Original article

A new third order convergent numerical solver for continuous dynamical systems

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ABSTRACT

Continuous dynamical systems are attempted to solve via proposed third order convergent numerical solver. The solver is shown to possess third order convergence on the basis of containing $\mathcal{O}\left(h^4\right)$ term in the leading coefficient of the local truncation error which has further been employed for the local and global error bounds whereas its increment function is found to be Lipchitz continuous. Consistency of the solver has extensively been discussed using Taylor's series expansion. The necessary and sufficient conditions for the solver to be stable have been proved accompanying the linear stability function obtained via Dahlquist test problem. In order to illustrate the performance of the solver, few scalar and vector-valued dynamical systems of both linear and nonlinear nature have numerically been solved in comparison to two well-known numerical solvers having same order of convergence as that of the proposed solver. Whereupon, the proposed solver is found to contain the smallest errors. The analysis of comparison is based upon three kinds of errors; namely, maximum absolute global relative error, absolute relative error determined at the final nodal point along the integration interval and the ℓ^2-error norm. MATLAB software having version 9.3.0.713579 (R2017b), using Intel(R) Core(TM) i3-4500U processor running on 1.70 GHz has been used.

1. Introduction

Mathematical models containing ordinary differential equations (ODEs) play a vital role in various fields of study. Areas including engineering, applied mathematics, bio-medical, statistical mechanics, quantum mechanics, economics, image processing, control theory, and many others are not considered to be complete without inclusion of subjects on differential equations and their applications in respective fields (Amann, 2011; Butcher, 2016; Hirsch et al., 2012; Jordan and Peter, 1999; Mermoud and Lausanne, 2015; Shampine, 2018; Stickler and Ewald, 2014; Teschl, 2000). Unfortunately, large quantity of such physical and natural models have no closed form solutions due to one or other type of non-linearity associated with the models.

Due to the universal acceptance of the importance of such mathematical models, many researchers have devised new solvers to get the desired solutions of the models under their study whereas others have substantially improved the performance of many existing solvers with regard to the convergence order, computational complexity, stability, consistency and error bounds.

In order to solve the scalar and continuous dynamical systems, the authors in (Ma and Simos, 2017; Rabiei and Fudziah, 2012; Rabiei et al., 2012a; Rabiei et al., 2012b) have improved the efficiency for the well-known family of explicit linear Runge-Kutta numerical solvers by decreasing the number of slope evaluations in its each application while maintaining the convergence order whereas others in (Qureshi and Ramos, 2018; Ramos et al., 2010; Ramos and Jesús, 2008) have devised new nonlinear numerical solvers to deal with the autonomous and non-autonomous continuous dynamical systems having singular or singularly perturbed solutions at some point of the time interval $[0, T]$. Rational block and trigonometrically fitted solvers have recently been proposed for the efficient solutions of initial value problems having oscillatory solutions, in particular (Ehigie et al., 2017; Jesús et al., 2017; Jesús et al., 2008; Kalogiratou, 2016; Tsitouras and Simos, 2018).

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Apart from this, the most recent work is discussed in (Bakodah, 2018; Liu, 2018; Moutsinga et al., 2018; Neštický et al., 2017; Nuruddeen and Nass, 2017; Qureshi and Emmanuel, 2018; Qureshi et al., 2018; Raza and Khan, 2019).

There are many reasons for one numerical solver to become superior over the other. One of the major reasons is the convergence order (a numerical solver is of order r if the error is proved to have $\mathcal{O}(h^r)$ as $h \to 0$ for a constant time step length $h > 0$) which shows the rate at which the numerical error in an application of a constant time step length solver along $[0, T]$ decreases as a consequence of decreasing the time step length h. It is therefore a numerical solver having first order convergence would experience halving of the numerical error when the time step length h is halved. Likewise, second order convergent numerical solver would yield the error being multiplied by a factor of $1/2^2$ when h is halved.

Nevertheless, the numerical errors observed in practice could be different (worse or better) from those predicted and it is because the convergence orders are asymptotic results and apply only when $h \to 0$. There is always an asymptotic error constant that depends on the solver and the equation which decides the size of the true error. This is the reason for two numerical solvers of same convergence order to have entirely different behavior in errors when they are used for solving the same initial value problem.

The present study aims to propose a new numerical solver having third order convergence. The proposed solver is of explicit linear single-step structure but the inclusion of partial derivative with respect to the second variable $y(t)$ of the slope u makes it well comparable with existing third order convergent solvers of similar characteristics. Thus, it can be included in the family of explicit single-step numerical solvers which are considered to be cost effective and easy to be implemented within a computer code.

