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Analytic solution to the pendulum equation for a given initial conditions

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ABSTRACT

In this paper we give the analytical solution for the undamped pendulum equation for a given arbitrary initial conditions. This solution is expressed in terms of the Jacobian elliptic functions. Approximated trigonometric solution is also provided. Three practical formulas for the period of oscillations are given. The results are illustrated with examples.

1. Introduction

The pendulum is a massless rod of length$l$with a point mass (bob) m at its end (Fig. 1).

When the bob performs an angular deflection $\phi$from the equilibrium downward position, the force of gravity mg provides a restoring torque $-mgl\sin\phi$. The rotational form of Newton's second law of motion states that this torque is equal to the product of the moment of inertia ml^2times the angular acceleration $\frac{d^2\phi}{dt^2}$ (Gitterman, 2010),

$$\frac{d^2\phi}{dt^2}+\frac{g}{l}\sin\phi=0 \tag{1}$$

For small angles, $\sin\phi\approx\phi$ Eq. (1) reduces to the equation of a harmonic oscillator. Other approximations are

$$\sin\phi\approx\phi-\frac{40}{243}\phi^3 \text{ for } |\phi|<\frac{7\pi}{18}=70^\circ. \tag{2}$$

$$\sin\phi\approx\phi-\frac{40}{243}\phi^3+\frac{1}{131}\phi^5 \text{ for } |\phi|\leqslant\frac{2\pi}{3}=120^\circ \tag{3}$$

These approximations are compared graphically in Fig. 2.

2. Analytic Solution to the Pendulum Equation

Our aim is to obtain the solution to the initial value problem

$$\frac{d^2\phi}{dt^2}+\omega^2\sin\phi=0 \text{ subjected to } \phi(0)=\phi_0 \text{ and}$$
$$\phi'(0)=\dot{\phi}_0,\ \text{where } \omega=\sqrt{\frac{g}{l}}>0. \tag{4}$$

To this end, we make the transformation

$$\phi(t)=2\arctan(u(t)), \tag{5}$$

where $u=u(t)$ is the solution to the Duffing equation (Duffing, 1918)

$$u''(t)+pu(t)+qu^3(t)=0, u(0)=u_0 \text{ and } u'(0)=\dot{u}_0. \tag{6}$$

so that

$$u'(t)^2+pu(t)^2+\frac{q}{2}u(t)^4=\dot{u}_0^2+pu_0^2+\frac{q}{2}u_0^4. \tag{7}$$

Inserting the ansatz (6) into the equation $\frac{d^2\phi}{dt^2}+\omega^2\sin\phi=0$ and taking into account (7) gives

$$(-p+q-\omega^2)u(t)^3+(2pu_0^2+p+qu_0^4+2\dot{u}_0^2-\omega^2)u(t)=0. \tag{8}$$

Equating to zero the coefficients of $u(t)$ and $u(t)^3$ we obtain the system to obtain the system

$$\begin{cases} 2p\dot{u}_0^2+p+q\dot{u}_0^4+2u_0^2-\omega^2=0,\\ -p+q-\omega^2=0 \end{cases} \tag{9}$$

Solving this system gives

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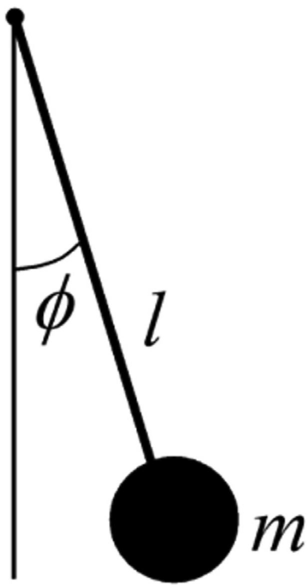

Fig. 1. Pendulum.

$$\begin{cases} p = -\frac{\left(\dot{u}_0^4 - 1\right)\omega^2 + 2u_0^2}{\left(\dot{u}_0^2 + 1\right)^2}, \\ q = \frac{2\left(\left(\dot{u}_0^2 + 1\right)\omega^2 - u_0^2\right)}{\left(\dot{u}_0^2 + 1\right)^2} \end{cases} \tag{10}$$

