreasing soft neighbourhood of H_E. Suppose that $V_E\widetilde{\bigcap}d((Cl(V_E))^c)\neq\widetilde{\emptyset}$. Then there exists $x\in X$ and there exists $e\in E$ such that $x\in V(e)$ and $x\in d((Cl(V))^c(e))$. So there exists $y\in(Cl(V))^c(e)$ satisfies that $x\preceq y$. This means that $y\in V(e)$. But this contradicts the disjointness between V_E and $(Cl(V_E))^c$. Thus $V_E\widetilde{\bigcap}d((Cl(V_E))^c)=\widetilde{\emptyset}$. This completes the proof.

A similar proof can be given for the case between parentheses. □

Proposition 4.23. *The following three properties are equivalent if $(X,\tau,E,\preceq)$ is p-soft regularly ordered:*

(i) *$(X,\tau,E,\preceq)$ is p-soft T_2-ordered;*
(ii) *$(X,\tau,E,\preceq)$ is p-soft T_1-ordered;*
(iii) *$(X,\tau,E,\preceq)$ is p-soft T_0-ordered.*

Proof. The direction (**i**) → (**ii**) → (**iii**) is obvious.

To prove that (**iii**) → (**i**), let $x,y\in X$ such that $x\npreceq y$. Since $(X,\tau,E,\preceq)$ is p-soft T_0-ordered, then it is lower p-soft T_1-ordered or upper p-soft T_1-ordered. Say, it is upper p-soft T_1-ordered. From Theorem (4.8), we have that $(i(x))_E$ is soft closed. Obviously, $(i(x))_E$ is increasing and $y\notin(i(x))_E$. Since $(X,\tau,E,\preceq)$ is p-soft regularly ordered, then there exist disjoint soft neighbourhoods W_E and V_E of y and $(i(x))_E$, respectively, such that W_E is decreasing and V_E is increasing. Thus $(X,\tau,E,\preceq)$ is p-soft T_2-ordered. □

Corollary 4.24. *The following three properties are equivalent if $(X,\tau,E,\preceq)$ is lower (resp. upper) p-soft regularly ordered:*

(i) *$(X,\tau,E,\preceq)$ is p-soft T_2-ordered;*
(ii) *$(X,\tau,E,\preceq)$ is p-soft T_1-ordered;*
(iii) *$(X,\tau,E,\preceq)$ is lower (resp. upper) p-soft T_1-ordered.*

Definition 4.25. An STOS $(X,\tau,E,\preceq)$ is said to be:

(i) Soft normally ordered if for each disjoint soft closed sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint soft neighbourhoods W_E of F_E and V_E of H_E such that W_E is increasing and V_E is decreasing.
(ii) p-soft T_4-ordered if it is soft normally ordered and p-soft T_1-ordered.

Theorem 4.26. *An STOS $(X,\tau,E,\preceq)$ is soft normally ordered if and only if for every decreasing (resp. increasing) soft closed set F_E and every decreasing (resp. increasing) soft open neighborhood U_E of F_E, there is a decreasing (resp. an increasing) soft neighborhood V_E of F_E satisfies that $Cl(V_E)\widetilde{\subseteq}U_E$.*

Proof. Necessity: let F_E be a decreasing soft closed set and U_E be a decreasing soft open neighborhood of F_E. Then U_E^c is an increasing soft closed set and $F_E\widetilde{\bigcap}U_E^c=\widetilde{\emptyset}$. Since $(X,\tau,E,\preceq)$ is soft normally ordered, then there exist disjoint a decreasing soft neighborhood V_E of F_E and an increasing soft neighborhood W_E of U_E^c. Since W_E is a soft neighborhood of U_E^c, then there exists a soft open set H_E such that $U_E^c\widetilde{\subseteq}H_E\widetilde{\subseteq}W_E$. Consequently, $W_E^c\widetilde{\subseteq}H_E^c\widetilde{\subseteq}U_E$ and $V_E\widetilde{\subseteq}W_E^c$. So it follows that $Cl(V_E)\widetilde{\subseteq}Cl(W_E^c)\widetilde{\subseteq}H_E^c\widetilde{\subseteq}U_E$. Thus $F_E\widetilde{\subseteq}Cl(V_E)\widetilde{\subseteq}Cl(W_E^c)\ \widetilde{\subseteq}H_E^c\widetilde{\subseteq}U_E$. Hence the necessary part holds.