2. Derivation of the proposed numerical solver

Suppose that a well-defined initial value problem is given as

$$\frac{dy(t)}{dt} = u(t, y(t)), \ y(t_0) = y_0, y, u(t, y(t)) \in \mathbb{R}, \ t \in [0, T] \subset \mathbb{R}, \ T > 0.$$
(1)

Further, it is assumed that the problem (1) possesses a unique continuously differentiable solution $y(t)$ where $y(t_i) \approx y_i$ for y_i is taken to be the approximation to the analytical (exact) solution $y(t)$ at $t = t_i$ over the associated interval of integration $[0, T]$ with $h = T/n, \ (i = 0, 1, \dots n)$ as a fixed time step length used at each step of the integration.

where $\Phi_u(t_i, y_i; h)$ can be expressed in terms of Taylor series expansion of an arbitrary function $u(t, y)$. Moreover, the Taylor series expansion of $y(t_i + h)$ is represented as:

$$y(t_i + h) = y(t_i) + hu + \frac{1}{2!}h^2(u_t + uu_y)$$
$$+ \frac{1}{3!}h^3(u_{tt} + 2uu_{ty} + u^2 u_{yy} + uu_y^2 + u_t u_y)$$
$$+ \frac{1}{4!}h^4(u_{ttt} + 3uu_{tty} + 3u^2 u_{tyy} + 5uu_y u_{ty} + 3u_t u_{ty}$$
$$+ u^3 u_{yyy} + 4u^2 u_y u_{yy} + 3uu_t u_{yy} + uu_y^3 + u_t u_y^2 + u_{tt} u_y)$$
$$+ \mathcal{O}(h^5).$$
(3)

Taking into consideration the general structure for the single-step numerical solver, we formulated the required solver as follows:

$$y_{i+1} = y_i + h(c_1 m_1 + c_2 m_2 + c_3 m_3).$$
(4)

The slopes in (4) are structured as follows:

$$m_1 = u(t_i, y_i),$$
$$m_2 = u(t_i + p_2 h, y_i + hm_1(q_{21} + hr_{21}u_y)),$$
$$m_3 = u(t_i + p_3 h, y_i + h(q_{31}m_1 + q_{32}m_2) + h^2 r_{31}m_1 u_y).$$
(5)

Expanding m_2 and m_3 in Taylor's series, we obtain

$$m_2 = u + (uu_y q_{21} + u_t p_2)h +$$
$$\left(\tfrac{1}{2}u_{yy}u^2 q_{21}^2 + u_{ty}up_2 q_{21} + uu_y^2 r_{21} + \tfrac{1}{2}u_{tt}p_2^2\right)h^2$$
$$+ (\tfrac{1}{6}u_{yyy}u^3 q_{21}^3 + \tfrac{1}{2}u_{tyy}u^2 p_2 q_{21}^2 + u_{yy}u^2 u_y q_{21} r_{21}$$
$$+ \tfrac{1}{2}u_{tty}up_2^2 q_{21} + u_{ty}uu_y p_2 r_{21} + \tfrac{1}{6}u_{ttt}p_2^3)h^3 + \mathcal{O}(h^4).$$

$$m_3 = u + (u_y uq_{32} + u_y uq_{31} + u_t p_3)h + (\tfrac{1}{2}u_{yy}u^2 q_{32}^2 + u_{yy}u^2 q_{31}q_{32}$$
$$+ \tfrac{1}{2}u_{yy}u^2 q_{31}^2 +$$
$$u_{ty}up_3 q_{32} + u_{ty}up_3 q_{31} + u_y^2 ur_{31} + \tfrac{1}{2}u_{tt}p_3^2 + u_y^2 uq_{21}q_{32} + u_t u_y p_2 q_{32})h^2$$
$$+ (\tfrac{1}{2}u_{yyy}u^3 q_{31}q_{32}^2 + \tfrac{1}{2}u_{yyy}u^3 q_{31}^2 q_{32} + \tfrac{1}{2}u_{tyy}u^2 p_3 q_{32}^2 + \tfrac{1}{2}u_{tyy}u^2 p_3 q_{31}^2$$
$$+ \tfrac{1}{2}u_{tty}up_3^2 q_{32} + \tfrac{1}{2}u_{tty}up_3^2 q_{31} + \tfrac{1}{2}u_y u^2 u_{yy}q_{21}^2 q_{32} + u_y^3 uq_{32}r_{21}$$
$$+ u_y uu_{ty}p_2 q_{21}q_{32} + \tfrac{1}{2}u_y u_{tt}p_2^2 q_{32} + \tfrac{1}{6}u_{yyy}u^3 q_{32}^3 + \tfrac{1}{6}u_{yyy}u^3 q_{31}^3$$
$$+ u_{tyy}u^2 p_3 q_{31}q_{32} + u_{yy}u^2 u_y q_{31}r_{31} + u_{ty}uu_y p_3 r_{31} + u_y u^2 u_{yy}q_{21}q_{31}q_{32}$$
$$+ u_t uu_{yy}p_2 q_{31}q_{32} + \tfrac{1}{6}u_{ttt}p_3^3 + u_y u^2 u_{yy}q_{21}q_{32}^2 + u_t uu_{yy}p_2 q_{32}^2$$
$$+ u_y uu_{ty}p_3 q_{21}q_{32} + u_t u_{ty}p_2 p_3 q_{32} + u_{yy}u^2 u_y q_{32}r_{31})h^3 + \mathcal{O}(h^4).$$

