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ORIGINAL ARTICLE

Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses

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Abstract In this paper, we consider a non-instantaneous impulsive system represented by second order nonlinear differential equation with deviated argument in a Banach space X. We used the strongly continuous cosine family of linear operators and Banach fixed point method to study the existence and uniqueness of the solution of the non-instantaneous impulsive system. Also, we study the existence and uniqueness of the solution of the nonlocal problem and stability of the non-instantaneous impulsive system. Finally, we give examples to illustrate the application of these abstract results.

1. Introduction

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disaster. These phenomena involve short term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. Sometimes time abrupt changes may stay for time intervals such impulses are called non-instantaneous impulses. The importance of the study of non-instantaneous impulsive differential equations lies in its diverse fields of applications such as in the theory of stage by stage rocket combustion, maintaining hemodynamical equilibrium etc. A very well known application of non-instantaneous impulses is the introduction of insulin in the bloodstream which is abrupt change and the consequent absorption which is a gradual process as it remains active for a finite interval of time. The theory of impulsive differential equations has found enormous applications in realistic mathematical modeling of a wide range of practical situations. It has emerged as an important area of research such as modeling of

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impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology and so forth.

Recently, Hernández and O'Regan (2013) studied mild and classical solutions for the impulsive differential equation with non-instantaneous impulses which is of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_i, t_{i+1}] \quad i = 0, 1, \ldots, m, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i], \quad i = 1, 2, \ldots, m, \\ x(0) = x_0 \in X \end{cases} \tag{1.1}$$

In Wang and Fečkan (2015) have a remark on the conditions in Eq. (1.1):

$$x(t) = g_i(t, x(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, m, \tag{1.2}$$

where $g_i \in C([t_i, s_i] \times X, X)$ and there are positive constants L_{g_i}, $i = 1, \ldots, m$ such that

$$\|g_i(t, x_1) - g_i(t, x_2)\| \leqslant L_{g_i}\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \ t \in [t_i, s_i]$$

It follows from Theorem 2.1 in Hernández and O'Regan (2013) that $\max\{L_{g_i} : i = 1, \ldots, m\} < 1$ is a necessary condition. Then Banach fixed point theorem gives a unique $z_i \in C([t_i, s_i], X)$ so that $z = g_i(t, z)$ if and only if $z = z_i(t)$. So Eq. (1.2) is equivalent to

$$x(t) = z_i(t), \quad t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \tag{1.3}$$

which does not depend on the state $x(.)$. Thus, it is necessary to modify Eq. (1.2) and consider the condition

$$x(t) = g_i(t, x(t_i^-)), \quad t \in (t_i, s_i], \ i = 1, 2, \ldots, m. \tag{1.4}$$

Of course then $x(t_i^+) = g_i(t_i, x(t_i^-)), i = 1, 2, \ldots, m$. The symbols $x(t_i^+) := \lim_{\epsilon \to 0^+} x(t_i + \epsilon)$ and $x(t_i^-) := \lim_{\epsilon \to 0^-} x(t_i + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$ respectively. Motivated by above remark, Wang and Fečkan Wang and Fečkan, 2015 have shown existence, uniqueness and stability of solutions of such general class of impulsive differential equations.

In this paper, we continue in this direction to study the second order nonlinear differential equation with non-instantaneous impulses and deviated argument in a Banach space X

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x[h(x(t), t)]), \\ \quad t \in (s_i, t_{i+1}), \ i = 0, 1, \ldots, m, \\ x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.5}$$

where $x(t)$ be a state function, $0 = s_0 = t_0 < t_1 < s_1 < t_2, \ldots, t_m < s_m < t_{m+1} = T < \infty$. We consider in Eq. (1.5) that $x \in C((t_i, t_{i+1}], X), i = 0, 1, \ldots, m$ and there exist $x(t_i^-)$ and $x(t_i^+)$, $i = 1, 2, \ldots, m$ with $x(t_i^-) = x(t_i)$. The functions $J_i^1(t, x(t_i^-))$ and $J_i^2(t, x(t_i^-))$ represent noninstantaneous impulses during the intervals $(t_i, s_i], i = 1, 2, \ldots, m$, so impulses at t_i^- have some duration, namely on intervals $(t_i, s_i]$. A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t \in \mathbb{R}}$ on X. J_i^1, J_i^2, h and f are suitable functions and they will be specified later.

