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Short communication

On the Jensen functional and superterzaticity

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ABSTRACT

In this note we describe some results concerning upper and lower bounds for the Jensen functional. We use several known and new results to shed light on the concept of superterzatic function. Particular cases of interest are also considered.

1. Introduction

The aim of this paper is to discuss new results concerning the Jensen functional in the framework of superterzatic functions.

For the reader's convenience, before going into details, we quote here some relevant results regarding the superterzaticity and the Jensen functional.

Definition 1. We consider a real valued function f defined on an interval $I, x_1, x_2, \ldots, x_n \in I$ and $p_1, p_2, \ldots, p_n \in (0,1)$ with $\sum_{i=1}^n p_i = 1$. The *Jensen functional* is defined by

$$\mathcal{J}(f, \mathbf{p}, \mathbf{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

(see Dragomir, 2006).

Definition 2. A function f defined on an interval $I = [0, a)$ is *superterzatic*, if for each $\bar{x} \in I$ there exists a real number $C(\bar{x})$ such that

$$\begin{aligned}\mathcal{J}(f, \mathbf{p}, \mathbf{x}) &\geqslant \sum_{i=1}^{n} p_i x_i \left[(x_i - \bar{x})C(\bar{x}) + \tfrac{f(|x_i - \bar{x}|)}{|x_i - \bar{x}|}\right] \\ &= \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 C(\bar{x}) + \sum_{i=1}^{n} p_i x_i \tfrac{f(|x_i - \bar{x}|)}{|x_i - \bar{x}|}\end{aligned}$$

for all $x_i \geqslant 0$, $i = 1, \ldots, n$, and $p_i \geqslant 0, i = 1, \ldots, n$, with $\sum_{i=1}^n p_i = 1$, such that $\bar{x} = \sum_{i=1}^n p_i x_i$. We use the convention $f(0)/0 = 0$.

This definition was mentioned by S. Abramovich in her talk given at Conference on Inequalities and Applications 10, (Abramovich et al., 2012).

The set of superterzatic functions is closed under addition and positive scalar multiplication.

Example 1. (Abramovich et al., 2012) Let $f(x) = x^p$, $p \geqslant 3$. This function is superterzatic with $C(\bar{x}) = p\bar{x}^{p-1}$. The function $g(x) = x^3 \log \frac{1}{x}$ is also subterzatic (i.e. the inequality in Definition 2 holds with reversed sign).

The paucity of literature on this topic motivated us to study the class of superterzatic functions and to emphasize some basic results connected to the Jensen functional and its behavior in the context of superterzaticity (See also Abramovich and Persson, 2013).

The Jensen functional has been already investigated under various assumptions: convexity, strong convexity, quasiconvexity, superquadraticity and so on. Therefore there is a wide literature about it, we only recommend several recent papers: (Abramovich, 2014, 2016; Dehghan, 2016; Franjić and Pecaric, 2015; Mitroi-Symeonidis and Minculete, 2016a,b; Moradi et al., in press). However there is no comparison between these estimates and we cannot indicate at this point *the best (the sharpest) one*. Throughout this paper we deal with the class of superterzatic functions which eventualy contains functions that also belong to other classes mentioned above and then the interested reader may find more convenient estimates using the same general technique.

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2. Main results

We introduce in a natural way a more general functional.

Definition 3. Assume that we have a real valued function f defined on an interval I, the real numbers p_{ij}, $i=1,\ldots,k$ and $j=1,\ldots,n_i$ such that $p_{ij}>0$, $\sum_{j=1}^{n_i}p_{ij}=1$ for all $i=1,\ldots,k$ (we denote $\mathbf{p}_i=\left(p_{i1},p_{i2},\ldots,p_{in_i}\right)$), $\mathbf{x}_i=(x_{i1},x_{i2},\ldots,x_{in_i})\in I^{n_i}$ for all $i=1,\ldots,k$ and $\mathbf{q}=(q_1,\ldots,q_k)$, $q_i>0$ such that $\sum_{i=1}^{k}q_i=1$. We define the *generalized Jensen functional* by

$$\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k) := \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\ldots p_{kj_k} f\left(\sum_{i=1}^{k}q_i x_{ij_i}\right) - f\left(\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}\right).$$

We notice that for $k=1$ this definition is reduced to Definition 1.

For more results concerning Jensen's functional the reader is referred to the papers (Mitroi, 2011; Mitroi, 2012).

Before announcing the main result, let us give the following lemma that describes the behaviour of the functional under the superterzaticity condition:

Lemma 1. *Let f, $\mathbf{p}_i$, $\mathbf{x}_i$, $\mathbf{q}$ be as in Definition 3. If f is superterzatic then we have*

$$\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k) \geqslant \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\ldots p_{kj_k}\sum_{i=1}^{k}q_i x_{ij_i} \times\left[\left(\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right)C(\bar{x})+\frac{f\left(\left|\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right|\right)}{\left|\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right|}\right],$$

where $\bar{x}=\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}$.