Sufficiency: Let F_{1_E} and F_{2_E} be two disjoint soft closed sets such that F_{1_E} is decreasing and F_{2_E} is increasing. Then $F_{2_E}^c$ is a decreasing soft open set containing F_{1_E}. By hypothesise, there exists a decreasing soft neighborhood V_E of F_{1_E} such that $Cl(V_E)\widetilde{\subseteq}F_{2_E}^c$. Setting $H_E=\widetilde{X}\setminus Cl(V_E)$. This means that H_E is a soft open set containing F_{2_E}. Obviously, $F_{2_E}\widetilde{\subseteq}H_E, F_{1_E}\widetilde{\subseteq}V_E$ and $H_E\widetilde{\bigcap}V_E=\widetilde{\emptyset}$. Now, $i(H_E)$ is an increasing soft neighborhood of F_{2_E}. Suppose that $i(H_E)\widetilde{\bigcap}V_E\neq\widetilde{\emptyset}$. Then there exists $e\in E$ such that $x\in i(H(e))$ and $x\in V(e)=d(V(e))$. This implies that there exist $a\in H(e)$ and $b\in V(e)$ such that $a\preceq x$ and $x\preceq b$. As $\preceq$ is transitive, then $a\preceq b$. Therefore $b\widetilde{\in}H_E\widetilde{\bigcap}V_E$. This contradicts the disjointness between H_E and V_E. Thus $i(H_E)\widetilde{\bigcap}V_E=\widetilde{\emptyset}$. Hence the proof is completed. □

Proposition 4.27. *Every p-soft T_i-ordered space $(X,\tau,E,\preceq)$ is p-soft T_{i-1}-ordered, for $i=3,4$.*

Proof. From Proposition (4.23), we obtain that every p-soft T_3-ordered space is p-soft T_2-ordered. To prove the proposition in case of $i=4$, let $a\in X$ and F_E be a decreasing soft closed set such that $a\notin F_E$. Since $(X,\tau,E,\preceq)$ is p-soft T_1-ordered, then $(i(a))_E$ is an increasing soft closed set and since $(X,\tau,E,\preceq)$ is soft normally ordered, then there exist disjoint soft neighborhoods W_E and V_E of $(i(a))_E$ and F_E, respectively, such that W_E is increasing and V_E is decreasing. Therefore $(X,\tau,E,\preceq)$ is lower p-soft regularly ordered. If F_E is an increasing soft set, then we prove similarly that $(X,\tau,E,\preceq)$ is upper p-soft regularly ordered. Thus $(X,\tau,E,\preceq)$ is p-soft regularly ordered. Hence $(X,\tau,E,\preceq)$ is p-soft T_3-ordered. □

The converse of the above proposition is not always true as illustrated in the following two examples.

Example 4.28. Let $E=\{e_1,e_2,e_3\}$ be a set of parameters, $\preceq=\triangle\bigcup\{(1,2)\}$ be a partial order relation on the set of natural numbers $\mathcal{N}$ and $\tau=\{G_E\widetilde{\subseteq}\widetilde{\mathcal{N}}$ such that $1\notin G_E$ or $[1\in G(e_2)$ and G_E^c is finite $]\}$ be a soft topology on $\mathcal{N}$. Obviously, $(\mathcal{N},\tau,E,\preceq)$ is p-soft T_2-ordered. In the following, we illustrate that $(\mathcal{N},\tau,E,\preceq)$ is p-soft regularly ordered. A soft subset H_E of $(\mathcal{N},\tau,E,\preceq)$ is soft closed if $1\in H_E$ or $[1\notin H(e_2)$ and H_E is finite$]$.

On the one hand, consider $\widetilde{\emptyset}\neq H_E\neq\widetilde{\mathcal{N}}$ is a decreasing soft closed set. Then we have the following two cases:

(i) Either $1\in H_E$. Then for each $x\in\mathcal{N}$ such that $x\notin H_E$, we define a soft set G_E as follows $G(e)=\{x\}$, for each $e\in E$. So G_E is an increasing soft open set containing x and its relative complement is a decreasing soft open set containing H_E.