Many partial differential equations that arise in several problems connected with the transverse motion of an extensible beam, the vibration of hinged bars, and many other physical phenomena can be formulated as the second order abstract differential equations in the infinite dimensional spaces. A useful tool for the study of second-order abstract differential equations is the theory of strongly continuous cosine families of operators. Existence and uniqueness of the solution of second-order nonlinear systems and controllability of these systems in Banach spaces have been investigated extensively by many authors (Chalishajar, 2009; Pandey et al., 2014; Acharya, 2013; Arthi and Balachandran, 2014; Sakthivel et al., 2009).

In certain real world problems, delay depends not only on the time but also on the unknown quantity. The differential equations with deviated arguments are generalization of delay differential equations. Gal (2007) has considered a nonlinear abstract differential equation with deviated arguments and studied the existence and uniqueness of solutions. Recently, Muslim et al. (2016) studied exact and trajectory controllability of second order impulsive nonlinear systems with deviated argument. There are only few papers discussing the second order differential equations with deviated arguments in infinite dimensional spaces. As per my knowledge, there is no paper discussing the existence, uniqueness and stability of the mild solution of the second order differential equation with non-instantaneous impulses and deviated argument in Banach space. In order to fill this gap, we consider a nonlinear second order differential equation with deviated argument. Moreover, the study of second order differential equations with noninstantaneous impulses has not only mathematical significance but also it has applications such as harmonic oscillator with impulses and forced string equation, which we present in examples.

2. Preliminaries and assumptions

We briefly review definitions and some useful properties of the theory of cosine family.

Definition 2.1 (*see, Travis and Webb, 1978*). A one parameter family $(C(t))_{t \in \mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$,
(ii) $C(0) = I$,
(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed point $x \in X$.

$(S(t))_{t \in \mathbb{R}}$: is the sine function associated to the strongly continuous cosine family, $(C(t))_{t \in \mathbb{R}}$: which is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X, \ t \in \mathbb{R}.$$

$D(A)$ be the domain of the operator A which is defined by

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in t}\}.$$

$D(A)$ is the Banach space endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$. We define a set

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in t}\}$$

which is a Banach space endowed with norm $\|x\|_E = \|x\| + \sup_{0 \leqslant t \leqslant 1}\|AS(t)x\|$ for all $x \in E$.

With the help of $C(t)$ and $S(t)$, we define a operator valued function

$$\bar{h}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}.$$

Operator valued function $\bar{h}(t)$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator

$$\overline{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined on $D(A) \times E$. It follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. If $x : [0, \infty) \to X$ is locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E valued continuous function which is a consequence of the fact that

$$\int_0^t \bar{h}(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)ds \\ \int_0^t C(t-s)x(s)ds \end{bmatrix}$$

defines an $(E \times X)$ valued continuous function.

Propostion 2.1 (*see, Travis and Webb, 1978*). Let $(C(t))_{t\in R}$ be a strongly continuous cosine family in X. The following are true:

(i) there exist constants $K \geqslant 1$ and $\omega \geqslant 0$ such that $|C(t)| \leqslant Ke^{\omega|t|}$ for all $t \in \mathbb{R}$.
(ii) $|S(t_2) - S(t_1)| \leqslant K|\int_{t_1}^{t_2} e^{\omega|s|}ds|$ for all $t_1, t_2 \in \mathbb{R}$.

For more details on cosine family theory, we refer to Fattorini (1985), Travis et al. (1977) and Travis and Webb (1978).

Let $PC([0, T], X)$ be the space of piecewise continuous functions.

$PC([0, T], X) = \{x : J = [0, T] \to X : x \in C((t_k, t_{k+1}], X)$, $k = 0, 1, \ldots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, \ldots, m$ with $x(t_k^-) = x(t_k)\}$. It can be seen easily that $PC([0, T], X)$ for all $t \in [0, T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{PCB} := \sup\{\|\psi(\eta)\|e^{-\Omega\eta}, \quad 0 \leqslant \eta \leqslant t\}$$

for some $\Omega > 0$. We set, $C_L(J, X) = \{y \in PC([0, T], X) : \|y(t) - y(s)\| \leqslant L|t - s|, \forall\ t, s \in [0, T]\}$, where L is a suitable positive constant. Clearly $C_L(J, X)$ is a Banach space endowed with PCB norm.