Substituting the result of m_1, m_2 and m_3 into (4) and equating the coefficients of powers of h up to h^3 with that of (3), one obtains:

$$c_1 + c_2 + c_3 = 1 \qquad p_2 c_3 q_{32} = \tfrac{1}{6} \qquad p_2 c_2 + p_3 c_3 = \tfrac{1}{2}$$
$$\tfrac{1}{2}(p_2^2 c_2 + p_3^2 c_3) = \tfrac{1}{6} \qquad c_2 q_{21} + c_3 q_{31} + c_3 q_{32} = \tfrac{1}{2} \qquad p_2 c_2 q_{21} + p_3 c_3 q_{31} + p_3 c_3 q_{32} = \tfrac{1}{3}$$
$$c_2 r_{21} + c_3 r_{31} + c_3 q_{21}q_{32} = \tfrac{1}{6} \qquad \tfrac{1}{2}(c_2 q_{21}^2 + c_3 q_{31}^2 + c_3 q_{32}^2) + c_3 q_{31}q_{32} = \tfrac{1}{6}.$$

A numerical solver having explicit nature for single-step structure is generally formulated as:

$$y_{i+1} = y_i + h\Phi_u(t_i, y_i; h),$$
(2)

This non-linear system contains eight equations and ten variables which has been solved using MATLAB to obtain one of its optimal solutions and finally we obtained the following numerical solver with third order accuracy:

$$m_1 = u(t_i, y_i),$$
$$m_2 = u\left(t_i + \tfrac{2}{3}h, y_i + hm_1\left(\tfrac{2}{3} + h\tfrac{1}{2}u_y\right)\right),$$
$$m_3 = u\left(t_i + \tfrac{2}{3}h, y_i + h\left(-\tfrac{5}{6}m_1 + \tfrac{3}{2}m_2\right) - h^2\tfrac{7}{4}m_1u_y\right), \quad (6)$$
$$y_{i+1} = y_i + \tfrac{h}{12}(3m_1 + 7m_2 + 2m_3).$$

3. Analysis for the proposed solver

For the required analysis, some definitions and the related theorems have been stated.

3.1. Consistency

Definition 1. A numerical solver used for finding the solution of an initial value problem in ODEs is said to be consistent if

$$\lim_{h \to 0} \Phi_u(t, y; h) = u(t, y(t)), \quad (7)$$

where Φ_u is the increment function for the numerical solver.

The increment function of the proposed solver is as follows:

$$\Phi_u(t, y; h) = \frac{1}{4}(3m_1 + 7m_2 + 2m_3), \quad (8)$$

which clearly satisfies the condition (7) of the above definition for the consistency. Thus, the proposed solver is found to be consistent with the given initial value problem (1).

3.2. Stability

Theorem 1. Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_n$ be real numbers satisfying

$$|\lambda_{j+1}| \leqslant (1 + \alpha)|\lambda_j| + \beta, \ \alpha > 0, \beta \geqslant 0, \ j = 0, 1, 2, \ldots n - 1. \quad (9)$$

Thus,

$$|\lambda_n| \leqslant \exp(n\alpha)|\lambda_0| + \frac{\exp(n\alpha) - 1}{\alpha}\beta. \quad (10)$$

Proof. Using the assumption presented in the above Theorem 1, we can write the following:

$$|\lambda_1| \leqslant (1 + \alpha)|\lambda_0| + \beta,$$
$$|\lambda_2| \leqslant (1 + \alpha)^2|\lambda_0| + (1 + \alpha)\beta + \beta,$$
$$|\lambda_3| \leqslant (1 + \alpha)^3|\lambda_0| + (1 + \alpha)^2\beta + (1 + \alpha)\beta + \beta,$$
$$\vdots$$
$$|\lambda_n| \leqslant (1 + \alpha)^n|\lambda_0| + \frac{(1+\alpha)^n - 1}{\alpha}\beta,$$
$$\leqslant \exp(n\alpha)|\lambda_0| + \frac{\exp(n\alpha) - 1}{\alpha}\beta, \text{ for } \alpha > -1; \ 0 < 1 + \alpha \leqslant \exp(\alpha).$$
$$(11)$$

Theorem 2. Suppose that y_{i+1} and $\hat{y}_{i+1}$ be two numerical solutions to the ODE $y'(t) = u(t, y(t))$ produced by a numerical solver with prescribed initial conditions $y(t_0) = y_0$ and $\hat{y}(t_0) = \hat{y}_0$ respectively such that $|y_0 - \hat{y}_0| < \epsilon, \epsilon > 0$. The following condition

$$|y_{i+1} - \hat{y}_{i+1}| \leqslant M|y_0 - \hat{y}_0|, \ M > 0, \quad (12)$$

is considered to be the necessary and sufficient condition for the numerical solver to be stable.