(ii) Or [$1 \notin H(e_2)$ and H_E is finite]. Suppose that $x \notin H_E$. Then we have the following two cases:
1. Either $x = 1$. Then $2 \notin H_E$. So we define a soft set G_E as follows $G(e) = \mathcal{N} \setminus H(e)$, for each $e \in E$. Thus G_E is an increasing soft open set containing 1 and its relative complement is a decreasing soft open set containing H_E.
2. Or $x \neq 1$. Then we define a soft set G_E as follows $G(e) = \{x\}$, for each $e \in E$. Thus G_E is an increasing soft open set containing x and its relative complement is a decreasing soft open set containing H_E.

Thus $(\mathcal{N}, \tau, E, \preceq)$ is lower p-soft regularly ordered.

On the other hand, consider $\widetilde{\varnothing} \neq H_E \neq \widetilde{\mathcal{N}}$ is an increasing soft closed set. Then we have the following two cases:

(i) Either $1 \in H_E$. Then $2 \in H_E$. So for each $x \in \mathcal{N}$ such that $x \notin H_E$, we define a soft set G_E as follows $G(e) = \{x\}$, for each $e \in E$. Thus G_E is a decreasing soft open set containing x and its relative complement is an increasing soft open set containing H_E.

(ii) Or [$1 \notin H(e_2)$ and H_E is finite]. Suppose that $x \notin H_E$. Then we have the following two cases:
1. Either $x = 1$. Then we define a soft set G_E as follows $G(e) = \mathcal{N} \setminus H(e)$, for each $e \in E$. Thus G_E is a decreasing soft open set containing 1 and its relative complement is an increasing soft open set containing H_E.
2. Or $x \neq 1$. If $x = 2$, then $1 \notin H_E$. So, by the definition of soft open sets in this soft topology, we obtain that H_E is an increasing soft open set. Obviously, its relative complement is a decreasing soft open set containing x. If $x \neq 1 \neq 2$, then we define a soft set G_E as follows $G(e) = \{x\}$, for each $e \in E$. Thus G_E is a decreasing soft open set containing x and its relative complement is an increasing soft open set containing H_E.

Thus $(\mathcal{N}, \tau, E, \preceq)$ is upper p-soft regularly ordered.

From the above discussion, we conclude that $(\mathcal{N}, \tau, E, \preceq)$ is p-soft regularly ordered. Hence $(\mathcal{N}, \tau, E, \preceq)$ is p-soft T_3-ordered. To illustrate that $(\mathcal{N}, \tau, E, \preceq)$ is not soft normally ordered, we define an increasing soft closed set H_E and a decreasing soft closed set F_E as follows:

$$H(e_1) = \{1,2\}, H(e_2) = \{3\}, H(e_3) = \{4\},$$

$$F(e_1) = \{3\}, F(e_2) = \{4\} \text{ and } F(e_3) = \{1,5\}.$$

Since the two soft closed set are disjoint and there do not exist disjoint soft neighborhoods W_E and V_E containing H_E and F_E, respectively, then $(\mathcal{N}, \tau, E, \preceq)$ is not soft normally ordered. Hence $(\mathcal{N}, \tau, E, \preceq)$ is not p-soft T_4-ordered.

Example 4.29. It can be considered that the p-soft T_i-ordered spaces are equivalent for T_i-ordered spaces if E is singleton. So by taking $E = \{e\}$, we consider Example 4 which given in McCartan (1968). It is p-soft T_2-ordered, but it is not p-soft T_3-ordered.

Definition 4.30. Let $\{(X_i, \tau_i, E_i, \preceq_i) : i \in \{1,2,\ldots,n\}\}$ be a finite family of soft topological ordered spaces. The product of these soft topological ordered spaces is given by $X = \prod_{i=1}^{i=n} X_i, \tau$ is the product topology on $X, E = \prod_{i=1}^{i=n} E_i$ and $\preceq = \{(x,y) : x,y \in X$ such that $(x_i, y_i) \in \preceq_i$ for every $i \in \{1,2,\ldots,n\}\}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

Lemma 4.31. *If $H_{E_1 \times E_2}$ is a decreasing (resp. an increasing) soft closed subset of a soft ordered product space $(X \times Y, \tau_1 \times \tau_2, E_1 \times E_2, \preceq)$, then $H_{E_1 \times E_2} = [G^c_{E_1} \times \widetilde{Y}] \widetilde{\bigcup} [\widetilde{X} \times F^c_{E_2}]$, for some increasing (resp. decreasing) soft open sets $G_{E_1} \in \tau_1$ and $F_{E_2} \in \tau_2$.*