In order to prove the existence, uniqueness and stability of the solution for the problem Eq. (1.5), we need the following assumptions:

(A1) A be the infinitesimal generator of a strongly continuous cosine family, $(C(t))_{t\in\mathbb{R}}$: of bounded linear operators.
(A2) $f : J_1 \times X \times X \to X$, $J_1 = \bigcup_{i=0}^m [s_i, t_{i+1}]$ is a continuous function and there exists a positive constant K_1 such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqslant K_1(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for every $x_1, x_2, y_1, y_2 \in X$, $t \in J_1$. Also there exists a positive constant N such that $\|f(t, x, y)\| \leqslant N$, $\forall\ t \in J_1$ and $x, y \in X$.
(A3) $h : X \times J_1 \to J$ is continuous and there exists a positive constant L_h such that

$$|h(x_1, s) - h(x_2, s)| \leqslant L_h\|x_1 - x_2\|, \forall x_1, x_2 \in X, t \in J_1$$

and it holds $h(., 0) = 0$.
(A4) $J_i^l \in C(I_i \times X, X)$, $I_i = [t_i, s_i]$ and there are positive constants $L_{J_i^l}$, $i = 1, 2, \ldots, m$, $l = 1, 2$, such that

$$\|J_i^l(t, x_1) - J_i^l(s, x_2)\| \leqslant L_{J_i^l}(|t - s| + \|x_1 - x_2\|),$$
$$\forall\ t, s \in I_i \text{ and } x_1, x_2 \in X.$$

(A5) There exist positive constants $C_{J_i^1}$ and $C_{J_i^2}$, $i = 1, 2, \ldots, m$ such that

$$\|J_i^1(t, x)\| \leqslant C_{J_i^1} \text{ and } \|J_i^2(t, x)\| \leqslant C_{J_i^2}, \ \forall\ t \in I_i \text{ and } x \in X.$$

In the following definition, we introduce the concept of mild solution for the problem Eq. (1.5).

Definition 2.2. A function $x \in C_L(J, X)$ is called a mild solution of the impulsive problem Eq. (1.5) if it satisfies the following relations:

$$x(0) = x_0,\ x'(0) = y_0,$$

the non-instantaneous impulse conditions $x(t) = J_i^1(t, x(t_i^-))$, $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, $x'(t) = J_i^2(t, x(t_i^-))$, $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$ and $x(t)$ is the solution of the following integral equations

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x[h(x(s), s)])ds,$$
$$t \in [0, t_1],$$

$$x(t) = C(t - s_i)(J_i^1(s_i, x(t_i^-))) + S(t - s_i)(J_i^2(s_i, x(t_i^-))) + \int_{s_i}^t S(t-s)f(s, x(s), x[h(x(s), s)])ds,$$
$$t \in [s_i, t_{i+1}], \quad i = 1, 2, \ldots, m.$$

3. Existence and uniqueness result

Theorem 3.1. *Let $x_0 \in D(A)$, $y_0 \in E$. If all the assumptions (A1)-(A5) are satisfied, then the second order problem Eq. (1.5) has a unique mild solution $x \in C_L(J, X)$.*

Proof. Since $AS(t)$ is a bounded linear operator therefore, we set $\rho = \sup_{t\in J}\|AS(t)\|$. For more details on $\|AS(t)\|$, we refer (Pandey et al., 2014; Sakthivel et al., 2009; Hernández and McKibben, 2005). By choosing

$$\delta = \max_{1\leqslant i\leqslant m}\left\{\left(Ke^{\omega T}C_{J_i^1} + KTe^{\omega T}C_{J_i^2} + \frac{NKT}{\omega}e^{\omega T}\right)e^{-\Omega s_i}, \right.$$
$$\left. Ke^{\omega T}\|x_0\| + KTe^{\omega T}\|y_0\| + \frac{NKT}{\omega}e^{\omega T}\, e^{-\Omega t_i}C_{J_i^1}\right\},$$

we set

$\mathcal{W} = \{x \in C_L(J,X) : \|x\|_{PCB} \leqslant \delta\}$.

