Journal of King Saud University – Science 32 (2020) 726–731

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Nonlinear integral models with delays: Recent developments and applications

Natali Hritonenko [a], Yuri Yatsenko [b,*]

[a] Department of Mathematics, Prairie View A&M University, PO Box 519, Prairie View, TX 77446, USA
[b] Dunham College of Business, Houston Baptist University, 7502 Fondren Road, Houston, TX 77074, USA

ARTICLE INFO

Article history:
Received 21 March 2018
Accepted 7 November 2018
Available online 10 November 2018

Keywords:
Integral dynamic models
Endogenous delay
Optimal control
Population models
Technological renovation

ABSTRACT

The authors examine contemporary integral dynamic models of biological, environmental, and technological systems with heterogeneous components. The models contain two-dimensional control functions, nonlinearities, and delays in system inputs. The paper demonstrates the versatility of integral equations and their importance to various applications. Connections and advantages of integral and differential models are discussed. A survey of optimal control strategies for such models is provided.

1. Introduction

Integral equations have been intensively used in modeling various processes in physics, biology, environmental sciences, engineering, economics, and operations research (Ahmed and Teo, 1981; Brokate, 1985; Boucekkine et al., 1997; Corduneanu, 1991; Hritonenko and Yatsenko, 1996; Jovanovic and Tse, 2010; Volterra, 1959). An integral dynamic model with delay was proposed by Boltzman in 1874 to describe elastic persistence in physics. Vito Volterra developed Boltzman theory and introduced integral models with delay to population ecology in 1900. Sharpe and Lotka (1911) derived the integral *renewal equation* for age-structured human populations, which remains a backbone of modern demography. Integral models of economic-technological growth, known as vintage capital models, were suggested in the 1960's to describe technological renovation of economic systems (Solow et al., 1966).

The motivation of this paper is to highlight recent developments in the contemporary theory of integral models and their applications. It explores models of dynamic systems with heterogeneous components that depend on certain structural parameters (Corduneanu, 1991; Kato et al., 2007; Webb, 1985; Hritonenko and Yatsenko, 2013a). Those parameters have various interpretations, such as the age or size of individuals in population models or a starting point of technological renovation in economic-environmental problems. The models under consideration include two-dimensional controls and nonlinearities and delays in the system inputs. Another related goal of this paper is to demonstrate the versatility of integral equations and how similar models are used to describe different applied phenomena that seem to be unrelated at the first sight.

The study of integral models enhances other areas. For instance, operations research studies equipment replacement problems, which are usually considered in discrete settings as integer programming problems (Hartman and Tan, 2014). Similar problems are also studied by integral vintage capital models. Establishing relevant links between two modeling tools, continuous and discrete replacement models, helps to overcome challenges of discrete analysis and address open issues in operations research. Thus, time-continuous vintage models (Yatsenko and Hritonenko, 2005, 2009) explain a paradox in equipment replacement under technological improvement raised in (Cheevaprawatdomrong and Smith, 2003).

Partial differential equations (PDEs) are an alternative modeling tool for heterogeneous dynamic systems with delay. Both integral

* Corresponding author.
E-mail addresses: nahritonenko@pvamu.edu (N. Hritonenko), yyatsenko@hbu.edu (Y. Yatsenko).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2018.11.001
1018-3647/

and PDE models have been applied to age-structured biological populations for centuries (Sharpe and Lotka, 1911; McKendrick, 1926; Brokate, 1985; Clark, 1976; Webb, 1985; Greenhalgh, 1987) and to size-structured populations since 1980's (Metz and Diekmann, 1986; Calsina and Saldaña, 1995; De Roos and Persson, 2001; Kato et al., 2007; Goetz et al., 2010, 2013). The integral equations have been used to study age-structured systems in economics since 1960's (Solow et al., 1966; Malcomson, 1975; Hritonenko and Yatsenko, 1995, 1996; Boucekkine et al., 1997), and more recently the PDEs joined this pool (Jovanovic and Yatsenko, 2012; Hritonenko and Yatsenko, 2010; Hritonenko et al., 2017). The advantage of integral models is they are more general and flexible and often allow for deeper research results compared to differential models. Connections between integral and PDE models in various applications are discussed in this paper.

Growing applications of integral models raise new questions and require their investigation. Optimal control is a major tool in applying integral models to practical problems. Optimality conditions lead to dual integral equations that bring new analytic challenges. Optimization problems in some models require solving nonlinear integral equations with unknowns in the lower (equations with delays) and upper (equations with leads) limits of integration. The paper offers a systematic survey of modeling outcomes, such as structure and asymptotics of optimal trajectories, turnpike properties, solution irregularities, and so on (Hritonenko and Yatsenko, 2013a).

The paper is organized as follows. Section 2 describes the integral dynamic models with delays, their applied interpretation, connection to PDE models, and challenges of their analysis. Section 3 presents the integral models with endogenous delays as a special case of general models of Section 2 and discusses their major investigation techniques and applications. Section 4 compares and summarizes advantages of integral and differential models.

