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On the exponential solutions to three extracts from extended fifth-order KdV equation

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A B S T R A C T

An extended fifth order Korteweg-de-Vries (efKdV) equation is an important equation in fluids dynamics for the description of nonlinear wave processes, and contains quite a number of KdV-type equations including the Sawada-Kotera equation, the Caudrey-Dodd-Gibbon equation, the Lax equation, the Kaup-Kuperschmidt equation and the Ito equation among others. However, in this paper, we examine the efKdV by extracting three different fifth order Korteweg-de-Vries (fKdV) equations. Solitary wave solutions to the extracts are found by means of an exponential function ansatz specifically constructed for the study. The obtained solutions can be used in description of shallow-water waves.

1. Introduction

Nonlinear partial differential equations are known for their roles in many nonlinear science applications including the hydrodynamics, fluid dynamics and mechanics, plasma physics and nonlinear dispersive wave among others. An important model in this regard is the Korteweg-de-Vries (KdV) equation (Olver, 1984) that has many applications such as in description of shallow water waves and ion-acoustic waves in plasmas. The KdV equation has undergone several modifications and extensions leading to various forms of KdV equations appearing in three, five, seven or more order forms and different dimensions; read (Hereman, 2000; Hereman, 2009; Helal and Seadawy, 2009; Bakodah, 2013; Adomian, 1996; Nuruddeen, 2018; Djidjeli et al., 1995; Kaya, 2005; Khater et al., 2000; Khater et al., 2006; Lei et al., 2002; Sawada and Kotera, 1974; El-Wakil et al., 2011; Pandir et al., 2013; Islam et al., 2015) for various forms and methods.

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Further, of recent, Yun-hu and Yong (Yun-hu and Yong, 2013) analyzed the extended fifth order Korteweg-de-Vries (efKdV) equation of the form

$$u_t + p_1 u^2 u_x + p_2 u_x u_{xx} + p_3 u u_{xxx} + p_4 u_{xxxxx} + p_5 u u_x + p_6 u_{xxx} + p_7 u_x = 0, \quad (1)$$

where $p_j (j = 1, 2, \ldots, 7)$ are constants. They (Yun-hu and Yong, 2013) established all the conservation laws through Lax pair and talked of many special cases of efKdV equations. One can also see the recent paper by Nikolay (2016) for the analysis of the generalized modified fKdV with dissipation.

However in this study, we aim to study three different fKdV equations to be extracted from the efKdV equation given in Eq. (1) as follows:

1). The first one is obtained by setting $P_5 = P_6 = P_7 = 0$ in Eq. (1), to yields

$$u_t + p_1 u^2 u_x + p_2 u_x u_{xx} + p_3 u u_{xxx} + p_4 u_{xxxxx} = 0. \quad (2)$$

2). The second by setting $P_1 = P_2 = P_6 = P_7 = 0$ in Eq. (1), which gives

$$u_t + p_3 u u_{xxx} + p_4 u_{xxxxx} + p_5 u u_x = 0. \quad (3)$$

3). The third by setting $P_1 = P_2 = P_5 = P_7 = 0$ in Eq. (1), that reads

$$u_t + p_3 u u_{xxx} + p_4 u_{xxxxx} + p_6 u_{xxx} = 0. \quad (4)$$

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1018-3647/

Further, different analytical methods have been proposed in the literature to determining exact solutions to various evolutions equations modelled in partial and or fractional differential equations, see Ali and Kalisch (2014), Grujic and Kalisch (2009), Aly Seadawy (2017), Seadawy (132 2017,), Seadawy (2017), Nuruddeen and Nass (2017), Seadawy and Lu (2017), Helal and Seadawy (2011), Alquran et al. (2017), Khalid and Nuruddeen (2017), Khalid et al. (2018), Bakodah et al. (2016), Seadawy (2015), Seadawy (2017), Aly (2018), Khalid et al. (2018), Khalid et al. (2018) for details. The paper is organized as follows: Section 2 presents the concept of the constructed method; Section 3 is for the application; Section 4 discusses the obtained results and Section 5 is for conclusion.