Proof. Using the above Theorem 2, the proposed solver would give us the following:

$$y_{i+1} - y_i = h\Phi_u(t_i, y_i; h),$$
$$\hat{y}_{i+1} - \hat{y}_i = h\Phi_u(t_i, \hat{y}_i; h). \quad (13)$$

Using the triangle inequality, we have

$$|y_{i+1} - \hat{y}_{i+1}| \leqslant |y_i - \hat{y}_i| + h|\Phi_u(t_i, y_i; h) - \Phi_u(t_i, \hat{y}_i; h)|,$$
$$\leqslant |y_i - \hat{y}_i| + h|y_i - \hat{y}_i|,$$
$$= \left(1 + \hat{L}h\right)|y_i - \hat{y}_i|, \quad (14)$$
$$= \left(1 + \hat{L}h\right)^i|y_0 - \hat{y}_0|.$$

Employing the Theorem 1 while taking $\alpha = \hat{L}h$ and $\beta = 0$, we obtain the following

$$|y_{i+1} - \hat{y}_{i+1}| \leqslant M|y_0 - \hat{y}_0|, \ M = \exp\left(\hat{L}hi\right). \quad (15)$$

This confirms the proof for the proposed solver to be stable.

Further, the Dahlquist (Dahlquist, 1956) test problem $y'(t) = \mu y(t), \ y(0) = 1, \ Re(\mu) < 0$ gives the following stability function for the proposed solver.

$$\Psi(z) = 1 + \frac{11}{12}z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{8}z^4, \ z = \mu h. \quad (16)$$

3.3. Error analysis

Theorem 3. Suppose that (t_i, y_i) and $(t_i, \hat{y}_i)$ be any two points in the region S defined by $S = \{(t, y) \in \mathbb{R}^2 | t_0 \leqslant t \leqslant t_n, -\infty < y < \infty\}$, and that u is a Lipschitz continuous function on S such that $|u(t_i, y_i) - u(t_i, \hat{y}_i)| \leqslant L|y_i - \hat{y}_i|$, then the increment function Φ is Lipschitz continuous, and $|\Phi(t_i, y_i; h) - \Phi(t_i, \hat{y}_i; h)| \leqslant \hat{L}|y_i - \hat{y}_i|$ where L and $\hat{L}$ are, respectively, the Lipschitz constants for u and Φ.

Proof. Employing the assumptions above, we get the following

$$|m_1 - \hat{m}_1| \leqslant L|y_i - \hat{y}_i|. \quad (17)$$

$$|m_2 - \hat{m}_2| \leqslant \left(1 + \frac{2}{3}hL + \frac{1}{2}h^2L^2\right)|y_i - \hat{y}_i|, \because |u_y| \leqslant L. \quad (18)$$

$$|m_3 - \hat{m}_3| \leqslant \left(1 + \frac{2}{3}hL - \frac{3}{4}h^2L^2 + \frac{3}{4}h^3L^3\right)|y_i - \hat{y}_i|. \quad (19)$$

Thus, the increment function Φ for the proposed solver becomes

$$|\Phi_u(t, y; h) - \Phi_u(t, \hat{y}; h)| \leqslant \hat{L}|y_i - \hat{y}_i|, \quad (20)$$

where $\hat{L} = \frac{3}{4} + \frac{1}{4}L + \frac{1}{2}hL + \frac{1}{6}h^2L^2 + \frac{1}{8}h^3L^3$.

This completes the proof for the increment function Φ_u for the proposed solver to be Lipchitz continuous.

3.3.1. Error bounds
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Definition 2. Suppose that $y(t_i)$ be the analytical (exact) solution of the underlying initial value problem at $t = t_0$ and y_i be its corresponding numerical (approximate) solution obtained via a numerical solver. The global error Ω_i at $t = t_i$ is defined as:

$$\Omega_i = y(t_i) - y_i. \quad (21)$$

Definition 3. The local error Ξ_{i+1} at $t = t_{i+1}$ for a numerical solver like the one being studied in the present study is defined as:

$$\Xi_{i+1} = y(t_{i+1}) - y_{i+1}, \quad (22)$$

where it should be assumed that the history of the solver is free from any error, that is, $y(t_i) = y_i$.

Definition 4. The local error Ξ_{i+1} of a numerical solver with order of accuracy r is also defined as

$$\Xi_{i+1} = \Psi(t_i, y_i)h^{r+1} + \mathcal{O}\left(h^{r+2}\right), \tag{23}$$

where the first term in Ξ_{i+1} is known as the leading local truncation error.