The values of the constants u_0 and $\dot{u}_0$are found from the initial conditions

$$\phi(0) = \phi_0 \text{ and } \phi'(0) = \dot{\phi}_0 \tag{11}$$

The general solution to the Duffing equation (Kovacic and Brennan, 2011; Salas and Castillo, 2014) $u''(t) + pu(t) + qu^3(t) = 0$ may be expressed in terms of the twelve Jacobian elliptic functions as shown below:

$$u(t) = c_1 \text{cd}\left(\sqrt{\frac{qc_1^2 + 2p}{2}}t + c_2, -\frac{qc_1^2}{qc_1^2 + 2p}\right)$$

$$u(t) = c_1 \text{cn}\left(\sqrt{qc_1^2 + pt} + c_2, \frac{qc_1^2}{2\left(qc_1^2 + p\right)}\right)$$

$$u(t) = c_1 \text{dc}\left(\sqrt{\frac{-qc_1^2}{2}}t + c_2, \frac{-qc_1^2 - 2p}{qc_1^2}\right)$$

$$u(t) = c_1 \text{dn}\left(\sqrt{\frac{qc_1^2}{2}}t + c_2, \frac{2\left(qc_1^2 + p\right)}{qc_1^2}\right)$$

$$u(t) = c_1 \text{nc}\left(\sqrt{-qc_1^2 - pt} + c_2, \frac{qc_1^2 + 2p}{2\left(qc_1^2 + p\right)}\right)$$

$$u(t) = c_1 \text{nd}\left(\sqrt{\frac{-qc_1^2 - 2p}{2}}t + c_2, \frac{2\left(qc_1^2 + p\right)}{qc_1^2 + 2p}\right)$$

$$u(t) = c_1 \text{sc}\left(\sqrt{\frac{qc_1^2 - 2p}{2}}t + c_2, \frac{2\left(p - qc_1^2\right)}{2p - qc_1^2}\right)$$

$$u(t) = c_1 \text{sd}\left(\sqrt{qc_1^2 + \sqrt{q^2c_1^4 + p^2}}t + c_2, \frac{qc_1^2 + p - \sqrt{q^2c_1^4 + p^2}}{2p}\right)$$

$$u(t) = c_1 \text{sn}\left(\sqrt{\frac{qc_1^2 + 2p}{2}}t + c_2, -\frac{qc_1^2}{qc_1^2 + 2p}\right)$$

The values of the constants c_1 and c_2 are obtained from the initial conditions

$$u(0) = u_0 \text{ and } u'(0) = \dot{u}_0 \tag{12}$$

Let

$$u(t) = c_1 \text{cn}\left(\sqrt{qc_1^2 + pt} + c_2, m\right), \text{ where } m = \frac{qc_1^2}{2\left(qc_1^2 + p\right)}. \tag{13}$$

We will make use if the addition formula

$$u(t) = \frac{c_1 \text{cn}(\sqrt{qc_1^2 + pt}, m)\text{cn}(c_2, m) - \text{sn}(\sqrt{qc_1^2 + pt}, m)\text{dn}(\sqrt{qc_1^2 + p}, m)\text{sn}(c_2, m)\text{dn}(c_2, m)}{1 - c\text{sn}^2(\sqrt{qc_1^2 + pt}, m)} \tag{14}$$

Let

$$a = c_1 \text{cn}(c_2) \text{ and } b = \text{sn}(c_2)\ \text{dn}\ (c_2) \tag{15}$$

Since

$$\text{sn}^2(c_2) + \text{cn}^2(c_2) = 1 \text{ and } \text{dn}^2(c_2) = 1 - m\text{sn}^2(c_2), \tag{16}$$

$$a^4 q + 2a^2 p + c_1^2\left(2b^2 p - 2p\right) + c_1^4\left(2b^2 q - q\right) = 0. \tag{17}$$