2. The methodology

In constructing exact solutions to the extracrted equations in Eqs. (2)–(4), we devise an exponential function ansatz and presented here by considering the general form of differential equation below:

$$P(u, D_t u, D_x u, D_t D_x, D_{tt} u, D_{xx} u, \ldots) = 0. \tag{5}$$

We then seek for an exponential function solution of Eq. (5) of the form:

$$u(x,t) = A_1 + A_2 \frac{e^{k(x-ct)}}{\left(1+e^{k(x-ct)}\right)^2}, \tag{6}$$

where k and c are nonzero constants, and A_1, and A_2, are determined with $A_2 \neq 0$ in any case.

Thus, substituting Eq. (6) and its possible derivatives into Eq. (5) gives a polynomial in $e^{jk(x-ct)}$ for $j = 0, 1, 2, \ldots, n$. Equating the coefficient of each power of $e^{jk(x-ct)}, (j = 0, 1, 2, \ldots, n)$ to zero, we obtain a system of algebraic equations which will then be solved for the unknowns (we use Mathematica software) to obtain the solution of Eq. (5).

3. Application

In this section, we present exact exponential function traveling wave solutions to the three extracts given in Eqs. (2)–(4) which we extracted from the extended fifth-order KdV (efKdV) equation given in (1) as follows:

3.1. The first extract

We consider the first extract of the extended fifth-order KdV (efKdV) equation given in Eq. (2);

$$u_t + p_1 u^2 u_x + p_2 u_x u_{xx} + p_3 u u_{xxx} + p_4 u_{xxxxx} = 0, \tag{7}$$

Substituting the exponential solution ansatz given in Eq. (6) into Eq. (7) and equating the coefficient of each power of $e^{jk(x-ct)}, (j = 0, 1, 2, \ldots, 6)$ to zero we get the following system of algebraic equations:

$$ckA_2 - kA_1^2 A_2 p_1 - k^3 A_1 A_2 p_3 - k^5 A_2 p_4 = 0,$$

$$-ckA_2 + kA_1^2 A_2 p_1 + k^3 A_1 A_2 p_3 + k^5 A_2 p_4 = 0,$$

$$3ckA_2 - 3kA_1^2 A_2 p_1 - 2kA_1 A_2^2 p_1 - k^3 A_2^2 p_2 + 9k^3 A_1 A_2 p_3 - k^3 A_2^2 p_3 + 57k^5 A_2 p_4 = 0,$$

$$-3ckA_2 + 3kA_1^2 A_2 p_1 + 2kA_1 A_2^2 p_1 + k^3 A_2^2 p_2 - 9k^3 A_1 A_2 p_3 + k^3 A_2^2 p_3 - 57k^5 A_2 p_4 = 0,$$

$$\vdots$$

Solving the above system, we get the following sets of solutions:

$$A_1 = \frac{-k^2 p_2 - 2k^2 p_3 - \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{4p_1},$$

$$A_2 = 3\left(\frac{k^2 p_2 + 2k^2 p_3 + \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{p_1}\right),$$

$$c = \tfrac{1}{8}\left(-12k^4 p_4 + \frac{k^4 p_2^2 + 2k^4 p_2 p_3 + k^2 p_2 \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{p_1}\right), \text{ and}$$

$$A_1 = \frac{-k^2 p_2 - 2k^2 p_3 + \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{4p_1},$$

$$A_2 = 3\left(\frac{k^2 p_2 + 2k^2 p_3 - \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{p_1}\right),$$

$$c = \tfrac{1}{8}\left(-12k^4 p_4 + \frac{k^4 p_2^2 + 2k^4 p_2 p_3 - k^2 p_2 \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{p_1}\right), \quad \text{where,}$$

$p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4 > 0.$

The first set gives the exact solution

$$u(x,t) = \frac{-k^2 p_2 - 2k^2 p_3 - \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{4p_1} + 3\frac{k^2 p_2 + 2k^2 p_3 + \sqrt{k^4 (p_2^2 + 4p_2 p_3 + 4p_3^2 - 40p_1 p_4)}}{p_1 (1 + e^{k(x-c_1 t)})^2} e^{k(x-c_1 t)}. \tag{8}$$

(see Fig. 1.)