In order to obtain the local truncation error of the proposed integrator, a usual functional associated to the integrator has been considered, that is given below:

$$\Xi(v(t), h) = v(t + h) - y_{i+1}, \tag{24}$$

where $v(t)$ is an arbitrary function defined along the integration interval $[0, T]$ and differentiable as many times as required. Having expanded it into Taylor series about t and collecting the terms in h, the local truncation error under local assumption of the following form has been obtained that ensures at least third order accuracy of the proposed solver:

$$\Xi_{i+1} = \frac{h^4}{216}\left(3uu_{tty} + 3u_t u_{ty} + u^2 u_{tyy} + uu_t u_{yy} - uu_y u_{ty} - u_{tt}u_y \right.$$
$$\left. - 18uu_y^3 + 9u_t u_y^2 + u^3 u_{yyy} + u_{ttt}\right) + \mathcal{O}\left(h^5\right). \tag{25}$$

Lotkin in (Lotkin, 1951) introduced the concept of determining the local bounds for the function u and its partial derivatives for $t \in [0, T]$ and $y \in (-\infty, \infty)$ as follows:

$$|u(t, y)| < P, \quad \left|\frac{\partial^{l+m}u}{\partial t^l \partial u^m}\right| < \frac{Q^{l+m}}{P^{m-1}}, \ (l + m) \leqslant r, \tag{26}$$

where P and Q are some positive real numbers and r is the order of accuracy of the numerical solver. Using this idea of the Lotkin's bounds, a bound for the leading local error of the proposed solver is determined to be as follows:

$$|\Psi(t_i, y_i)| < \frac{47}{216}h^4 PQ^3. \tag{27}$$

A bound for the leading local error of a single step explicit linear numerical solver is also considered to be a bound for its local truncation error. Using the fact, the local error bound for the proposed solver is finalized as follows:

$$|\Xi_{i+1}| < Ah^4, \ A = \frac{47}{216}PQ^3. \tag{28}$$

3.4. Convergence

Theorem 4. *The convergence for a numerical solver is guaranteed if for every initial value problem $y'(t) = u(t, y(t)), y(t_0) = y_0$ with u being a Lipchitz continuous function, one obtains*

$$\limsup_{h \to 0 \ 0 \leqslant i \leqslant N} \|y(t_i) - y_i\| \to 0, \forall \ t > 0. \tag{29}$$

Proof. Let us consider the proposed solver to be

$$y_{i+1} - y_i - h\Phi_u(t_i, y_i; h) = 0. \tag{30}$$

Using the definition of the local error in conjunction with the above theorem (4), one obtains

$$|\Omega_{i+1}| \leqslant |\Omega_i| + h|\Phi_u(t_i, y(t_i); h) - \Phi_u(t_i, y_i; h)| + |\Xi_i|, \tag{31}$$

where Ω_i is the global error at $t = t_i$.

Employing the Theorems 1 and 3, we obtain

$$|\Omega_{i+1}| \leqslant \left(1 + h\widehat{L}\right)|\Omega_i| + |\Xi_i|. \tag{32}$$

Suppose that $R = \sup_{0 \leqslant i \leqslant N}|\Xi_i|$, then from Theorem 1 one can obtain

$$|\Omega_i| \leqslant \exp\left(i\widehat{L}h\right)|\Omega_0| + \frac{R}{\widehat{L}h}\left(\exp\left(i\widehat{L}h\right) - 1\right). \tag{33}$$

This finally yielded the global error bound for the proposed solver on the basis of having $\Omega_0 = 0$ and $h = t_i/i$.

$$|\Omega_i| \leqslant \frac{R}{\widehat{L}h}\left(\exp\left(\widehat{L}t_i\right) - 1\right), \tag{34}$$

where R is the local error bound as found in (28).

Thus, the convergence of the proposed solver is verified for

$$\lim_{h \to 0}|\Omega_i| = 0 \Rightarrow \lim_{h \to 0}|y(t_i) - y_i| = 0. \tag{35}$$

4. Numerical experiments

Example 1. As shown in the Table 1 where the proposed solver shows the three-order decreasing behavior in the maximum absolute global relative errors $\left(E_{max} = \max_{0 \leqslant i \leqslant T}|y(t_{i+1}) - y_{i+1}|\right)$ and in the errors computed at final nodal point $(E(t = T) = |y(T) - y_N|)$ over $[0, T]$ for every one-order decrease in the time step length h. The ℓ^2-error norm $\left(\ell^2 = \sqrt{\sum_{k=0}^n |y(t_{i+1}) - y_{i+1}|^2}\right)$ is shown in the last row. In addition, the computed errors are smallest in every case upon comparison with Heun and RK3HM (Wazwaz, 1990).

$$y'(t) = ty(t)^3 - y(t), \ y(0) = 1, \ t \in [0, 2]$$
$$y(t) = \frac{2}{\sqrt{2 + 4t + 2\exp(2t)}} \tag{36}$$

Example 2. The observation for the linear problem with exponentially increasing solution based upon the Table 2 is more or less same as was in the former example.

$$y'(t) = t^2 y(t), \ y(0) = 1, \ t \in [0, 1]$$
$$y(t) = \exp(t^3/3) \tag{37}$$

Example 3. The third initial value problem is structured in the form of a well-known nonlinear first order Riccati ODE. The proposed solver as shown in the Table 3 provides smallest magnitude of errors when compared with Heun and RK3HM methods. With one-order decrease in the time step length h, there are three-order decrease in the maximum absolute global relative errors and in the absolute relative errors determined at the final nodal point over $[0, 0.5]$.