We have

$$u(t) = \frac{a\text{cn}(\sqrt{qc_1^2 + pt}, m) - b\text{sn}(\sqrt{qc_1^2 + pt}, m)\text{dn}(\sqrt{qc_1^2 + p}, m)}{1 - c\text{sn}^2(\sqrt{qc_1^2 + pt}, m)} \tag{18}$$

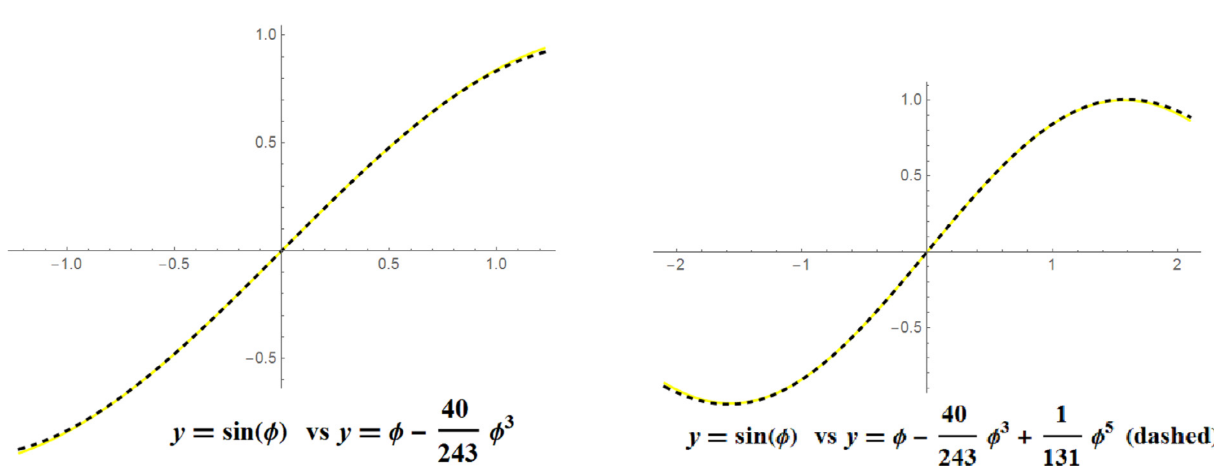

Fig. 2. Polynomial approximations to sine function.

Define the residual $R(t)$

$$R(t) = \frac{d^2\phi}{dt^2} + \omega^2 \sin\phi, \text{where} \phi(t) = 2\arctan(u(t)) \tag{19}$$

To determine the constants a, b and c we solve the system

$$\phi(0) = \phi_0, \phi'(0) = \dot{\phi}_0 \text{ and } R(0) = 0 \tag{20}$$

which reads

$$\begin{aligned} 2\tan^{-1}(a) &= \phi_0, 2b\sqrt{w} \\ &= (a^2+1)\dot{\phi}_0, w\left(2(a^2+1)c + a^2 + 2b^2 + 1\right) \\ &= (a^2+1)\omega^2. \end{aligned} \tag{21}$$

Solving system (21) and taking into account (17) we obtain, after some algebraic simplifications, the following expression for the solution to the pendulum Eq. (4):

$$\phi(t) = 2\tan^{-1}\left(\frac{\tan\left(\frac{\phi_0}{2}\right)\text{cn}(i\omega t|m) - \frac{i}{2\omega}\dot{\phi}_0 \sec^2\left(\frac{\phi_0}{2}\right)\text{dn}(i\omega t|m)\text{sn}(i\omega t|m)}{1 + \left(\frac{\dot{\phi}_0^2}{2\omega^2(\cos(\phi_0)+1)} - 1\right)\text{sn}(i\omega t|m)^2}\right),$$
$$m = \frac{1}{4}\left(2(1+\cos(\phi_0)) - \frac{\dot{\phi}_0^2}{\omega^2}\right) \tag{22}$$