Proof. Since

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\ldots p_{kj_k}=1$$

and

$$\bar{x}=\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k} p_{1j_1}\ldots p_{kj_k}\sum_{i=1}^{k}q_i x_{ij_i},$$

we use Definition 2 and the conclusion follows. □

Theorem 1. *Let f, $\mathbf{p}_i$, $\mathbf{x}_i$ and $\mathbf{q}$ be as in Definition 3 and the positive real numbers r_{ij}, $i=1,\ldots,k$ and $j=1,\ldots,n_i$ such that $\sum_{j=1}^{n_i}r_{ij}=1$ for all $i=1,\ldots,k$. We denote*

$$\mathbf{r}_i=\left(r_{i1},r_{i2},\ldots,r_{in_i}\right) \text{ for all } i=1,\ldots,k,$$

$$m=\min_{1\leqslant j_1\leqslant n_1\ldots 1\leqslant j_k\leqslant n_k}\left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\},$$

$$M=\max_{1\leqslant j_1\leqslant n_1\ldots 1\leqslant j_k\leqslant n_k}\left\{\frac{p_{1j_1}\ldots p_{kj_k}}{r_{1j_1}\ldots r_{kj_k}}\right\}.$$

If f is superterzatic, then:

$$\begin{aligned}&\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)-m\mathcal{J}_k(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\ &\geqslant \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k}-mr_{1j_1}\ldots r_{kj_k}\right)\sum_{i=1}^{k}q_i x_{ij_i}\\ &\quad\times\left[\left(\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right)C(\bar{x})+\frac{f\left(\left|\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right|\right)}{\left|\sum_{i=1}^{k}q_i x_{ij_i}-\bar{x}\right|}\right]+m\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}\\ &\quad\times\left[\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}(r_{ij}-p_{ij})x_{ij}C(\bar{x})+\frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right]\end{aligned}$$

and

$$\begin{aligned}&M\mathcal{J}_k(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)-\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\ &\geqslant \sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(Mr_{1j_1}\ldots r_{kj_k}-p_{1j_1}\ldots p_{kj_k}\right)\sum_{i=1}^{k}q_i x_{ij_i}\\ &\quad\times\left[\left(\sum_{i=1}^{k}q_i x_{ij_i}-\check{x}\right)C(\check{x})+\frac{f\left(\left|\sum_{i=1}^{k}q_i x_{ij_i}-\check{x}\right|\right)}{\left|\sum_{i=1}^{k}q_i x_{ij_i}-\check{x}\right|}\right]+\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}\\ &\quad\times\left[\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}(r_{ij}-p_{ij})x_{ij}C(\check{x})+\frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right].\end{aligned}$$

where $\bar{x}=\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}$ *and* $\check{x}=\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}$.

Proof. *The first inequality*. Obviously

$$\begin{aligned}&\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)-m\mathcal{J}_k(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\ &=\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k}-mr_{1j_1}\ldots r_{kj_k}\right)f\left(\sum_{i=1}^{k}q_i x_{ij_i}\right)\\ &\quad+mf\left(\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}\right)-f\left(\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}\right).\end{aligned}$$

Since

$$\begin{aligned}\bar{x}&=\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}\\ &=\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k}-mr_{1j_1}\ldots r_{kj_k}\right)\sum_{i=1}^{k}q_i x_{ij_i}+m\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}\end{aligned}$$

and

$$\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(p_{1j_1}\ldots p_{kj_k}-mr_{1j_1}\ldots r_{kj_k}\right)+m=1,$$

the conclusion follows by Lemma 1.

The second inequality. One has

$$\begin{aligned}&M\mathcal{J}_k(f,\mathbf{r}_1,\ldots,\mathbf{r}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)-\mathcal{J}_k(f,\mathbf{p}_1,\ldots,\mathbf{p}_k,\mathbf{q},\mathbf{x}_1,\ldots,\mathbf{x}_k)\\ &=\sum_{j_1,\ldots,j_k=1}^{n_1,\ldots,n_k}\left(Mr_{1j_1}\ldots r_{kj_k}-p_{1j_1}\ldots p_{kj_k}\right)f\left(\sum_{i=1}^{k}q_i x_{ij_i}\right)\\ &\quad+f\left(\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}p_{ij}x_{ij}\right)-Mf\left(\sum_{i=1}^{k}q_i\sum_{j=1}^{n_i}r_{ij}x_{ij}\right).\end{aligned}$$

Since

$$\check{x} = \sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} r_{ij} x_{ij} = \sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} \frac{M r_{1j_1}\dots r_{kj_k} - p_{1j_1}\dots p_{kj_k}}{M} \sum_{i=1}^{k} q_i x_{ij_i} + \frac{1}{M}\sum_{i=1}^{k} q_i \sum_{j=1}^{n_i} r_{ij} x_{ij}$$

and

$$\sum_{j_1,\dots,j_k=1}^{n_1,\dots,n_k} \frac{M r_{1j_1}\dots r_{kj_k} - p_{1j_1}\dots p_{kj_k}}{M} + \frac{1}{M} = 1,$$

we may apply again Lemma 1 and the conclusion follows. □

The particular case $\mathbf{p}_1 = \ldots = \mathbf{p}_k = \mathbf{p}$ and $\mathbf{x}_1 = \ldots = \mathbf{x}_k = \mathbf{x}$ is of interest.