2. Integral models with distributed controls and their applications

The nonlinear integral model

$$\boldsymbol{x}(s,t)=\int_0^t K(s,t,\boldsymbol{u}(\phi(s,t,\tau),\tau))\boldsymbol{x}(\phi(s,t,\tau),\tau)d\tau+\boldsymbol{f}(s,t,\boldsymbol{u}(s,t)),$$
$$s_{\min}\le s\le s_{\max},\quad 0\le t\le\infty, \tag{1}$$

describes a deterministic dynamic system with heterogeneous elements, whose performance depends on the current time t and a certain structural parameter s. The kernel matrix $K(s,t,\mathbf{u},\tau)$, function $\phi(s,t,\tau)$, and n-vector $\mathbf{f}(s,t,\mathbf{u})$ are given, while some or all components of the n-vector $\mathbf{x}(s,t)$ and l-vector $\mathbf{u}(s,t)$ are unknown (Volterra, 1959; Barnett, 1975; Ljung, 1987). The model (1) considers the distributed delay (*after effect, or hereditary effect*) in dynamic systems, when a continuous sequence of past states of the system affects its future evolution.

The structural parameter s is often referred to as a *size*, and the given function $s=\phi(s_0,t,t_0)$ describes the change of the size s over time t starting with the initial size $s_0=s(t_0)$. An example of nonlinear $\phi(s_0,t,t_0)$ is provided in Section 2.2.

If the size s of system elements linearly depends on their age $z=t-t_0$, then $\phi(s_0,t,t_0)=k\cdot(t-t_0)+s_0$, where k is a constant, and the size-structured model (1) can be transformed to the age-structured model (De Roos and Persson, 2001; Kato, 2004; Hritonenko and Yatsenko, 2013a):

$$\boldsymbol{x}(z,t)=\int_0^t \bar{K}\,t,\tau,\boldsymbol{u}(z,\tau))\boldsymbol{x}(z,\tau)d\tau+\bar{\boldsymbol{f}}(z,t,\boldsymbol{u}(z,t)),$$
$$-\infty<z\le t,\ 0\le t<\infty \tag{2}$$

in which the performance of elements depends on their age z.

In the control theory (Barnett, 1975; Ahmed and Teo, 1981; Caputo, 2005; Hritonenko and Yatsenko, 2013a), the input vector $\mathbf{x}$ manages the use of various system resources. The inputs of heterogeneous resources produce a certain aggregate output $\mathbf{y}(t)$. This process is nonlinear, involves additional delays, and is determined by various physical, economic, and environmental factors. The corresponding output balance can be written as

$$\boldsymbol{y}(t)=G\left(\int_{a_{\min}}^{a_{\max}} G_1(z,t,\boldsymbol{x}(t,z))dz\right),$$
$$0\le a_{\min}<a_{\max}<\infty,\quad 0\le t<\infty, \tag{3}$$

where G and G_1 are given nonlinear functions.

The rational control of the dynamic system (2) and (3) is often described by an optimization objective

$$\min_{\boldsymbol{y},\boldsymbol{x},z,\boldsymbol{u}}\int_{t_0}^T F\left(t,\boldsymbol{y}(t),\int_{a_{\min}}^{a_{\max}}F_1(z,t,\boldsymbol{x}(z,t),\boldsymbol{u}(z,t))dz)\right)dt,\quad T\le\infty, \tag{4}$$

subject to different constraints and initial conditions. A similar objective makes sense for the size-structured model (1) as well.

Special cases of the age-structured model (2)–(4) and size-structured model (1) are widely used in applications. They are written as PDE-based or integral optimal control problems. Connections and comparative advantages of these two different modelling tools are discussed below.

2.1. Population biology and demography

Modern age-structured models of population dynamics are mostly versions of the PDE-based Lotka-McKendrik or Gurtin-MacCamy population models (McKendrick, 1926; Clark, 1976; Brokate, 1985; Webb, 1985; Barbu and Iannelli, 1999; Anita, 2000; Fister and Lenhart, 2004). Harvesting is among classic applications of such models. In particular, rational harvesting of a fully manageable biological population can be described by the following optimization problem (Hritonenko and Yatsenko, 2010, 2012):

Find $U(t)$, $u(z,t)$, and $x(z,t)$, $z\in[0,A]$, $t\in[0,T]$, that maximize

$$\int_0^T\left[\int_0^A g(z,t,x(z,t),u(z,t))dz+\phi(U(t),t)\right]dt\to\max_{U,u,x} \tag{5}$$

subject to

$$\frac{\partial x(z,t)}{\partial t}+\frac{\partial x(z,t)}{\partial z}=u(z,t)-d(z)x(z,t),\quad t\in[0,T],\quad z\in[0,A], \tag{6}$$

with the initial and boundary conditions

$$x(0,t)=U(t),\quad t\in[0,T],\qquad x(z,0)=x_0(z),\quad z\in[0,A], \tag{7}$$

$$-u_{min}\le u(z,t)\le u_{max},\quad 0\le U(t)\le U_{max},$$
$$x(z,t)\ge 0,\ z\in[0,A],t\in[0,T], \tag{8}$$

where $x(z,t)$ is the unknown *population density* of individuals of age z at time t, $d(z)$ is the age-specific mortality rate, A is the maximum age, and $x_0(z)$ is the initial age-distribution of individuals at $t=0$. The control variables are the inflow $U(t)$ of newborns and the number of individuals $u(z,t)$ of age z brought to ($u(z,t)>0$) or taken from ($u(z,t)<0$) the population at time t.