Proof. Suppose that $H_{E_1 \times E_2}$ is a decreasing soft closed subset of a soft product space $(X \times Y, \tau_1 \times \tau_2, E_1 \times E_2, \preceq)$. Then from Lemma (2.28), there exist soft open sets $G_{E_1} \in \tau_1$ and $F_{E_2} \in \tau_2$ such that $H_{E_1 \times E_2} = [G^c_{E_1} \times \widetilde{Y}] \widetilde{\bigcup} [\widetilde{X} \times F^c_{E_2}]$.

To prove that G_{E_1} and F_{E_2} are increasing, consider that at least one of them is not increasing. Without lose of generality, consider that G_{E_1} is not increasing. Then $G^c_{E_1}$ is not decreasing. It follows that there exist $e \in E_1$ and $x \in X$ such that $P^x_e \in d(G^c_{E_1})$ and $P^x_e \notin G^c_{E_1}$. By choosing $P^y_k \notin F_{E_2}$, we obtain that $P^{(x,y)}_{(e,k)} \in d[G^c_{E_1} \times \widetilde{Y}]$ and $P^{(x,y)}_{(e,k)} \notin [G^c_{E_1} \times \widetilde{Y}] \widetilde{\bigcup} [\widetilde{X} \times F^c_{E_2}]$. This implies that $H_{E_1 \times E_2}$ is not a decreasing soft set. But this contradicts the given condition. Hence G_{E_1} and F_{E_2} are increasing soft sets.

A similar proof is given for the case between parentheses. □

Now, we are in a position to verify the following main theorem in this section.

Theorem 4.32. *The finite product of p-soft T_i-ordered spaces is p-soft T_i-ordered, for $i = 0,1,2,3,4$.*

Proof. We prove the theorem in case of $i = 2$ and $i = 3$, and the other follow similar lines.

(i) Consider $(X \times Y, \tau, E, \preceq)$ is the soft ordered product space of two p-soft T_2-ordered spaces $(X, \tau_1, E_1, \preceq_1)$ and $(Y, \tau_2, E_2, \preceq_2)$ and let (x_1, y_1) and (x_2, y_2) be two distinct points in $X \times Y$ such that $(x_1, y_1) \npreceq (x_2, y_2)$. Then $x_1 \npreceq_1 x_2$ or $y_1 \npreceq_2 y_2$. Without lose of generality, say $x_1 \npreceq_1 x_2$. Since $(X, \tau_1, E_1, \preceq_1)$ is p-soft T_2-ordered, then there exist disjoint soft neighborhoods W_{E_1} and V_{E_1} of x_1 and x_2, respectively, such that W_{E_1} is increasing and V_{E_1} is decreasing. So $W_{E_1} \times \widetilde{Y}$ is an increasing soft neighborhood of (x_1, y_1) and $V_{E_1} \times \widetilde{Y}$ is a decreasing soft neighborhood of (x_2, y_2) such that $[W_{E_1} \times \widetilde{Y}] \widetilde{\bigcap} [V_{E_1} \times \widetilde{Y}] = \widetilde{\varnothing}_{E_1 \times E_2}$. Hence the proof is completed.