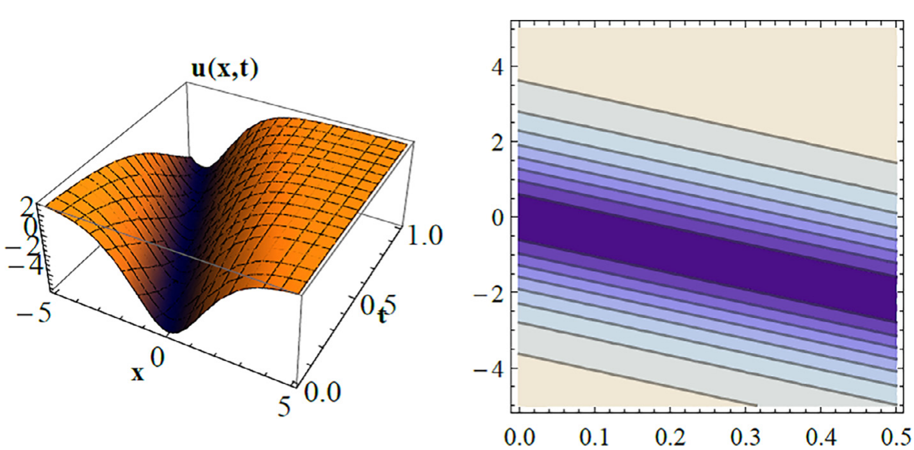

Fig. 1. The 3D and the contour plots for Eq. (8).

Further, the second set gives the solution:

$$u(x,t)=\frac{-k^2p_2-2k^2p_3+\sqrt{k^4(p_2^2+4p_2p_3+4p_3^2-40p_1p_4)}}{4p_1}+3\frac{k^2p_2+2k^2p_3-\sqrt{k^4(p_2^2+4p_2p_3+4p_3^2-40p_1p_4)}}{p_1(1+e^{k(x-c_2t)})^2}e^{k(x-c_2t)}. \tag{9}$$

where k is left as a free parameter in Eqs. (8) and (9) and c_1 is the c in the first set and c_2 is the c in the second set. (see Fig. 2.)

3.2. The second extract

We consider the second extract of the extended fifth-order KdV (efKdV) equation given in Eq. (3);

$$u_t+p_3uu_{xxx}+p_4u_{xxxxx}+p_5uu_x=0, \tag{10}$$

Substituting the exponential solution ansatz given in Eq. (6) into Eq. (10) and equating the coefficient of each power of $e^{jk(x-ct)}, (j=0,1,2,\dots,6)$ to zero we get the following system of algebraic equations:

$$-ckA_2+k^3A_1A_2p_3+k^5A_2p_4+kA_1A_2p_5=0,$$

$$-3ckA_2-9k^3A_1A_2p_3+k^3A_2^2p_3-57k^5A_2p_4+3kA_1A_2p_5+kA_2^2p_5=0,$$

$$-2ckA_2-10k^3A_1A_2p_3-11k^3A_2^2p_3+302k^5A_2p_4+2kA_1A_2p_5+kA_2^2p_5=0,$$

$$2ckA_2+10k^3A_1A_2p_3+11k^3A_2^2p_3-302k^5A_2p_4-2kA_1A_2p_5-kA_2^2p_5=0,$$

$$3ckA_2+9k^3A_1A_2p_3-k^3A_2^2p_3+57k^5A_2p_4-3kA_1A_2p_5-kA_2^2p_5=0,$$

$$ckA_2-k^3A_1A_2p_3-k^5A_2p_4-kA_1A_2p_5=0.$$

Solving the above system, we get the following set of solution:
$A_1=-\frac{5(k^2p_3p_4-p_4p_5)}{2p_3^2}$,
$A_2=\frac{30k^2p_4}{p_3}$,
$c=-\frac{3k^4p_3^2p_4-5p_4p_5^2}{2p_3^2}$; which produces a solution:

$$u(x,t)=-\frac{5(k^2p_3p_4-p_4p_5)}{2p_3^2}+\frac{30k^2p_4e^{k\left(x+\frac{3k^4p_3^2p_4-5p_4p_5^2}{2p_3^2}t\right)}}{p_3\left(1+e^{k\left(x+\frac{3k^4p_3^2p_4-5p_4p_5^2}{2p_3^2}t\right)}\right)^2}, \tag{11}$$

where k is left as a free parameter. (see Fig. 3.)

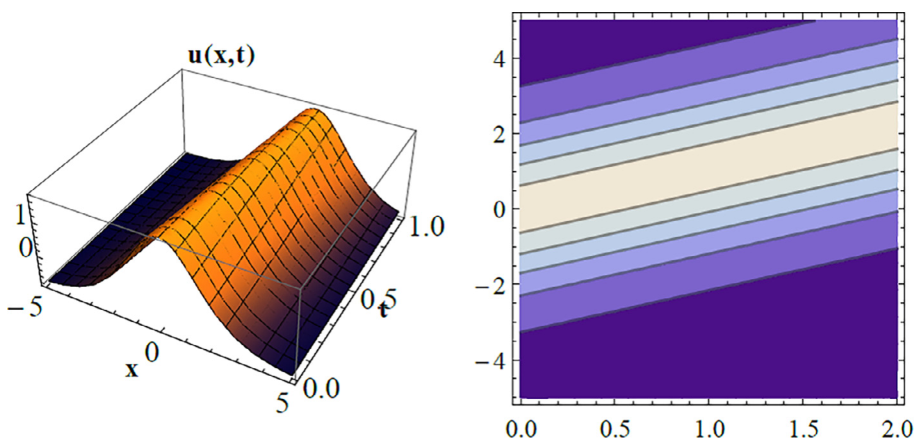

Fig. 2. The 3D and the contour plots for Eq. (9).

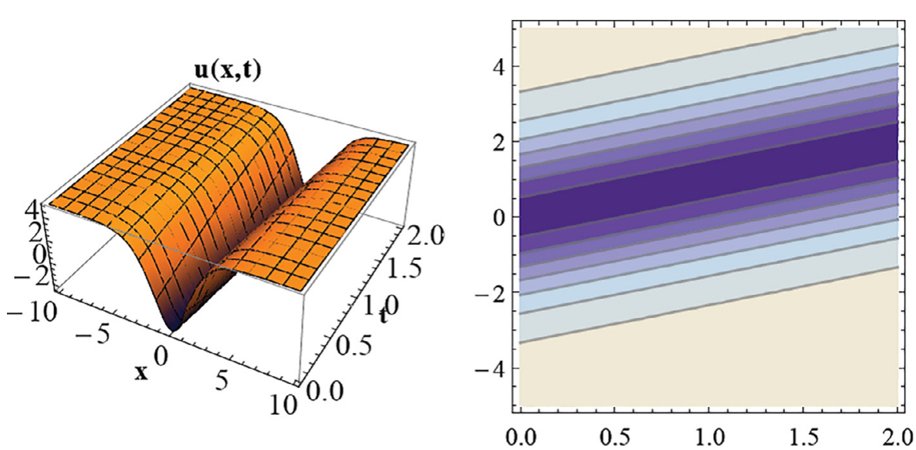

Fig. 3. The 3D and the contour plots for Eq. (11).