It is important to recognize, that in harvesting models (Clark, 1976; Murphy and Smith, 1990; Hritonenko and Yatsenko, 2012, 2013a), the control $u(z,t)$ can be introduced as the harvesting rate (density) as in (6) or as the harvesting effort (then, the PDE (6) contains the product $u(z,t)x(z,t)$ instead of $u(z,t)$).

The model (5)–(8) has been employed to find optimal harvesting strategies and explore sustainable development. It can be used as a demographic block to analyze the dependence of human lifespan on medical expenditures (Hritonenko and Yatsenko, 2010), explore links between technological progress and endogenous retirement age and other problems in demographic and biomedical sciences.

Mathematical investigation of the problem (5)–(8) includes optimality conditions, steady-state analysis, convergence of optimal trajectories to the steady state, closed-form solutions, a long-term balanced growth and transition dynamics, just to name a few. A simple transformation (Hritonenko and Yatsenko, 2013a, pp. 151–153) shows that the linear age-structured PDE model (6)–(8) is a special case of the linear integral model (2)–(4), known as the Lotka model (Sharpe and Lotka, 1911). The advantage of PDE models is that they are easier to solve.

2.2. Environmental protection and natural resources

Carbon and other pollutants contaminate the environment and negatively impact human well-being. Environmental protection is among important global issues (Bréchet et al. 2013, Kharrazi et al. 2013). Biological, chemical, and engineering techniques have been developed to reduce the environmental pollution. Ocean and forest are major biological means to decrease pollutions and clean the environment. Forest sequestrates a great amount of carbon dioxide and provides valuable timber. Here we consider a forestry model used in environmental applications (Hritonenko et al., 2009; Goetz et al., 2013).

As established by forest scientists, the link between the size and age of a tree is weak and corresponding mathematical models should use the tree size rather than its age as a structured parameter. First size-structured forestry models were introduced in the 80's (Metz and Diekmann, 1986; Calsina and Saldaña, 1995; De Roos and Persson, 2001; Kato et al., 2007). The carbon sequestration in timber was added later. The following problem determines the optimal planting/logging regime that maximizes the joint benefits from timber production and carbon sequestration (Hritonenko et al., 2009):

$$\max_{u,U,x,b,s} J = \int_0^T e^{-rt}\left\{\int_{l_0}^{lm} B(x(l,t),u(l,t))dl + B_2(t)\left[\frac{db(t)}{dt}+\frac{dc(t)}{dt}\right] - B_3(t)U(t)\right\}dt \tag{9}$$

under restrictions

$$\frac{\partial x(l,t)}{\partial t}+\frac{\partial[g(l,E(t))x(l,t)]}{\partial l} = -\mu(l,E(t))x(l,t)-u(l,t),\quad t\in[0,T),\quad l\in[l_0,l_m], \tag{10}$$

$$\frac{ds(t)}{dt} = h\left(\frac{dV(t)}{dt},c(t)\right),\qquad E(t)=\chi\int_{l_0}^{lm} l^2 x(l,t)dl,$$
$$b(t)=\gamma_0\int_{l_0}^{lm} v(l)l^{\beta}x(l,t)dl, \tag{11}$$

$$s(0) = s_0,\quad v'(l)>0,\quad 0\le u(l,t)\le u_{max}(l,t),\quad 0\le p(t)\le p_{max}(t), \tag{12}$$

$$x(l,0)=x_0(l),\ l\in[l_0,l_m],\quad x(l_0,t)=U(t),\quad t\in[0,T), \tag{13}$$

where l is a tree diameter, $l\in[l_0, l_m]$, $x(l,t)$ is the density of trees, $u(l, t)$ is the flux of logged trees, $U(t)$ is the flux of new trees planted at t with the diameter l_0, $g(l,E(t))$ is their growth rate, $\mu(l,E(t))$ is a mortality rate, $E(t)$ reflects the intra-species competition, $b(t)$ and $c(t)$ are the amount of carbon sequestered in timber and soil. The Eq. (11) describes the dynamics of carbon in timber and soil. The condition $v'(l) > 0$ in (12) means that the carbon sequestered in the long run is higher for larger trees. The given positive parameters χ, β, and γ_0 are specific for tree species and are estimated from empirical data.