We define a map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ given by

$$(\mathcal{F}x)(t) = J_i^1\big(t, C(t_i - s_{i-1})(J_i^1(s_{i-1}, x(t_{i-1}^-))) + S(t_i - s_{i-1})(J_i^2(s_{i-1}, x(t_{i-1}^-))) + \int_{s_{i-1}}^{t_i} S(t_i - s)f(s, x(s), x[h(x(s), s)])ds\big),$$
$$t \in (t_i, s_i] \quad i = 1, 2, \ldots, m;$$

$$(\mathcal{F}x)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \quad t \in [0, t_1];$$

$$(\mathcal{F}x)(t) = C(t - s_i)(J_i^1(s_i, x(t_i^-))) + S(t - s_i)(J_i^2(s_i, x(t_i^-))) + \int_{s_i}^t S(t-s)f(s, x(s), x[h(x(s), s)])ds,$$
$$t \in (s_i, t_{i+1}] \quad i = 1, 2, \ldots, m.$$

First, we need to show that $\mathcal{F}x \in C_L(J,X)$ for any $x \in C_L(J,X)$ and some $L > 0$. If $t_{i+1} \geqslant \tilde{t}_2 > \tilde{t}_1 > s_i$, then we get

$$\begin{aligned}|(\mathcal{F}x)(\tilde{t}_2) - (\mathcal{F}x)(\tilde{t}_1)\| &\leqslant \|(C(\tilde{t}_2 - s_i) - C(\tilde{t}_1 - s_i))(J_i^1(s_i, x(t_i^-)))\| \\ &+\|(S(\tilde{t}_2 - s_i) - S(\tilde{t}_1 - s_i))(J_i^2(s_i, x(t_i^-)))\| \\ &+N\textstyle\int_{s_i}^{\tilde{t}_1} \|(S(\tilde{t}_2 - s) - S(\tilde{t}_1 - s))\|ds \\ &+N\textstyle\int_{\tilde{t}_1}^{\tilde{t}_2} \|S(\tilde{t}_2 - s)\|ds \\ &\leqslant I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.1}$$

We have,

$$\begin{aligned}I_1 &= \|(C(\tilde{t}_2 - s_i) - C(\tilde{t}_1 - s_i))(J_i^1(s_i, x(t_i^-)))\| \\ &= \|\textstyle\int_{\tilde{t}_1 - s_i}^{\tilde{t}_2 - s_i} AS(\tau)(J_i^1(s_i, x(t_i^-)))d\tau\| \\ &\leqslant C_1(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.2}$$

where $C_1 = \rho C_{J_i^1}$.

Similarlly, we have

$$\begin{aligned}I_2 &= \|(S(\tilde{t}_2 - s_i) - S(\tilde{t}_1 - s_i))(J_i^2(s_i, x(t_i^-)))\| \\ &= \|K\textstyle\int_{\tilde{t}_1 - s_i}^{\tilde{t}_2 - s_i} e^{\omega\tau}(J_i^2(s_i, x(t_i^-)))d\tau\| \\ &\leqslant C_2(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.3}$$

where $C_2 = Ke^{\omega t_{i+1}} C_{J_i^2}$.

Similarly, we calculate third and fourth parts of inequality Eq. (3.1) as follows

$$\begin{aligned}I_3 &= N\textstyle\int_{s_i}^{\tilde{t}_1} \|(S(\tilde{t}_2 - s) - S(\tilde{t}_1 - s))\|ds \\ &\leqslant N\textstyle\int_{s_i}^{\tilde{t}_1} \|K\int_{\tilde{t}_1 - s}^{\tilde{t}_2 - s} e^{\omega\tau}d\tau\|ds \\ &\leqslant C_3(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.4}$$

where $C_3 = KNt_{i+1}e^{\omega t_{i+1}}$ and

$$I_4 = N\int_{\tilde{t}_1}^{\tilde{t}_2} \|S(\tilde{t}_2 - s)\|ds \leqslant C_4(\tilde{t}_2 - \tilde{t}_1), \tag{3.5}$$

where $C_4 = KNt_{i+1}e^{\omega t_{i+1}}$.

We use the inequalities Eqs. (3.2)–(3.5) in inequality Eq. (3.1) and get the following inequality

$$\|(\mathcal{F}x)(\tilde{t}_2) - (\mathcal{F}x)(\tilde{t}_1)\| \leqslant L|\tilde{t}_2 - \tilde{t}_1|, \tag{3.6}$$

where $L \geqslant C_1 + C_2 + C_3 + C_4$.