(ii) Consider $(X \times Y, \tau, E, \preceq)$ is the soft ordered product space of two p-soft T_3-ordered spaces $(X, \tau_1, E_1, \preceq_1)$ and $(Y, \tau_2, E_2, \preceq_2)$ and let $H_{E_1 \times E_2}$ be a decreasing soft closed set. Then $H_{E_1 \times E_2} = (G^c_{E_1} \times \widetilde{Y}) \widetilde{\bigcup} (\widetilde{X} \times U^c_{E_2})$, for some increasing soft open sets $G_{E_1} \in \tau_1$ and $U_{E_2} \in \tau_2$. For every $(x,y) \notin H_{E_1 \times E_2}$, we have $(x,y) \notin G^c_{E_1} \times \widetilde{Y}$ and $(x,y) \notin \widetilde{X} \times U^c_{E_2}$. It follows that $x \notin G^c_{E_1}$ and $y \notin U^c_{E_1}$. Since $(X, \tau_1, E_1, \preceq_1)$ and $(Y, \tau_2, E_2, \preceq_2)$ are p-soft regularly ordered, then there exist disjoint soft neighbourhoods $F_{1_{E_1}}$ and $F_{2_{E_1}}$ of x and $G^c_{E_1}$, respectively, such that $F_{1_{E_1}}$ is increasing and $F_{2_{E_1}}$ is decreasing, and there exist disjoint soft neighbourhood $F_{3_{E_2}}$ and $F_{4_{E_2}}$ of y and $U^c_{E_2}$, respectively, such that $F_{3_{E_2}}$ is increasing and $F_{4_{E_2}}$ is decreasing. Thus $(F_{2_{E_1}} \times \widetilde{Y}) \widetilde{\bigcup} (\widetilde{X} \times F_{4_{E_2}})$ is a decreasing soft neighbourhood of $H_{E_1 \times E_2}$ in $(X \times Y, \tau, E, \preceq)$ and $(F_{1_{E_1}} \times F_{3_{E_2}})$is an increasing soft neighbourhood of (x,y) in $(X \times Y, \tau, E, \preceq)$. Since $[F_{1_{E_1}} \times F_{3_{E_2}}] \widetilde{\bigcap} [(F_{2_{E_1}} \times \widetilde{Y}) \widetilde{\bigcup} (\widetilde{X} \times F_{4_{E_2}})] = \widetilde{\varnothing}_{E_1 \times E_2}$, then $(X \times Y, \tau, E, \preceq)$ is lower p-soft regularly ordered. Similarly, one can prove that $(X \times Y, \tau, E, \preceq)$ is upper p-soft regularly ordered. Hence $(X \times Y, \tau, E, \preceq)$ is p-soft T_3-ordered. □

Definition 4.33. A soft ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of an STOS $(X, \tau, E, \preceq)$ is called soft compatibly ordered provided that for each increasing (resp. decreasing) soft closed subset H_E of $(Y, \tau_Y, E, \preceq_Y)$, there exists an increasing (resp. a decreasing) soft closed subset $H^{\star}_E$ of $(X, \tau, E, \preceq)$ such that $H_E = \widetilde{Y} \widetilde{\bigcap} H^{\star}_E$.

Theorem 4.34. *Every soft compatibly ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of a p-soft regularly ordered space $(X, \tau, E, \preceq)$ is p-soft regularly ordered.*

Proof. Let $y \in Y$ and H_E be a decreasing soft closed subset of $(Y, \tau_Y, E, \preceq_Y)$ such that $y \notin H_E$. As the soft ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of $(X, \tau, E, \preceq)$ is soft compatibly ordered, then there exists a decreasing soft closed subset $H^{\star}_E$ of $(X, \tau, E, \preceq)$ such that $H_E = \widetilde{Y} \widetilde{\bigcap} H^{\star}_E$. So that by hypothesis, there exist disjoint soft neighborhoods V_E and W_E of y and $H^{\star}_E$, respectively, such that V_E is increasing and W_E is decreasing. It follows, by Lemma (4.18) that $\widetilde{Y} \widetilde{\bigcap} V_E$ is an increasing soft neighborhood of y and $\widetilde{Y} \widetilde{\bigcap} W_E$ is a decreasing soft neighborhood of H_E in $(Y, \tau_Y, E, \preceq_Y)$ such that $(\widetilde{Y} \widetilde{\bigcap} V_E) \widetilde{\bigcap} (\widetilde{Y} \widetilde{\bigcap} W_E) = \widetilde{\varnothing}_Y$. Consequently, $(Y, \tau_Y, E, \preceq_Y)$ is lower p-soft regularly ordered. Similarly, one can prove that $(Y, \tau_Y, E, \preceq_Y)$ is upper p-soft regularly ordered. Hence the proof is completed. □

Corollary 4.35. *Every soft compatibly ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of a p-soft T_3-ordered space $(X, \tau, E, \preceq)$ is p-soft T_3-ordered.*

One can easily verify the following proposition and so the proof will be omitted.

Proposition 4.36. *Every soft closed compatibly ordered subspace $(Y, \tau_Y, E, \preceq_Y)$ of a p-soft T_4-ordered space $(X, \tau, E, \preceq)$ is p-soft T_4-ordered.*

Definition 4.37. A soft topological ordered property or soft topological ordered invariant is a property of a soft topological ordered space which is invariant under ordered embedding soft homeomorphism maps.