The model (9)–(13) is a special case of the integral model (1), (3), (4). To illustrate that, let us define characteristic curves of the Eq. (10). For any continuous $E(t)$, $t\in[0,T)$, the *characteristic curve* $\varphi_E(t;\, l_1, t_1)$ through a point $(l_1, t_1)\in(0,\infty)$x$[0,T)$ is the solution of the differential equation

$$\varphi'(t) = g(\varphi(t),E(t)),\qquad \varphi(t_1)=l_1. \tag{14}$$

For a given E, the function $\varphi_E(t;l_1,t_1)$ describes the tree size $\varphi(t)$ reached at time t if $\varphi(t_1) = l_1$. For clarity, let us choose a commonly accepted nonlinear growth rate

$$g(l,E) = (\bar{l}-l)\widehat{g}(E),\qquad \widehat{g}(E)>0, \tag{15}$$

which means that a tree cannot grow indefinitely and asymptotically reaches its maximal size $\bar{l}$. Solving the initial value problem (14) under the growth law (15), we obtain the characteristic curve

$$\varphi_E(t;l_1,t_1) = \bar{l}-(\bar{l}-l_1)e^{-\int_{t_1}^{t}\widehat{g}(E(\tau))d\tau},\qquad t\in[0,T]. \tag{16}$$

At a constant $E(t) = E$, the characteristic $\varphi_E(t;l_1,t_1) = \bar{l}-(\bar{l}-l_1)e^{-\hat{g}(E)(t-t_1)}$ depends only on the difference between the initial time t_1 and current time t.

Using (15) and (16), the PDE (10) can be rewritten as the nonlinear integral equation

$$x(l,t) = f(l,t) - \int_0^t \tilde{\mu}(\varphi_E(\zeta;l,t),E(\zeta),)x(\varphi_E(\zeta;l,t),\zeta)d\zeta,\quad 0<l<l_m,\ \ 0<t<\infty, \tag{17}$$

where the function f also depends on p, E, or x_0, and

$$\tilde{\mu}(\varphi_E(\zeta;l,t),E(\zeta)) = \mu(\varphi_E(\zeta;l,t),E(\zeta)) + u(\varphi_E(\zeta;l,t),\zeta) + \frac{\partial g}{\partial l}(\varphi_E(\zeta;l,t),E(\zeta)) \tag{18}$$

(Kato, 2004). The integral Eq. (17) is of the form (1). Qualitative and numeric analysis of the model (9)–(13) for various tree species under different climate scenarios helps to estimate changes in the forest caused by the environment and produce recommendations for sustainable forest management under climate change (Hritonenko et al., 2012; Goetz et al., 2013). The major advantage of size-structured models compared to age-structured models (5) and (6) is that they better describe real processes in forestry and fishery. A drawback is in increased analytic and computational complexity.

2.3. Technological innovations and equipment replacement

Equipment ranging from computers to industrial machines is among key production inputs of any business, but it ages with time and should be periodically replaced before it becomes obsolete because new better equipment appears on market (Hartman and Tan, 2014; Yatsenko and Hritonenko, 2017). Finding the optimal replacement time in multi-disciplinary settings involves not only financial and technological, but also social and environmental aspects (contamination and protection, shortage of energy and natural resources). Alongside with new equipment, the old one is still on market because of its lower prices, increased performance (learning-by-doing), and limited substitutability of vintages.

From a mathematical viewpoint, any production system is a system with memory implemented in technological structures and, as such, can be described by the model (2)–(4). The memory is implemented in the existing structure of productive equipment. Economic applications of models (2)–(4) are known as the *vintage capital models* and describe the optimal renovation of heterogeneous assets under technological progress (Solow et al., 1966;

Malcomson, 1975; Hritonenko and Yatsenko, 1995, 2005; Boucekkine et al., 1997; Boucekkine and Pommeret, 2004; Jovanovic and Tse, 2010). They describe the equipment (capital assets) as a collection of heterogeneous vintages that differ by their age and installation time.

The vintage model with investments into new and old vintages (Jovanovic and Yatsenko, 2012) considers rational capital management in a firm that faces the integral production function

$$y(t) = \left(\int_{-\infty}^{t} A(t-z)\xi^{\beta}(z)x^{\beta}(z,t)dz\right)^{\alpha/\beta}, \tag{19}$$

where $x(z,t)$ is the amount of vintage z at time t, $\xi(z)$ is the unit efficiency of vintage z, $A(t-z)$ is the age-dependent "learning curve", $t\in[0,\infty)$, $z\in(-\infty,t]$, the parameter $0<\alpha\le 1$ describes returns to scale, and $\beta<1$ reflects limited substitutability among different vintages. Let $U(t)$ denote the investment in the new vintage t, $u(z,t)$ be the investment in the old vintage $z<t$ (of the age $t-z$), and $\delta>0$ be the rate of physical depreciation. The law of motion of vintages x is described by the PDE population model

$$\frac{\partial x(z,t)}{\partial t} + \frac{\partial x(z,t)}{\partial z} = -\delta x(z,t) + u(z,t), \qquad x(0,t) = U(t), \tag{20}$$

with a given initial distribution $x(v,0) = x_0(v)$ of existing vintages, $v\in(-\infty,0]$. The firm aims to maximize the discounted profit on the infinite horizon:

$$\max_{u,U,x,y} \int_0^{\infty} e^{-rt}\left(y(t) - U(t) - \int_{-\infty}^{t} u(z,t)dv\right)dz, \qquad x\ge 0,\ \ u\ge 0, \tag{21}$$

subject to (19) and (20) and initial conditions. The unknown control variables are U and u, while the state variables x and y are determined from (20) and (19). Similarly to Section 2.2, the PDE model (20) and (21) is a special case of the integral model (2)–(4).