This solution may also be expressed in the forms

$$\phi(t) = 2\tan^{-1}\left(\frac{\tan\left(\frac{\phi_0}{2}\right)\text{cn}(\omega t|m) + \frac{\dot{\phi}_0}{2\omega}\sec^2\left(\frac{\phi_0}{2}\right)\text{dn}(\omega t|m)\text{sn}(\omega t|m)}{1 - \frac{\dot{\phi}_0^2}{4\omega^2}\sec^2\left(\frac{\phi_0}{2}\right)\text{sn}(\omega t|m)^2}\right),$$
$$m = \frac{1}{4}\left(2(1-\cos(\phi_0)) + \frac{\dot{\phi}_0^2}{\omega^2}\right). \tag{23}$$

and

$$\phi(t) = 2\tan^{-1}\left(\frac{2\omega\left(\omega\sin(\phi_0)\text{nc}(\omega t|m) + \dot{\phi}_0\text{dc}(\omega t|m)\text{sc}(\omega t|m)\right)}{2\omega^2(1+\cos(\phi_0)) + \left(2\omega^2(1+\cos(\phi_0)) - \dot{\phi}_0^2\right)\text{sc}(\omega t|m)^2}\right),$$
$$m = \frac{1}{4}\left(2(1-\cos(\phi_0)) + \frac{\dot{\phi}_0^2}{\omega^2}\right). \tag{24}$$

The period of oscillations is given by

$$T = \frac{K(m)}{\omega} = \frac{1}{\omega}K\left(\frac{1}{4}\left(2(1-\cos(\phi_0)) + \frac{\dot{\phi}_0^2}{\omega^2}\right)\right) \tag{25}$$

This number may be approximated by means of the formulas

$$T = \frac{\pi}{2}\sqrt{\frac{l}{g}\frac{5m-16}{9m-16}} \text{ for } |m| \leqslant \frac{1}{3} \tag{26}$$

and also

$$T = \frac{\pi}{2}\sqrt{\frac{l}{g}\frac{409m^2 - 3984m + 4864}{1025m^2 - 5200m + 4864}} \text{ for } |m| \leqslant \frac{3}{4}. \tag{27}$$

$$T = \frac{\pi}{2}\sqrt{\frac{l}{g}\frac{540191m^3 - 10617024m^2 + 32584960m - 24653824}{1708091m^3 - 16837184m^2 + 38748416m - 24653824}}$$
$$\text{for } |m| < 1 \tag{28}$$

Example 1. Fig. 3.

Example 2. Fig. 4.

3. Trigonometric approximations

In this section we provide some approximations that solve in a reasonable way the pendulum equation in terms of trigonometric functions. From (3), we see that the equation of the pendulum $\frac{d^2\phi}{dt^2} + \frac{g}{l}\sin\phi = 0$may be approximated by means of the equation

$$\frac{d^2\phi}{dt^2} + \frac{g}{l}\left(\phi(t) - \frac{40}{243}\phi^3(t) + \frac{1}{131}\phi^5(t)\right) = 0 \text{ for } |\phi| \leqslant \frac{2\pi}{3} = 120^\circ. \tag{29}$$

Let us consider the case when

$$\phi(0) = \phi_0 \text{ and } \phi'(0) = 0$$

$$\phi(t) = \frac{\phi_0\cos(kt)}{\sqrt{1+\lambda\sin^2(kt)}} \tag{30}$$

Define the residual $R(t)$ as

$$R(t) = \frac{d^2\phi}{dt^2} + \omega^2\left(\phi(t) - \frac{40}{243}\phi^3(t) + \frac{1}{131}\phi^5(t)\right), \omega = \sqrt{\frac{g}{l}}. \tag{31}$$

Direct calculations show that $R'(0) = 0$. In order to find the values of the constants k and λwe solve the system