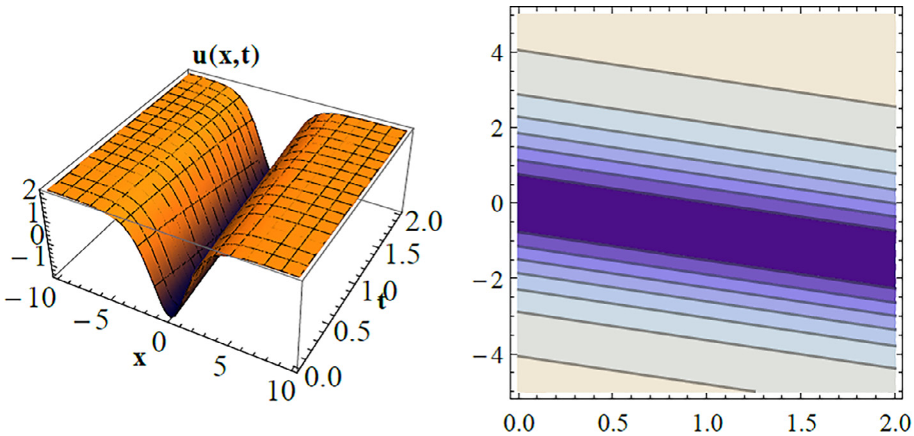

Fig. 4. The 3D and the contour plots for Eq. (13).

3.3. The third extract

We consider the third extract of the extended fifth-order KdV (efKdV) equation given in Eq. (4);

$$u_t + p_3 u u_{xxx} + p_4 u_{xxxxx} + p_6 u_{xxx} = 0. \quad (12)$$

Proceeding as above, we get the following system of algebraic equations:

$$-ckA_2 + k^3 A_1 A_2 p_3 + k^5 A_2 p_4 + k^3 A_2 p_6 = 0,$$

$$-3ckA_2 - 9k^3 A_1 A_2 p_3 + k^3 A_2^2 p_3 - 57k^5 A_2 p_4 - 9k^3 A_2 p_6 = 0,$$

$$-2ckA_2 - 10k^3 A_1 A_2 p_3 - 11k^3 A_2^2 p_3 + 302k^5 A_2 p_4 - 10k^3 A_2 p_6 = 0,$$

$$2ckA_2 + 10k^3 A_1 A_2 p_3 + 11k^3 A_2^2 p_3 - 302k^5 A_2 p_4 + 10k^3 A_2 p_6 = 0,$$

$$3ckA_2 + 9k^3 A_1 A_2 p_3 - k^3 A_2^2 p_3 + 57k^5 A_2 p_4 + 9k^3 A_2 p_6 = 0,$$

$$ckA_2 - k^3 A_1 A_2 p_3 - k^5 A_2 p_4 - k^3 A_2 p_6 = 0.$$

Solving the above system, we get the following set of solution:

$$A_1 = -\frac{5k^2 p_4 + 2p_6}{2p_3},$$

$$A_2 = \frac{30k^2 p_4}{p_3},$$

$c = -\frac{3}{2}k^4 p_4$; which produces a solution:

$$u(x,t) = -\frac{5k^2 p_4 + 2p_6}{2p_3} + \frac{30k^2 p_4 e^{k\left(x+\frac{3}{2}k^4 p_4 t\right)}}{p_3\left(1 + e^{k\left(x+\frac{3}{2}k^4 p_4 t\right)}\right)^2}, \quad (13)$$

where k is left as a free parameter. (see Fig. 4.)