$$y'(t) = \frac{2\cos^2(t) - \sin^2(t) + y^2(t)}{2\cos(t)}, \ y(0) = -1, \ t \in [0, 0.5]$$
$$y(t) = \sin(t) - \frac{1}{0.5\sin(t) + \cos(t)}. \tag{38}$$

Example 4. Here, the errors are computed for each unknown function $x(t)$ and $y(t)$ of the system as shown in the Tables 4–6 where decreasing behavior is clearly seen for all the solvers with the proposed solver having the smallest errors almost for every time step length h.

$$x'(t) = x(t) - 10y(t), \ x(0) = 0,$$
$$y'(t) = 15x(t) + y(t), \ y(0) = 1$$
$$x(t) = -\sqrt{\frac{2}{3}}\exp(t)\sin\left(5\sqrt{6}t\right) \tag{39}$$
$$y(t) = \exp(t)\cos\left(5\sqrt{6}t\right).$$

Example 5. The angular displacement $x(t)$ for an undamped pendulum with a driving force $f(t) = \cos(4t)$ can be modeled in terms of a second order ODE as shown below. (See Fig 1)

Table 1
Maximum absolute global relative errors on $[0, 2]$ (first row), absolute relative errors at $t = 2$ (second row), and ℓ^2-error norm (third row) for the Example-1.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	1.3048e−04 1.3048e−04 4.2260e−04	1.2425e−07 1.2425e−07 1.2441e−06	1.2352e−10 1.2352e−10 3.9015e−09	1.5984e−13 1.5970e−13 1.5036e−11
RK3HM	6.6502e−04 6.6502e−04 2.2030e−03	5.9464e−06 5.9464e−06 6.0417e−05	5.9024e−08 5.9024e−08 1.8898e−06	5.8979e−10 5.8979e−10 5.9693e−08
Proposed	2.3861e−05 8.2608e−06 8.1340e−05	2.6075e−08 1.3196e−08 2.8703e−07	2.6284e−11 1.3664e−11 9.1636e−10	5.8001e−14 4.9848e−14 5.6582e−12

Table 2
Maximum absolute global relative errors on $[0, 1]$ (first row), absolute relative errors at $t = 1$ (second row), and ℓ^2-error norm (third row) for the Example-2.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	6.4697e−05 6.4697e−05 8.3000e−05	6.8998e−08 6.8998e−08 2.2883e−07	6.9400e−11 6.9400e−11 7.1171e−10	1.3111e−13 1.3110e−13 4.4807e−12
RK3HM	2.2335e−03 2.2335e−03 4.1400e−03	2.4716e−05 2.4716e−05 1.3587e−04	2.4971e−07 2.4971e−07 4.3116e−06	2.4998e−09 2.4998e−09 1.3639e−07
Proposed	2.0183e−05 2.0183e−05 2.8573e−05	1.8702e−08 1.8702e−08 7.7040e−08	1.8535e−11 1.8535e−11 2.3974e−10	7.3701e−14 7.3664e−14 2.6931e−12

Table 3
Maximum absolute global relative errors on $[0, 0.5]$ (first row), absolute relative errors at $t = 0.5$ (second row), and ℓ^2-error norm (third row) for the Example-3.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	5.4644e−05 5.4644e−05 7.9834e−05	5.1896e−08 5.1896e−08 2.0943e−07	5.1603e−11 5.1603e−11 6.4899e−10	5.9233e−14 5.9172e−14 2.1293e−12
RK3HM	3.1635e−04 3.1635e−04 3.5014e−04	3.5178e−06 3.5178e−06 9.1739e−06	3.5477e−08 3.5477e−08 2.8310e−07	3.5505e−10 3.5505e−10 8.9296e−09
Proposed	6.4731e−06 3.2754e−06 1.0836e−05	8.3861e−09 1.9656e−09 4.2872e−08	8.3674e−12 2.1622e−12 1.3480e−10	1.2274e−14 1.2155e−14 4.2406e−13

Table 4
Maximum absolute global relative errors on $[0, 10]$ for each unknown in the Example-4.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	9.3411e+01 2.0974e+01	8.1516e−01 3.9783e+00	1.8376e−03 4.0405e−02	1.9256e−04 6.1855e−05
RK3HM	2.5717e+07 4.1029e+07	1.5990e+01 9.2617e+01	1.3092e+00 2.4304e+01	1.3639e+00 4.2652e−01
Proposed	2.6958e+01 1.7027e+01	3.6696e−01 1.7810e+00	5.1390e−04 1.1450e−02	5.3860e−05 1.7773e−05

Table 5
Absolute relative errors at $t = 10$ for each unknown in the Example-4.

h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	3.5681e+00 1.0769e+00	8.9169e−02 8.3767e−03	7.4569e−05 8.8139e−06	7.5540e−08 8.8355e−09
RK3HM	2.5571e+07 1.8402e+07	2.7526e+00 1.8522e−01	4.8144e−02 1.6401e−03	4.9232e−04 2.1372e−05
Proposed	1.3021e+00 1.0083e+00	4.5361e−02 9.2884e−03	2.8136e−05 9.5089e−06	2.8907e−08 9.5067e−09

Table 6
ℓ^2-error norms on $[0, 10]$ for each unknown in the Example-4 where NaN stands for Not a Number.