$$R(0) = 0, R''(0) = 0. \tag{32}$$

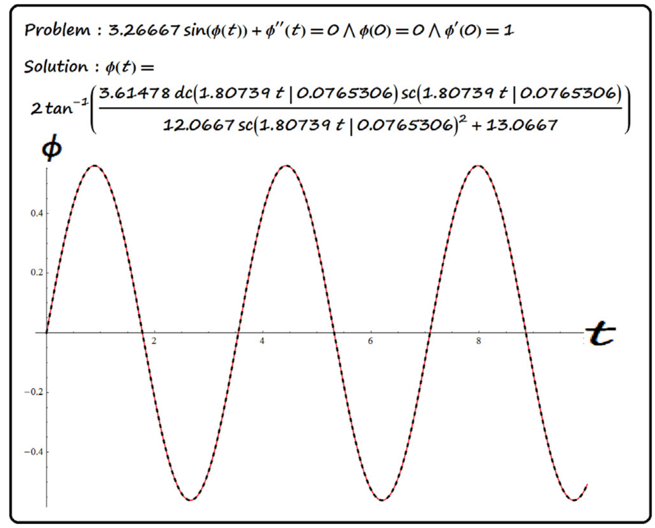

Fig. 3. The dashed curve is that of the numerical solution.

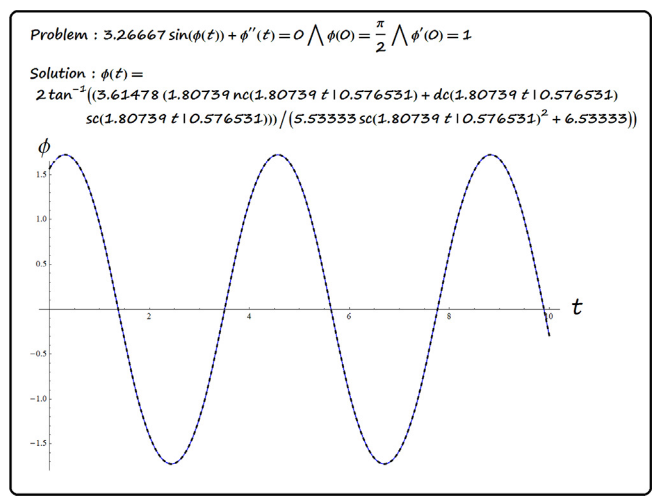

Fig. 4. The dashed curve is that of the numerical solution.

This system reads

$$\begin{cases} 31833k^2\lambda + 31833k^2 - 31833\omega^2 - 243\omega^2\phi_0^4 + 5240\omega^2\phi_0^2 = 0. \\ 95499k^2\lambda + 10611k^2 - 10611\omega^2 - 405\omega^2\phi_0^4 + 5240\omega^2\phi_0^2 = 0. \end{cases} \tag{33}$$

A solution is

$$k = \frac{1}{9}\sqrt{\frac{g}{l}}\sqrt{\frac{81}{262}\phi_0^4 - 10\phi_0^2 + 81}, \quad \lambda = 1 + \frac{262(20\phi_0^2 - 243)}{243\phi_0^4 - 7860\phi_0^2 + 63666} \tag{34}$$

so that an approximated trigonometric solution is given by

$$\phi(t) = \frac{\phi_0 \cos\left(\frac{1}{9}\sqrt{\frac{g}{l}}\sqrt{\frac{81}{262}\phi_0^4 - 10\phi_0^2 + 81}\,t\right)}{\sqrt{1 + \left(1 + \frac{262(20\phi_0^2-243)}{243\phi_0^4 - 7860\phi_0^2 + 63666}\right)\sin^2\left(\frac{1}{9}\sqrt{\frac{g}{l}}\sqrt{\frac{81}{262}\phi_0^4 - 10\phi_0^2 + 81}\,t\right)}}$$

$$\text{for } |\phi(t)| \leqslant \frac{2\pi}{3} = 120^\circ \tag{35}$$

Example 3. Compare the exact and approximated trigonometric solution for the pendulum equation

$$\frac{d^2\phi}{dt^2} + \sin\phi = 0 \text{ subjected to } \phi(0) = \frac{\pi}{2} \text{ and } \phi'(0) = 0. \tag{36}$$

Figs. 5 and 6.