4. Results and discussion

The present study effectively constructs exponential traveling wave solutions to three different fifth order Korteweg-de-Vries equations (fKdV) extracted from the extended fifth order Korteweg-de-Vries equation (efKdV) that has recently been studied by Yun-hu and Yong in Yun-hu and Yong (2013). The first extract Eq. (1) which was once numerically considered by Bakodah (2013) was found to have two different exponential function exact solutions. The fractional version of the second extract Eq. (2) was also recently studied by Khalid et al. (2018) using generalized *exp* expansion method, while the third extract Eq. (3) is first extracted and studied in this study to our knowledge with one exponential function exact solutions each. Since KdV equations play vital roles in the study of shallow-water waves, thus the present obtained traveling wave solutions would not be an exception in the description of shallow-water waves and other nonlinear wave processes. Finally, Three-dimensional (3D) and contour plots are given for the obtained solution.

5. Conclusion

In conclusion, exponential functions traveling wave solutions to three different fifth order Korteweg-de-Vries equations have been constructed. The three fKdV equations considered are extracted from the extended fifth order Korteweg-de-Vries equation that has recently been studied in the literature and found to be of various applications in nonlinear sciences. Interesting and reliable exponential function ansatz was proposed and implemented on Mathematica software in the study. We provide three-dimensional and contour plots for the obtained solutions for easier interpretations of results.

References

Adomian, G., 1996. The fifth-order Korteweg-De Vries equation. Int. J. Math. Math. Sci. 19.
Ali, A., Kalisch, H., 2014. On the formulation of mass, momentum and energy conservation in the KdV equation. Acta Appl. Math. 133, 113–131.
Alquran, M., Jaradat, H.M., Syam, M.I., 2017. Analytical solution of the time-fractional Phi-4 equation using modified power series method. Nonlinear Dyn.
Bakodah, H.O., 2013. Modified Adomain decomposition method for the generalized fifth order KdV equations. Am. J. Comput. Math. 3, 53–58.
Bakodah, H.O., Al-Zaid, N.A., Mirzazadeh, M., Zhou, Q., 2016. Decomposition method for solving Burger's equation with Dirichlet and Neumann boundary conditions. Int. J. Light Electron Opt. 30.
Djidjeli, K., Price, W.G., Twizell, E.H., Wang, Y., 1995. Numerical methods for the soltution of the third and fifth-order disprsive Korteweg-De Vries equations. J. Comput. Appl. Math. 58, 307–336.
El-Wakil, S.A., Abulwafa, E.M., Zahran, M.A., Mahmoud, A.A., 2011. Time-fractional KdV equation: formulation and solution using variational methods. Nonlinear Dyn. 65, 55–63.
Grujic, Z., Kalisch, H., 2009. Gevrey regularity for a class of water-wave models. Nonlinear Anal.: Theory Methods Appl. 71 (3–4), 1160–1170.
Helal, M.A., Seadawy, A.R., 2009. Variational method for the derivative nonlinear Schr-dinger equation with computational applications. Phys. Scr. 80, 350–360.
Helal, M.A., Seadawy, A.R., 2011. Exact soliton solutions of an D-dimensional nonlinear Shrodinger equation with damping and difusive terms. Z. Angew. Math. Phys. 62, 839–847.
Hereman, W., 2000. Wolfram Research Academic Intern Program Inc., Champaign. Illinois, pp. 1–14.
Hereman, W., 2009. Springer Verlag. Heibelberg, Germany, pp. 1620–1536.
Islam, M.T., Akbar, M.A., Azad, M.A.K., 2015. A rational (G/G)-expansion method and its application to modified KdV-Burgers equation and the (2+1)-dimensional Boussineq equation. Nonlinear Stud. 6, 4.
Kaya, D., 2005. An Application for the higher order modified KdV equation by decomposition method. Commun. Nonlinear Sci. Numer. Simul. 10, 693–702.