Theorem 4.38. *The property of being a p-soft T_i-ordered space is a soft topological ordered property, for $i = 0, 1, 2, 3, 4$.*

Proof. We prove the theorem in case of $i = 2$ and $i = 4$, and the other follow similar lines.

(i) Suppose that f_ϕ is an ordered embedding soft homeomorphism map of a p-soft T_2-ordered space $(X, \tau, A, \preceq_1)$ onto an STOS $(Y, \theta, B, \preceq_2)$ and let $x, y \in Y$ such that $x \npreceq_2 y$. Then $P^x_\beta \npreceq_2 P^y_\beta$, for each $\beta \in B$. Since f_ϕ is bijective, then there exist P^a_α and P^b_α in $\widetilde{X}$ such that $f_\phi(P^a_\alpha) = P^x_\beta$ and $f_\phi(P^b_\alpha) = P^y_\beta$ and since f_ϕ is an ordered embedding, then $P^a_\alpha \npreceq_1 P^b_\alpha$. So $a \npreceq_1 b$. By hypothesis, there exist disjoint soft neighborhoods W_E and V_E of a and b, respectively, such that W_E is increasing and V_E is decreasing. Since f_ϕ is bijective soft open, then $f_\phi(W_E)$ and $f_\phi(V_E)$ are disjoint soft neighborhoods of x and y, respectively. It follows, by Proposition (3.15), that $f_\phi(W_E)$ is increasing and $f_\phi(V_E)$ is decreasing. This completes the proof.

(ii) Suppose that f_ϕ is an ordered embedding soft homeomorphism map of a soft normally ordered space $(X, \tau, A, \preceq_1)$ onto an STOS $(Y, \theta, B, \preceq_2)$ and let H_E and F_E be two disjoint soft closed sets such that H_E is increasing and F_E is decreasing. Since f_ϕ is bijective soft continuous, then $f^{-1}_\phi(H_E)$ and $f^{-1}_\phi(F_E)$ are disjoint soft closed sets and since f_ϕ is ordered embedding, then $f^{-1}_\phi(H_E)$ is increasing and $f^{-1}_\phi(F_E)$ is decreasing. By hypothesis, there exist disjoint soft neighborhoods W_E and V_E of $f^{-1}_\phi(H_E)$ and $f^{-1}_\phi(F_E)$, respectively, such that W_E is increasing and V_E is decreasing. So $H_E \widetilde{\subseteq} f_\phi(W_E)$ and $F_E \widetilde{\subseteq} f_\phi(V_E)$. The disjointness of the soft neighborhoods $f_\phi(W_E)$ and $f_\phi(V_E)$ completes the proof. □

In the rest of this section, we present some results that connect between soft compactness and some ordered soft separation axioms.

Theorem 4.39. *If D_E is a stable soft compact subset of a p-soft T_2-ordered space $(X, \tau, E, \preceq)$, then $i(D_E)$ $(d(D_E))$ is a soft closed set.*

Proof. Consider D_E is a stable soft compact subset of a p-soft T_2-ordered space $(X, \tau, E, \preceq)$ and let $a \in (i(D_E))^c$. Then for all $b \in D_E$, we have $b \npreceq a$. Therefore there exist an increasing soft neighborhood G_{i_E} of b and a decreasing soft neighborhood H_{i_E} of a such that $G_{i_E} \widetilde{\bigcap} H_{i_E} = \widetilde{\varnothing}$. Thus $D_E \widetilde{\subseteq} \bigcup_{i \in I} G_{i_E}$. Since D_E is soft compact, then $D_E \widetilde{\subseteq} \widetilde{\bigcup}_{i=1}^{i=n} G_{i_E}$. Also, $a \in \widetilde{\bigcap}_{i=1}^{i=n} H_{i_E}$. Since $(\widetilde{\bigcup}_{i=1}^{i=n} G_{i_E}) \widetilde{\bigcap} (\widetilde{\bigcap}_{i=1}^{i=n} H_{i_E}) = \widetilde{\varnothing}$, then $i(D_E) \widetilde{\bigcap} (\widetilde{\bigcap}_{i=1}^{i=n} H_{i_E}) = \widetilde{\varnothing}$. So $a \in (\widetilde{\bigcap}_{i=1}^{i=n} H_{i_E}) \widetilde{\subseteq} (i(D_E))^c$ and this means that $a \in Int[(i(D_E))^c]$. Since a is chosen arbitrary, then $(i(D_E))^c$ is a soft open set. Hence $i(D_E)$ is soft closed. A similar proof can be given for the case between parentheses. □