Complete dynamics of such models combines a long-term balanced growth (a steady-state solution, turnpike trajectory) and short-term transition dynamics before a solution reaches balanced growth (Boucekkine et al., 1997; Hritonenko and Yatsenko, 2008, 2013a). The balance growth in (19)–(21) is studied by Jovanovic and Yatsenko (2012), who explore how learning affects buying older technologies. Qualitative properties of the model (19)–(21) and conditions for the convergence of its optimal trajectories to the balanced growth are obtained in (Hritonenko et al., 2017).

3. Integral models with controlled memory

Integral dynamic models can contain a special type of nonlinearities that arise when certain obsolete elements of a dynamic system are abruptly removed from the system. It leads to the appearance of specific nonlinear controls that can be introduced in the general age-structured dynamic model (2). Let us consider its linear version:

$$\boldsymbol{x}(t) = \int_{-\infty}^{t} K(\tau,t)A(\tau,t)\boldsymbol{x}(\tau)d\tau + \boldsymbol{f}(t), \qquad -\infty\le s<t, \quad 0\le t<\infty, \tag{22}$$

where $\mathbf{x}$ is an n-vector of inputs, the given nxn matrix $K(\tau,t)$ defines possible positive and negative feedbacks among different inputs, the $n\times l$ matrix A controls the feedback intensities, and n-vector function $\mathbf{f}$ reflects external impacts on the system.

The new features of the model (22) compared to (2) are that:

- the infinite delay over $(-\infty,0]$ is necessary to depict existing structure of inputs at the initial instant $t=0$,
- the inputs $\mathbf{x}(t)$ of the age-structured model (22) are not differentiated by their age, which occurs when $\mathbf{x}(t)$ contain only the system elements of age zero, such as newborns in biology or newest technologies in industry.

Those features make the model (22) suitable to technological applications. In control theory, the matrix function $A(\tau,t)$ is a flexible *two-dimensional* control that changes the *intensities of input-output channels* (Hritonenko and Yatsenko, 2005, 2013a). It can take more specific forms in some applications. In particular, production systems under improving technology (at $\partial K(\tau,t)/\partial\tau>0$) acquire only the newest and most efficient equipment vintages and scrap only the oldest obsolete vintages. The scrapping process is described by the special form of the control A:

$$A_{ij}(\tau,t) = \begin{cases} 1, & z_j(t)\le\tau\le t, \\ 0, & \tau<z_j(t), \end{cases} \qquad i,j=1,\dots,n, \tag{23}$$

which is now determined by the *one-dimensional delay control* $z_j(t)$ that describes the instant when a vintage should be removed from service. Then, substituting (23) into the model (2)–(4), (22) leads to the following *integral model with endogenous (controlled) delays*:

$$\text{Maximize } I = \int_{t_0}^{T} \Phi(\mathrm{x},z,t)dt, \qquad t_0<T\le\infty, \tag{24}$$

subject to $x_i(t) = \sum_{j=1}^{n}\int_{z_j(t)}^{t} K_{ij}(\tau,t,x_1\dots,x_n)x_j(\tau)d\tau + f_i(t)$, $i=1,\dots m$,

$$x_i(t)\ge 0,\ z_j(t)<t,\ z_j(t)\ge 0,\ t\in[t_0,T), \qquad j=1,\dots,n, \quad 0<m\le n,\ T\le\infty,$$

$$\boldsymbol{x}(\tau) = \boldsymbol{x}^0(\tau), \quad \tau\in[\tau_0, t0], \quad \tau_0 = \inf_{t\in[t_0,T),j=1,n} z_j(t)$$

The unknown lower limits $z_j(t)$ of integration in (25) reflect delays in a dynamic system and describe the unknown lifetimes t-$z_j(t)$ of system elements. In (24) and (25), some or all components of functions $\mathbf{x}(t) = \{x_i(t),\ i=1,\dots,n\}$, and $\mathbf{z}(t) = \{z_i(t),\ i=1,\dots,n\}$, $t\in[t_0,T)$, are unknown. The functions $K_{ij}(\tau,t,\mathbf{x})\ge 0$ and $f_i(t)\ge 0$, $i=1,\dots,m$, $j=1,\dots,n$, $\tau\in[\tau_0,\ T)$, $t\in[t_0,\ T)$, and the initial vector $\mathbf{x}^0(\tau)$, $\tau\in[\tau_0,\ t_0]$, are given.