Corollary 1. *Let us consider* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$, $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (0,1)^n$ *with* $\sum_{i=1}^{n} p_i = 1, \mathbf{q} = (q_1, q_2, \ldots, q_k)$*such that* $q_i > 0$, $\sum_{i=1}^{k} q_i = 1$ *(* $1 \leqslant k \leqslant n$*) and* $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in (0,1)^n$ *with* $\sum_{i=1}^{n} r_i = 1$. *We put*

$$m = \min_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\frac{p_{i_1}\cdots p_{i_k}}{r_{i_1}\cdots r_{i_k}}\right\},$$
$$M = \max_{1\leqslant i_1,\dots,i_k\leqslant n}\left\{\frac{p_{i_1}\cdots p_{i_k}}{r_{i_1}\cdots r_{i_k}}\right\}.$$

We define

$$\mathcal{J}_k(f,\mathbf{p},\mathbf{q},\mathbf{x}) := \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

If f is superterzatic, then

$$\mathcal{J}_k(f,\mathbf{p},\mathbf{q},\mathbf{x}) - m\mathcal{J}_k(f,\mathbf{r},\mathbf{q},\mathbf{x}) \geqslant \sum_{i_1,\dots,i_k=1}^{n}\left(p_{i_1}\dots p_{i_k} - m r_{i_1}\dots r_{i_k}\right) \times \sum_{j=1}^{k} q_j x_{i_j} \times \left[\left(\sum_{j=1}^{k} q_j x_{i_j} - \bar{x}\right) C(\bar{x}) + \frac{f\left(\left|\sum_{j=1}^{k} q_j x_{i_j} - \bar{x}\right|\right)}{\left|\sum_{j=1}^{k} q_j x_{i_j} - \bar{x}\right|}\right] + m\sum_{i=1}^{n} r_i x_i\left[\sum_{j=1}^{n}(r_j - p_j)x_j C(\bar{x}) + \frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right]$$

and

$$M\mathcal{J}_k(f,\mathbf{r},\mathbf{q},\mathbf{x}) - \mathcal{J}_k(f,\mathbf{p},\mathbf{q},\mathbf{x}) \geqslant \sum_{i_1,\dots,i_k=1}^{n}\left(M r_{i_1}\dots r_{i_k} - p_{i_1}\dots p_{i_k}\right) \times \sum_{j=1}^{k} q_j x_{i_j} \times \left[\left(\sum_{j=1}^{k} q_j x_{i_j} - \check{x}\right) C(\check{x}) + \frac{f\left(\left|\sum_{j=1}^{k} q_j x_{i_j} - \check{x}\right|\right)}{\left|\sum_{j=1}^{k} q_j x_{i_j} - \check{x}\right|}\right] + \sum_{i=1}^{n} r_i x_i\left[\sum_{j=1}^{n}(r_j - p_j)x_j C(\check{x}) + \frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right],$$

where $\bar{x} = \sum_{j=1}^{n} p_j x_j$ *and* $\check{x} = \sum_{j=1}^{n} r_j x_j$.

For the particular case $k = 1$ we get:

Corollary 2. *For* $i = 1, \ldots, n$, *we consider* $x_i \in I, p_i > 0$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $r_i > 0$ *with* $\sum_{i=1}^{n} r_i = 1$. *We denote*

$$m = \min_{i=1,\dots,n}\left\{\frac{p_i}{r_i}\right\},\ M = \max_{i=1,\dots,n}\left\{\frac{p_i}{r_i}\right\}.$$

If f is a superterzatic function, then we have:

$$\mathcal{J}(f,\mathbf{p},\mathbf{x}) - m\mathcal{J}(f,\mathbf{r},\mathbf{x}) \geqslant \sum_{i=1}^{n}(p_i - m r_i)x_i\left[(x_i - \bar{x})C(\bar{x}) + \frac{f(|x_i - \bar{x}|)}{|x_i - \bar{x}|}\right] + m\sum_{i=1}^{n} r_i x_i\left[\sum_{i=1}^{n}(r_i - p_i)x_i C(\bar{x}) + \frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right]$$

and

$$M\mathcal{J}(f,\mathbf{r},\mathbf{x}) - \mathcal{J}(f,\mathbf{p},\mathbf{x}) \geqslant \sum_{i=1}^{n}(M r_i - p_i)x_i\left[(x_i - \check{x})C(\check{x}) + \frac{f(|x_i - \check{x}|)}{|x_i - \check{x}|}\right] + \sum_{i=1}^{n} r_i x_i\left[\sum_{i=1}^{n}(r_i - p_i)x_i C(\check{x}) + \frac{f(|\check{x}-\bar{x}|)}{|\check{x}-\bar{x}|}\right],$$

where $\bar{x} = \sum_{j=1}^{n} p_j x_j$ *and* $\check{x} = \sum_{j=1}^{n} r_j x_j$.

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