Theorem 4.40. *Let F_E be a decreasing (resp. an increasing) soft compact subset of a p-soft T_2-ordered space $(X, \tau, E, \preceq)$. If $x \notin F_E$, then there exist a decreasing (resp. an increasing) soft neighborhood W_E of x and an increasing (resp. a decreasing) soft neighborhood V_E of F_E with $W_E \widetilde{\bigcap} V_E = \widetilde{\varnothing}$.*

Proof. Let F_E be a decreasing soft compact set such that $x \notin F_E$ and $y \Subset F_E$. Since F_E is decreasing, then $x \npreceq y$ and since $(X, \tau, E, \preceq)$ is p-soft T_2-ordered, then there exist disjoint soft neighborhoods W_{i_E} and V_{i_E} of x and y, respectively, such that W_{i_E} is increasing and V_{i_E} is decreasing. Therefore $\{V_{i_E}\}$ forms a decreasing soft neighborhood cover of F_E. By hypothesis, F_E is soft compact, it follows that $F_E \subseteq \widetilde{\bigcup}_{i=1}^{i=n} V_{i_E}$. Now, $\widetilde{\bigcup}_{i=1}^{i=n} V_{i_E}$ is a decreasing soft neighborhood of F_E and $\widetilde{\bigcap}_{i=1}^{i=n} W_{i_E}$ is an increasing soft neighborhood of x. In view of disjointness of the soft neighborhoods $\widetilde{\bigcup}_{i=1}^{i=n} V_{i_E}$ and $\widetilde{\bigcap}_{i=1}^{i=n} W_{i_E}$, the theorem holds. A similar proof is given in case of F_E is increasing soft compact. □

Corollary 4.41. *Every soft compact p-soft T_2-ordered space $(X, \tau, E, \preceq)$ is p-soft T_3-ordered.*

5. Conclusion

The concept of topological ordered spaces was first presented by Nachbin (1965). The idea of soft sets was given by Molodtsov (1999) for dealing with uncertain objects and then the notion of soft topological spaces was formulated depend on the soft sets notion by Shabir and Naz (2011). In this work, we present a notion of monotone soft sets and establish some properties associated with it such as the relative complement of an increasing (resp. a decreasing) soft set is decreasing (resp. increasing) and the finite product of increasing (resp. decreasing) soft sets is increasing (resp. decreasing). In the last section, we generate an STOS $(X, \tau, E, \preceq)$ which is finer than the given STS (X, τ, E) by adding a partial order relation on the universe set X and then we define new ordered soft separation axioms namely, soft T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ which are strictly stronger than soft T_i (Shabir and Naz, 2011) and p-soft T_i (El-Shafei et al., 2018) in case of $i = 0, 1, 2$. By analogy with the equivalent conditions of T_1-ordered and regularly ordered spaces on topological ordered spaces, we give the equivalent conditions for p-soft T_1-ordered and p-soft regularly ordered spaces on soft topological ordered spaces. In Proposition (4.23), we investigate the conditions under which such p-soft T_i-ordered spaces $(i = 0, 1, 2)$ are equivalent, and in Theorem (4.32), we point out that the finite product of p-soft T_i-ordered spaces is p-soft T_i-ordered, for $i = 0, 1, 2, 3, 4$. By using ordered embedding soft homeomorphism maps we define soft topological ordered properties and then verify that the property of being a p-soft T_i-ordered space is a topological ordered property, for $i = 0, 1, 2, 3, 4$. The important role which soft compactness play with some of the initiated ordered soft separation axioms are studied. From this study, it can be seen that an STOS $(X, \tau, E, \preceq)$ consider an STS if $\preceq$ is an equality relation and consider a topological ordered space if E is a singleton set. Finally, the concepts introduced and results obtained herein form an introductory platform and open scopes for studying further important topics related to soft topological ordered spaces. We plan in an upcoming paper, to introduce and study new ordered soft separation axioms by utilizing total belong $\in$ and partial non belong $\notin$.

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