If $t_1 \geqslant \tilde{t}_2 > \tilde{t}_1 \geqslant 0$, then we get

$$\begin{aligned}\|(\mathcal{F}x)(\tilde{t}_2) - (\mathcal{F}x)(\tilde{t}_1)\| &\leqslant \|(C(\tilde{t}_2) - C(\tilde{t}_1))x_0\| + \|(S(\tilde{t}_2) - S(\tilde{t}_1))y_0\| \\ &+N\textstyle\int_0^{\tilde{t}_1} \|(S(\tilde{t}_2 - s) - S(\tilde{t}_1 - s))\|ds \\ &+N\textstyle\int_{\tilde{t}_1}^{\tilde{t}_2} \|S(\tilde{t}_2 - s)\|ds \\ &\leqslant I_5 + I_6 + I_7 + I_8.\end{aligned} \tag{3.7}$$

We have,

$$\begin{aligned}I_5 = \|(C(\tilde{t}_2) - C(\tilde{t}_1))x_0\| &= \|\textstyle\int_{\tilde{t}_1}^{\tilde{t}_2} AS(\tau)x_0 d\tau\| \\ &\leqslant C_5(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.8}$$

where $C_5 = \rho\|x_0\|$.

Similarly, we have

$$\begin{aligned}I_6 = \|(S(\tilde{t}_2) - S(\tilde{t}_1))y_0\| &= \|K\textstyle\int_{\tilde{t}_1}^{\tilde{t}_2} e^{\omega\tau}y_0 d\tau\| \\ &\leqslant C_6(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.9}$$

where $C_6 = Ke^{\omega t_1}\|y_0\|$.

Similarly, we calculate third and fourth part of inequality Eq. (3.7) as follows

$$\begin{aligned}I_7 &= N\textstyle\int_0^{\tilde{t}_1} \|(S(\tilde{t}_2 - s) - S(\tilde{t}_1 - s))\|ds \\ &\leqslant N\textstyle\int_0^{\tilde{t}_1} \|K\int_{\tilde{t}_1 - s}^{\tilde{t}_2 - s} e^{\omega\tau}d\tau\|ds \\ &\leqslant C_7(\tilde{t}_2 - \tilde{t}_1),\end{aligned} \tag{3.10}$$

where $C_7 = KNt_1e^{\omega t_1}$ and

$$I_8 = N\int_{\tilde{t}_1}^{\tilde{t}_2} \|S(\tilde{t}_2 - s)\|ds \leqslant C_8(\tilde{t}_2 - \tilde{t}_1), \tag{3.11}$$

where $C_8 = KNt_1e^{\omega t_1}$.

We use the inequalities Eqs. (3.8)–(3.11) in inequality Eq. (3.7) and get the following inequality

$$\|(\mathcal{F}x)(\tilde{t}_2) - (\mathcal{F}x)(\tilde{t}_1)\| \leqslant L|\tilde{t}_2 - \tilde{t}_1|, \tag{3.12}$$

where $L \geqslant C_5 + C_6 + C_7 + C_8$.

Finally, if $s_i \geqslant \tilde{t}_2 > \tilde{t}_1 > t_i$, then we get

$$\|(\mathcal{F}x)(\tilde{t}_2) - (\mathcal{F}x)(\tilde{t}_1)\| \leqslant L_{J_i^1}.(\tilde{t}_2 - \tilde{t}_1). \tag{3.13}$$

Summarizing, we see that $\mathcal{F}x \in C_L(J,X)$ for any $x \in C_L(J,X)$ and some $L > 0$.