Problem : $\sin(\phi(t)) + \phi''(t) = 0 \wedge \phi(0) = \frac{2\pi}{3} \wedge \phi'(0) = 0$

Exact Solution : $\phi(t) = 2\tan^{-1}\left(\frac{\sqrt{3}\,\mathrm{nc}\left(t\,\middle|\,\frac{3}{4}\right)}{\mathrm{sc}\left(t\,\middle|\,\frac{3}{4}\right)^2 + 1}\right)$

Approx. Trigonometric Solution : $\theta(t) = \dfrac{2\pi\cos\left(\frac{1}{9}\sqrt{81 - \frac{40\pi^2}{9} + \frac{8\pi^4}{131}}\;t\right)}{3\sqrt{\left(1 + \frac{262\left(-243 + \frac{80\pi^2}{9}\right)}{63666 - \frac{10480\pi^2}{3} + 48\pi^4}\right)\sin^2\left(\frac{1}{9}\sqrt{81 - \frac{40\pi^2}{9} + \frac{8\pi^4}{131}}\;t\right) + 1}}$

Square Mean Error : $\sqrt{\int_0^{10}(\phi(t) - \theta(t))^2\,dt} = 0.0322714$

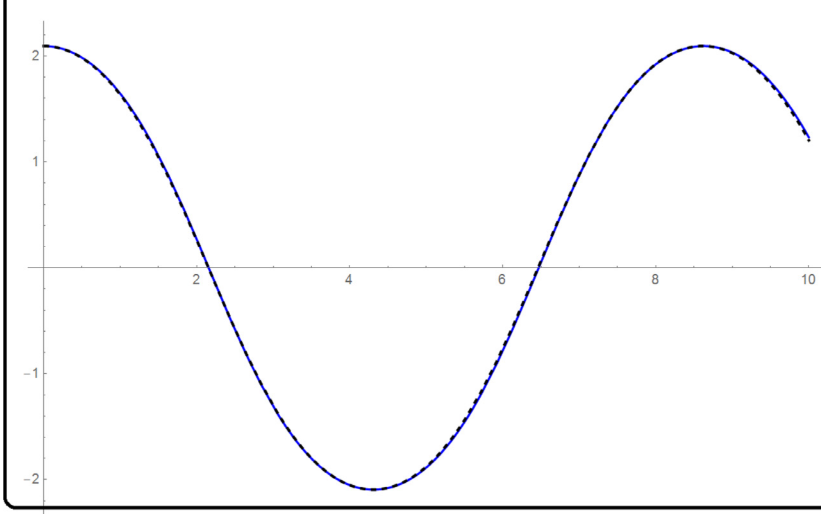

Fig. 5. The dashed curve is that of the trigonometric solution.

$$\phi(t) = 2\tan^{-1}\Bigg(\left(\frac{\omega}{\left(16\left(\dot{\phi}_0^2\sin^2(\omega\, t) - 2\omega^2(\cos(\phi_0)+1)\right)^2\right)}\right)\Bigg(16\,\omega^3\sin\left(\frac{\phi_0}{2}\right)\cos^3\left(\frac{\phi_0}{2}\right)\left(m\left(4\,\omega\, t\sin(\omega\, t) + \cos(3\,\omega\, t)\right) - (m-16)\cos(\omega\, t)\right) +$$
$$\dot{\phi}_0\Big(\dot{\phi}_0\sin(\omega\, t)\big(\omega\sin(\phi_0)\left(-12\,m\,\omega\, t + 2(3\,m-8)\sin(2\,\omega\, t) + m\sin(4\,\omega\, t) - 4\,m\,\omega\, t\cos(2\,\omega\, t)\right) -$$
$$2\,\dot{\phi}_0\sin(\omega\, t)\left(4\,m\,\omega\, t\cos(\omega\, t) + 2\sin(\omega\, t)\left(m\cos(2\,\omega\, t) - 3\,m + 8\right)\right)\big) -$$
$$8\,\omega^2\cos^2\left(\frac{\phi_0}{2}\right)\left(4\,m\,\omega\, t\cos(\omega\, t) + 2\sin(\omega\, t)\left(-3\,m\cos(2\,\omega\, t) + m - 8\right)\right)\Big)\Bigg)\Bigg)$$

Fig. 6. The dashed curve is that of the trigonometric solution.