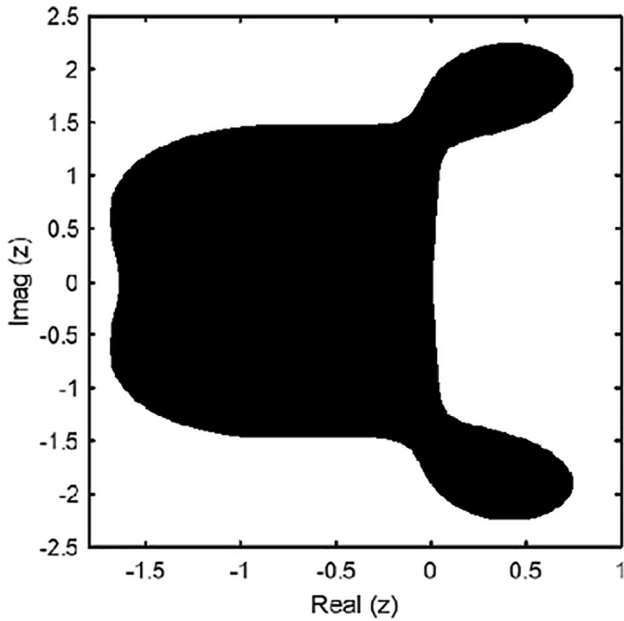| h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Heun	NaN 2.9205e+01	NaN 4.1695e+00	NaN 4.3136e−02	NaN 9.9840e−05
RK3HM	NaN 4.9949e+07	NaN 9.6513e+01	NaN 2.6034e+01	NaN 6.9477e−01
Proposed	NaN 2.1094e+01	NaN 1.8718e+00	NaN 1.2236e−02	NaN 2.7782e−05

Fig. 1. Stability region (shaded) for the proposed solver with $|\Psi(z)| \leqslant 1$.

$$x''(t) + \sin(x) = \cos(4t), \; x(0) = 1, \; x'(0) = 0. \tag{40}$$

or

$$
\begin{aligned}
&x_1'(t) = x_2(t), \; x_1(0) = 1, \\
&x_2'(t) = -\sin(x_1) + \cos(4t), \; x_2(0) = 0, \\
&t \in [0, 20].
\end{aligned}
\tag{41}
$$

Due to the nonavailability of the exact solution for the above nonlinear dynamical system, we have made some graphical comparisons of the results obtained through the numerical solvers with that of the numerical solution obtained through ode45 solver while taking the integration steps $N = 100$ and $N = 200$. However, for $N = 200$ it is difficult to observe the discrepancy among the curves for which a zooming plot at the bottom right corner for the Figs. 4 and 5 help to see the performance of the solvers over a small domain. Upon closer examination of the Figs. 2–5, it can be seen that the proposed and Heun solvers agree well with each other.

5. Conclusion

A numerical solver (6) having third order convergence has been proposed in the present study. Its error analysis led us to compute both local and global error bounds wherein the leading term $\mathcal{O}\left(h^4\right)$ in the local truncation error ensures the third order convergence of the solver. The increment function of the solver is shown to be
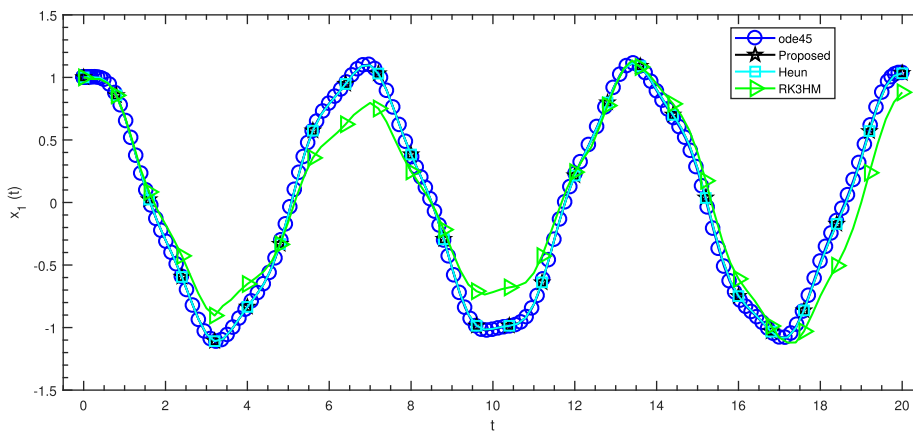