Compared to the model (22), the benefit of the integral model (24) with controlled delay is that it decreases the dimension of a control problem from two to one. However, its drawback is that it creates a new essential nonlinearity which leads to increased analytic complexity. Indeed, the integral equations with unknown delays are always nonlinear even if all other model functions are given. Moreover, the problem (24) and (25) involves the state-dependent delays $x(z(t))$ and the state constraints (25) on the derivative of the unknown $z_j(t)$. Challenges of state-dependent delays are well known in the modeling theory, e.g., (Carl et al., 2007, Desch and Turi, 1996; Domoshnitsky et al., 2002; Dmitruk and Vdovina, 2017). Furthermore, the integral equations with unknown upper integration limits (leads) appear in corresponding dual problems (Hritonenko and Yatsenko, 2009; Motamedi et al., 2014; Yatsenko 1995, Yatsenko, 2004). Finally, a finite interval $[t_0,T]$, $T<\infty$, causes solution irregularities compared to the infinite case $T=\infty$. To illustrate features of such models, we consider two special cases of (24) and (25) below.

3.1. Technological and industrial models

The optimal control problem

$$\max_{x,y,z} I = \int_{t_0}^{T} \rho(t)[y(t) - p(t)x(t)]dt, \qquad T\le\infty, \tag{26}$$

under the constraints

$$y(t) = \int_{z(t)}^{t} \beta(\tau,t)x(\tau)d\tau, \qquad R(t) = \int_{z(t)}^{t} x(\tau)d\tau, \tag{27}$$

$$p(t)x(t) \geq y(t), \quad z(t) \leq t, \quad z'(t) \geq 0, \quad x(t) \geq 0, \quad t \in [t_0, T), \tag{28}$$

and the initial conditions

$$z(t_0) = z_0 < t_0, \quad x(\tau) = x^0(\tau) \geq 0, \qquad \tau \in [z_0, t_0] \tag{29}$$

with the unknown x, y, z describes the rational replacement of obsolete equipment under improving technology and a constrained resource R. The efficiency function $\beta(\tau,t)$ is the output of product y produced by one vintage, that is, by one unit of equipment created at time τ. It increases in τ due to the *embodied technological change*: newer vintages are more efficient because of technological improvements. The endogenous variables are the scrapping time $z(t)$ of obsolete vintages and the number $x(t)$ of new vintages placed in service. The discount rate ρ, efficiency β, cost p, resource R, and initial state x^0 are positive and given.

The problem (26)–(29) has been intensively studied. Hritonenko and Yatsenko (2005, 2008) developed a technique to overcome state constraints (28) on z and z', obtained the necessary and sufficient condition for an extremum, and found asymptotic (turnpike) properties and exact structure of solutions to (26)–(29). In particular, the optimal control x at $T < \infty$ is affected by two groups of irregularities caused by *replacement echoes* and *anticipation echoes* (Boucekkine et al., 1997; Hritonenko and Yatsenko, 1996, 2005, 2008). The first group of echoes is disseminated from left to right and the second one from right to left. The cumulative effect of two echoes leads to a sophisticated behavior of the optimal control x. Such investment echoes have been observed in real economies (Jovanovich and Tse, 2010).

Some other integral models with one scalar endogenous delay have been investigated in (Malcomson, 1975; Hritonenko and Yatsenko, 1995; Boucekkine et al., 1997), and others. The model (26)–(29) has been extended to consider sustainable management of energy and natural resources, physical and human capital, environmental quotas and restrictions (Boucekkine and Pommeret, 2004; Boucekkine et al., 2014; Hritonenko and Yatsenko, 2013b; Hritonenko et al., 2015). Hritonenko and Yatsenko (2013c) study nonlinear integral Eq. (24) with several endogenous delays. A principal drawback of vintage models with endogenous delays compared to the vintage model (19)–(21) is that they cannot describe investments into older vintages.

3.2. Biological models

Integral models with delays of type (24) and (25) can be effectively used to describe controlled harvesting of biological populations (Brokate, 1985; Kato et al., 2007; Sharpe and Lotka, 1911; Webb, 1985). Hritonenko and Yatsenko (2006) describe the dynamics of a harvested population by the integral dynamic model with controlled delay

$$x(t) = \int_{t-z(t)}^{t} K(u,t,x)h(u,t)x(u)du + \int_{t-T}^{t} K(u,t,x)(1-h(u,t))x(u)du, \tag{30}$$

where z is the unknown harvesting age, h is the unknown harvesting intensity, x is the birth intensity of individuals, T is their maximal lifespan, and K reflects given population productivity (fertility). A major difference between integral models (22)–(25) in economics and biology is that biological populations reproduce themselves, while the economy is fully managed (all new elements are introduced into the system). As in PDE-based population models of Section 2, the objective is to maximize harvesting profit. The advantage of the integral model (30) is that the state variable x is *one-dimensional* compared to the two-dimensional population density x in the analogous PDE model (6), which simplifies analysis (Hritonenko and Yatsenko, 2010, 2007).

4. Conclusion

The choice of differential or integral equations as a modeling tool is ambiguous. In some situations, researchers transform original integral models to their differential analogues (ODEs or PDEs), and vice versa. A crucial benefit of differential equations is that they are easier to solve, so, the transition to them is natural in many cases. However, the major advantage of integral equations is that they are more general and can describe global situations that cannot be modeled by the differential equations. Indeed, although the derived equations of motion are often differential, the original general physical laws usually have an integral form. Even a slight modification of an applied problem may often require going back to an integral form of the model. Examples include problems of viscoelasticity, creep theory, super fluidity, aeroelasticity, coagulation and meteorology, electromagnetism, radiation transfer, radio physics, electronic lithography, and so on (Hritonenko and Yatsenko, 2013a). In principle, all differential models can be described with integral equations.