Problem : $9.8\sin(\phi(t)) + \phi''(t) = 0 \wedge \phi(0) = \frac{\pi}{4} \wedge \phi'(0) = 0.25$

Exact Solution : $\phi(t) =$

$$2\tan^{-1}\left(\frac{6.26099\,(2.21359\,\mathrm{nc}(3.1305\,t\,|\,0.148041) + 0.25\,\mathrm{dc}(3.1305\,t\,|\,0.148041)\,\mathrm{sc}(3.1305\,t\,|\,0.148041))}{33.3968\,\mathrm{sc}(3.1305\,t\,|\,0.148041)^2 + 33.4593}\right)$$

Approx. Trigonometric Solution : $\theta(t) =$

$$2\tan^{-1}\left(\left(0.782624\left(-18.1669\,t\cos^3(3.1305\,t) + 12.522\left(0.148041\left(\frac{0.0625\cos(6.26099\,t) - 39.0125}{\sqrt{2}} - 9.83125\,t\right)\sin^2(3.1305\,t) + 2\sqrt{2}\,(0.0625\cos(6.26099\,t) + 39.1375)\right)\cos(3.1305\,t) + \left(-1.31081\,(0.0625\cos(6.26099\,t) - 39.0125)\,t^2 + 8.93631\cos(6.26099\,t) + 0.00231314\cos(12.522\,t) + 153.665\right)\sin(3.1305\,t) + 55.4372\,\left(6.26099\,(2\sqrt{2} - 0.0370102\,t)\cos(3.1305\,t) + \sin(3.1305\,t)\,(0.25\,(0.444123\cos(6.26099\,t) + 7.85196) + 0.327703\,(6.26099\,t - \sin(6.26099\,t)))\right)\right)\right) \Big/ (0.0625\cos(6.26099\,t) + 66.8561)^2\right)$$

Square Mean Error on $0 \le t \le 2.08828$: $\sqrt{\int_0^T (\phi(t) - \varepsilon(t))^2\,dt} = 0.0404721$

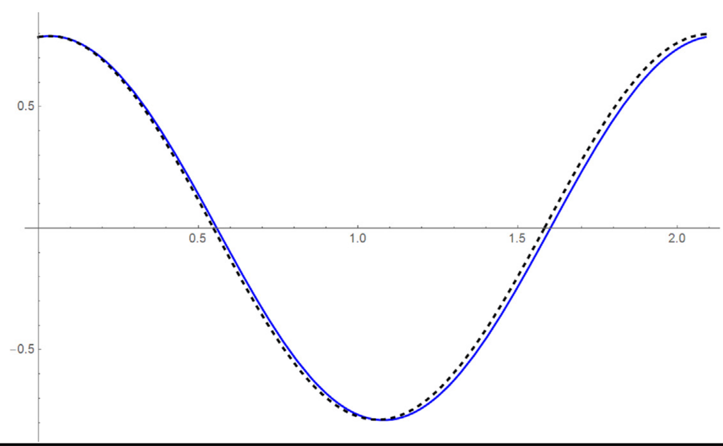

Fig. 7. The dashed curve is that of the trigonometric solution.

If we are interested in a trigonometric solution to the pendulum equation for a given initial conditions, we may approximate the exact solution by means of trigonometric functions. Let us consider solution given by Eq. (24). The trigonometric approximation is good when the modulus *m*is a small number. This approximation reads.

Other approximations may be found in Belendez et al. (2012) and Belendez et al. (2016).

Example 4. Fig. 7

4. Conclusions

We have derived the exact solution to the undamped pendulum equation for arbitrary given initial conditions. We conclude that the pendulum equation is closely related to both cubic and cubic-quintic Duffing oscillator equations. Two kinds of trigonometric approximate solutions were provided. We may say that trigonometric approximations are good when the modulus of the Jacobian elliptic functions is a small number say, a number in absolute value less that 0.2.The techniques used are applicable for solving the Duffing and the cubic-quintic Duffing equations (Belendez et al., 2012; Belendez et al., 2016) by means of trigonometric ansatze. We think that some of the formulas in this work are new in the literature. Other approaches to nonlinear pendulum may be found in Gitterman (2010), Johannessen (2014) and Salas and Castillo (2014)

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