Fig. 2. Comparison of solvers with $N = 100$ for the angular displacement $x_1(t)$ in the Example – 5.
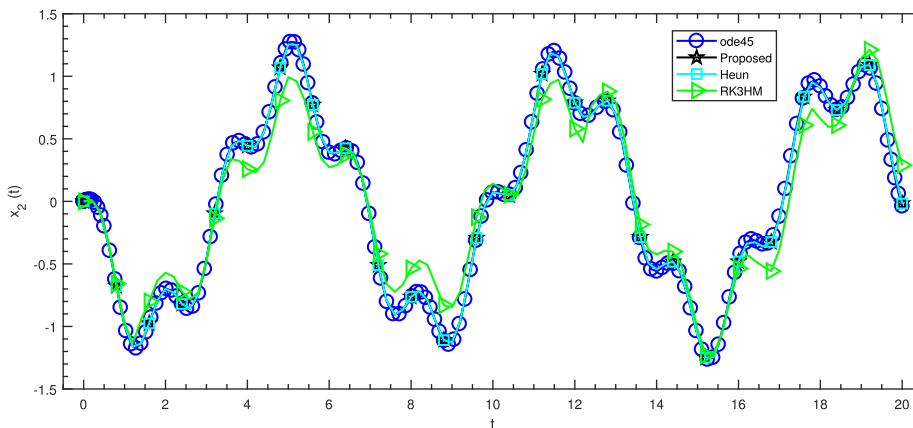

Fig. 3. Comparison of solvers with $N = 100$ for the angular velocity $x_2(t)$ in the Example – 5.
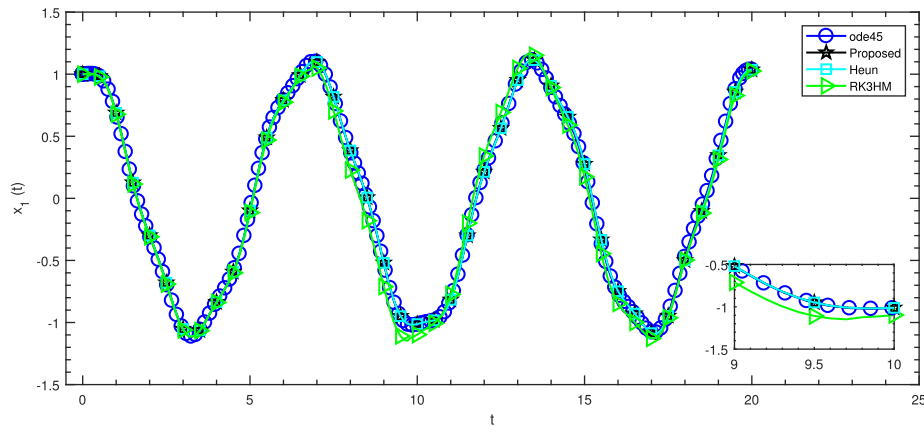
Fig. 4. Comparison of solvers with $N = 200$ for the angular displacement $x_1(t)$ in the Example – 5.
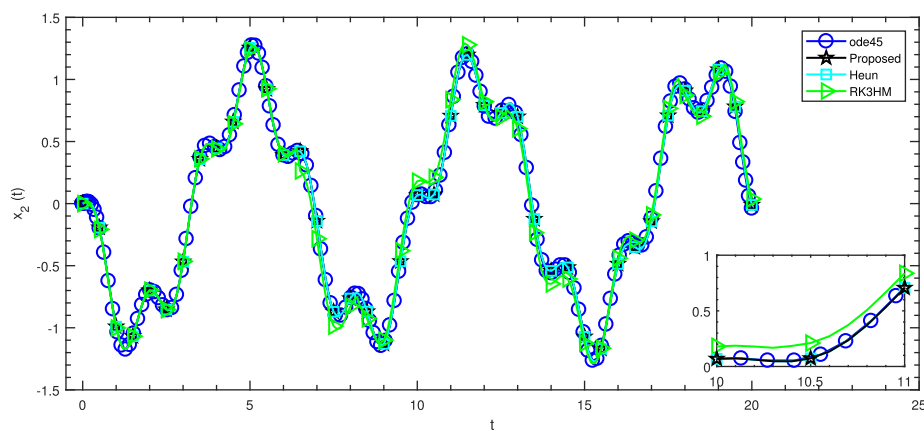

Fig. 5. Comparison of solvers with $N = 200$ for the angular velocity $x_2(t)$ in the Example – 5.

Lipchitz continuous and the linear stability analysis has been carried out to get the stability region of the solver satisfying the condition $|\Psi(z)| \leqslant 1$.

Upon comparison with two well-known solvers (Heun and RK3HM) having similar characteristics as that of the proposed solver, it has been observed that the latter yielded smallest errors in both scalar and continuous dynamical systems of first and higher order as shown in the Tables 1–6. Thus, the proposed solver can be considered a member for the family of explicit linear single-step numerical solvers used to solve various continuous dynamical systems in the scientific fields.

Conflict of Interest

The authors declare no conflict of interest. Both authors have read and approved the final version of the present research work.

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