Acknowledgements

This paper outlines research supported by the Ministry of Education and Science of Kazakhstan, Republic of Kazakhstan, under Grant AP05131784. The authors are grateful to two anonymous referees for valuable remarks.

References

Ahmed, N.U., Teo, K.L., 1981. Optimal Control of Distributed Parameter Systems. Elsevier North Holland, New York.
Anita, S., 2000. Analysis and Control of Age-Dependent Population Dynamics. Kluwer Academic Publishers, Dordrecht.
Barbu, V., Iannelli, M., 1999. Optimal control of population dynamics. J. Opt. Theory Appl. 102, 1–14.
Barnett, S., 1975. Introduction to Mathematical Control Theory. Oxford University Press, New York.
Brokate, M., 1985. Pontryagin's principle for control problems in age-dependent population dynamics. J. Math. Biol 23, 75–101.
Boucekkine, R., Germain, M., Licandro, O., 1997. Replacement echoes in the vintage capital growth model. J. Econ. Theory 74, 333–348.
Boucekkine, R., Hritonenko, N., Yatsenko, Y., 2014. Optimal investment in heterogeneous capital and technology under restricted natural resource. J. Opt. Theory Appl. 163, 310–331.
Boucekkine, R., Pommeret, A., 2004. Energy saving technical progress and capital stock: the role of embodiment. Econ. Model. 21, 429–444.
Bréchet, T., Hritonenko, N., Yatsenko, Y., 2013. Adaptation and mitigation in long-term climate policy. Environ. Res. Econ. 55, 217–243.
Calsina, À., Saldaña, J., 1995. A model of physiologically structured population dynamics with a nonlinear growth rate. J. Math. Biol. 33, 335–364.
Caputo, M.R., 2005. Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications. Cambridge University Press, New York.
Carl, S., Khoi Le, V., Motreanu, D., 2007. Nonsmooth Variational Problems and Their Inequalities. Springer, New York.
Cheevaprawatdomrong, T., Smith, R.L., 2003. A paradox in equipment replacement under technological improvement. Oper. Res. Lett. 31, 77–82.
Clark, C.W., 1976. Mathematical Bioeconomics: The Optimal Management of Renewable Resources. John Wiley and Sons, New York.
Corduneanu, C., 1991. Integral Equations and Applications. Cambridge University Press, Cambridge.
De Roos, A.M., Persson, L., 2001. Physiologically structured models – from versatile technique to ecological theory. Oikos 94, 51–71.
Desch, W., Turi, J., 1996. Asymptotic theory for a class of functional differential equations with state-dependent delays. J. Math. Anal. Appl. 199, 75–87.
Dmitruk, A.V., Vdovina, A.K., 2017. Study of a one-dimensional optimal control problem with a purely state-dependent cost. Diff. Equations Dyn. Syst. 25, 519–526. https://doi.org/10.1007/s12591-016-0306-x.
Domoshnitsky, A., Drakhlin, M., Litsyn, E., 2002. On equations with delay depending on solution. Nonlinear Anal. Theory Methods Appl. 49, 689–701.