Khalid, K.A., Nuruddeen, R.I., 2017. Analytical treatment for the conformable space-time fractional Benney-Luke equation via two reliable methods. Int. J. Phys. Res. 5, 109–114.
Khalid, K.A., Nuruddeen, R.I., Raslan, K.R., 2018. New structures for the space-time fractional simplified MCH and SRLW equations. Chaos Solitons Fractals 106, 304–309.
Khalid, K.A., Nuruddeen, R.I., Adel, R.H., 2018. New exact solitary wave solutions for the extended (3+1)-dimensional Jimbo-Miwa equations. Results Phys. 9, 12–16.
Khalid, K.A., Nuruddeen, R.I., Raslan, K.R., 2018. New hyperbolic structures for the conformable time-fractional variant bussinesq equations. Opt Quant Electron. 50.
Khater, A.H., Callebaut, D.K., Seadawy, A.R., 2000. General soliton solutions of an n-dimensional Complex Ginzburg-Landau equation. Phys. Scr. 62, 353–357.
Khater, A.H., Callebaut, D.K., Helal, M.A., Seadawy, A.R., 2006. Variational method for the nonlinear dynamics of an elliptic magnetic stagnation line. Eur. Phys. J. D 39, 237–245.
Lei, Y., Fajiang, Z., Yinghai, W., 2002. The homogeneous balance method, Lax Pair Hirota transformation and a general fifth order KdV equation. Chaos Solutions Fractals 13.
Nikolay, A.K., 2016. On solutions of generalized modified Korteweg-de-Vries equation of the fifth order with dissipation. Appl. Math. Comput. 230, 39–45.
Nuruddeen, R.I., 2018. Multiple soliton solutions for the (3+1) conformable space-time fractional modified Korteweg-de-Vries equations. Ocean J. Eng. Sci. 3, 11–18.
Nuruddeen, R.I., Nass, A.M., 2017. Exact solutions of wave-type equations by the Aboodh decomposition method. Stochastic Model Appl. 21, 23–30.
Olver, P.J., 1984. Hamiltonian and non-Hamiltonian models for water waves. In: Lecture Notes in Physics, 195. Springer-Verlag, New York, pp. 273–290.
Pandir, Y., Gurefe, Y., Misirli, E., 2013. New exact solutions of the time fractional nonlinear dispersive KdV equation. Int. J. Model. Optim. 3.
Sawada, K., Kotera, T., 1974. A method for finding N-soliton solutions of the KdV equation and the KdV-like equation. Progr. Theor. Phys. 51, 1355–1367.
Seadawy, Aly, 132 2017,. Two-dimensional interaction of a shear flow with a free surface in a stratified fluid and its a solitary wave solutions via mathematical methods. Eur. Phys. J. Plus 518 (12), 1–11.
Seadawy, A.R., 2015. Fractional solitary wave solutions of the nonlinear higher-order extended KdV equation in a stratified shear flow: Part I. Comp. Math. Appl. 70, 345–352.
Seadawy, A.R., 2017. Ion acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili-Burgers equation in quantum plasma. Math. Methods Appl. Sci. 40 (5), 1598–1607.
Seadawy, A.R., 2017. Ionic acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili Burgers equations in quantum plasma. Math. Methods Appl. Sci. 40, 1598–1607.
Seadawy, Aly R., 2018. Three-dimensional weakly nonlinear shallow water waves regime and its travelling wave solutions. Int. J. Comput. Methods 15 (1).
Seadawy, A.R., Lu, D., 2017. Bright and dark solitary wave soliton solutions for the generalized higher order nonlinear Shrodinger equation and its stability. Results Phys. 7, 43–48.
Seadawy, Aly, Arshad, M., Lu, Dianchen, 2017. Stability analysis of new exact traveling Wave Solutions of new coupled KdV and new coupled Zakharov-Kuznetsov systems. Eur. Phys. J. Plus 132 (162), 1–20.
Yun-hu, W., Yong, C., 2013. The integrability of an extended fifth-order KdV equation with Riccati-type pseudopotential. Pramana-J. Phys. 81, 737–746.