Fister, K.R., Lenhart, S., 2004. Optimal control of a competitive system with age-structure. J. Math. Anal. Appl. 291, 526–537.
Goetz, R., Hritonenko, N., Mur, R., Xabadia, A., Yatsenko, Y., 2010. Forest management and carbon sequestration in size-structured forests: The case of Pinus Sylvestris in Spain. Forest Sci. 56 (3), 242–256.
Goetz, R., Hritonenko, N., Xabadia, A., Yatsenko, Y., 2013. Forest management for timber and carbon sequestration in the presence of climate change: the case of *Pinus sylvestris*. Ecol. Econ. 88, 86–96.
Greenhalgh, D., 1987. Analytical results on the stability of age-structured recurrent epidemic models. Math. Med. Biol. 4: 2, 109–144.
Hartman, J., Tan, C.H., 2014. Equipment replacement analysis: a literature review and directions for future research. Eng. Econ. 59, 136–153.
Hritonenko, N., Kato, N., Yatsenko, Y., 2017. Optimal control of investments in old and new capital under improving technology. J. Opt. Theory Appl. 172, 247–266.
Hritonenko, N., Yatsenko, Y., 1995. Optimization of the lifetime and cost of equipment under technological progress. J. Automat. Inform. Sci. 28, 117–128.
Hritonenko, N., Yatsenko, Y., 1996. Modeling and Optimization of the Lifetime of Technologies. Kluwer Academic Publishers, Dordrecht.
Hritonenko, N., Yatsenko, Y., 2005. Turnpike properties of optimal delay in integral dynamic models. J. Opt. Theory Appl. 127, 109–127.
Hritonenko, N., Yatsenko, Y., 2006. Optimization of harvesting return from age-structured population. J. Bioecon. 8, 167–179.
Hritonenko, N., Yatsenko, Y., 2007. The structure of optimal time- and age-dependent harvesting in the Lotka-McKendrik population model. Math. Biosci. 208, 48–62.
Hritonenko, N., Yatsenko, Y., 2008. Anticipation echoes in vintage capital models. Math. Comp. Model. 48, 734–748.
Hritonenko, N., Yatsenko, Y., 2009. Integral equation of optimal replacement: analysis and algorithms. Appl. Math. Model. 33, 2737–2747.
Hritonenko, N., Yatsenko, Y., 2010. Age-structured PDEs in economics, ecology, and demography: optimal control and sustainability. Math. Popul. Stud. 17 (4), 191–214.
Hritonenko, N., Yatsenko, Y., 2012. Bang-bang, impulse, and sustainable harvesting in age-structured populations. J. Biol. Syst. 20, 133–153.
Hritonenko, N., Yatsenko, Y., 2013a. Mathematical Modeling in Economics, Ecology and the Environment. Springer, New York/Heidelberg.
Hritonenko, N., Yatsenko, Y., 2013b. Modeling of environmental adaptation: amenity vs productivity and modernization. Clim. Change Econ. 4 (2), 1–24.
Hritonenko, N., Yatsenko, Y., 2013c. Solvability of integral equations with endogenous delays. Acta Appl. Math. 128 (1), 49–66.
Hritonenko, N., Yatsenko, Y., Boranbayev, S., 2015. Environmentally sustainable industrial modernization and resource consumption: is the Hotelling's rule too steep? Appl. Math. Model. 39, 4365–4377.
Hritonenko, N., Yatsenko, Y., Goetz, R., Xabadia, A., 2009. A bang-bang regime in optimal harvesting of size-structured populations. Nonlinear Anal. Theory Methods Appl. 71, e2331–e2336.
Hritonenko, N., Yatsenko, Y., Goetz, R., Xabadia, A., 2012. Optimal harvesting in forestry: steady-state analysis and climate change impact. J. Biol. Dyn. 7, 41–58.
Jovanovic, B., Tse, C.-Y., 2010. Entry and exit echoes. Rev. Econ. Dyn. 13 (3), 514–536.
Jovanovic, B., Yatsenko, Y., 2012. Investment in vintage capital. J. Econ. Theory 147, 551–569.
Kato, N., 2004. A general model of size-structured population dynamics with nonlinear growth rate. J. Math. Anal. Appl. 297, 234–256.
Kato, N., Oharu, S., Shitaoka, K., 2007. Size-structured plant population models and harvesting problems. J. Comput. Appl. Math. 204, 114–123.
Kharrazi, A., Rovenskaya, E., Yarime, M., Kraines, S., 2013. Quantifying the sustainability of economic resource networks: an ecological information-based approach. Ecol. Econ. 90, 177–186.
Ljung, L., 1987. System Identification: Theory for the User. Prentice-Hall, Englewood Cliffs, New Jersey.
Malcomson, J.M., 1975. Replacement and the rental value of capital equipment subject to obsolescence. J. Econ. Theory 10, 24–41.
McKendrick, A.G., 1926. Applications of mathematics to medical problems. Proc. Edinburgh Math. Soc. 44, 98–130.
Metz, J., Diekmann, O., 1986. The Dynamics of Physiologically Structured Populations. Springer Lecture Notes in Biomathematics, Springer-Verlag, Heidelberg.
Motamedi, N., Hadizadeh, M., Peyghami, M., 2014. A nonlinear integral model of optimal replacement: numerical viewpoint. Commun. Numer. Anal. 2014, 1–9.
Murphy, L.F., Smith, S.J., 1990. Optimal harvesting of an age structured population. J. Math. Biol. 29, 77–90.
Sharpe, F.R., Lotka, A.J., 1911. A problem in age-distribution. Philos. Mag. 21, 435–438.
Solow, R., Tobin, J., von Weizsacker, C., Yaari, M., 1966. Neoclassical growth with fixed factor proportions. Rev. Econ. Stud. 33, 79–115.
Volterra, V., 1959. Theory of Functionals and of Integral and Integral-differential Equations. Dover Publications, New York.
Webb, G.F., 1985. Theory of Nonlinear Age-Dependent Population Dynamics. M. Dekker, New York, Basel.
Yatsenko, Y., 1995. Volterra integral equations with unknown delay time. Methods Appl. Anal. 2, 408–419.
Yatsenko, Y., 2004. Maximum principle for Volterra integral equations with controlled delay time. Optimization 53, 177–187.
Yatsenko, Y., Hritonenko, N., 2005. Optimization of the lifetime of capital equipment using integral models. J. Ind. Manage. Opt. 1, 415–432.
Yatsenko, Y., Hritonenko, N., 2009. Technological breakthroughs and asset replacement. Eng. Econ. 54, 81–100.
Yatsenko, Y., Hritonenko, N., 2017. Machine replacement under evolving deterministic and stochastic costs. Int. J. Prod. Econ. 193, 491–501.