Next, we need to show that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$. Now for $t \in (s_i, t_{i+1}]$ and $x \in \mathcal{W}$, we have

$$\begin{aligned}\|(\mathcal{F}x)(t)\| &\leqslant \|C(t - s_i)(J_i^1(s_i, x(t_i^-)))\| + \|S(t - s_i)(J_i^2(s_i, x(t_i^-)))\| \\ &+ \textstyle\int_{s_i}^t \|S(t-s)f(s, x(s), x[h(x(s), s)])\|ds \\ &\leqslant Ke^{\omega(t - s_i)}C_{J_i^1} + K(t - s_i)e^{\omega(t - s_i)}C_{J_i^2} \\ &+N\textstyle\int_{s_i}^t K(t-s)e^{\omega(t-s)}ds.\end{aligned}$$

Hence,

$$\|(\mathcal{F}x)\|_{PCB} \leqslant \left(Ke^{\omega T}C_{J_i^1} + KTe^{\omega T}C_{J_i^2} + \tfrac{NKT}{\omega}e^{\omega T}\right)e^{-\Omega s_i}.$$

Now for $t \in [0, t_1]$ and $x \in \mathcal{W}$, we have

$$\begin{aligned}\|(\mathcal{F}x)(t)\| &\leqslant \|C(t)x_0\| + \|S(t)y_0\| + N\int_0^t \|S(t-s)\|ds \\ &\leqslant Ke^{\omega t}\|x_0\| + Kte^{\omega t}\|y_0\| + N\int_0^t K(t-s)e^{\omega(t-s)}ds.\end{aligned}$$

Hence,

$$\|(\mathcal{F}x)\|_{PCB} \leqslant Ke^{\omega T}\|x_0\| + KTe^{\omega T}\|y_0\| + \frac{NKT}{\omega}e^{\omega T}.$$

Similarly for $t \in (t_i, s_i]$ and $x \in \mathcal{W}$, we have

$$\|(\mathcal{F}x)\|_{PCB} \leqslant e^{-\Omega t_i}C_{J_i^1}.$$

After summarizing the above inequalities, we get

$$\|(\mathcal{F}x)\|_{PCB} \leqslant \delta.$$

Therefore $\mathcal{F} : \mathcal{W} \to \mathcal{W}$. For any $x, y \in \mathcal{W}$, $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\leqslant Ke^{\omega(t-s_i)}L_{J_i^1}\|x(t_i^-) - y(t_i^-)\| \\ &+K(t-s_i)e^{\omega(t-s_i)}L_{J_i^2}\|x(t_i^-) - y(t_i^-)\| \\ &+KK_1(2+LL_h)\int_{s_i}^t (t-s)e^{\omega(t-s)}\|x(s) - y(s)\|ds \\ &\leqslant Ke^{\omega(t-s_i)+\Omega t_i}L_{J_i^1}\|x-y\|_{PCB} \\ &+K(t-s_i)e^{\omega(t-s_i)+\Omega t_i}L_{J_i^2}\|x-y\|_{PCB} \\ &+KK_1(2+LL_h)\int_{s_i}^t (t-s)e^{\omega(t-s)+\Omega s}ds\|x-y\|_{PCB} \\ &\leqslant Ke^{\omega(t-s_i)+\Omega t_i}L_{J_i^1}\|x-y\|_{PCB} \\ &+K(t-s_i)e^{\omega(t-s_i)+\Omega t_i}L_{J_i^2}\|x-y\|_{PCB} \\ &+\tfrac{KK_1(2+LL_h)te^{\Omega t}}{(\Omega-\omega)}\|x-y\|_{PCB}.\end{aligned}$$

Hence,

$$\begin{aligned}\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|e^{-\Omega t} &\leqslant Ke^{\omega(t-s_i)+\Omega(t_i-t)}L_{J_i^1}\|x-y\|_{PCB} \\ &+K(t-s_i)e^{\omega(t-s_i)+\Omega(t_i-t)}L_{J_i^2}\|x-y\|_{PCB} \\ &+\tfrac{KK_1(2+LL_h)t_{i+1}}{(\Omega-\omega)}\|x-y\|_{PCB} \\ &\leqslant \Big(Ke^{\Omega(t_i-s_i)}L_{J_i^1} + Kt_{i+1}e^{\Omega(t_i-s_i)}L_{J_i^2} \\ &+\tfrac{KK_1(2+LL_h)t_{i+1}}{(\Omega-\omega)}\Big)\|x-y\|_{PCB}.\end{aligned}$$

For $t \in [0, t_1]$, we obtain

$$\begin{aligned}\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\leqslant KK_1(2+LL_h)\int_0^t (t-s)e^{\omega(t-s)}\|x(s) - y(s)\|ds \\ &\leqslant KK_1(2+LL_h)\int_0^t (t-s)e^{\omega(t-s)+\Omega s}ds\|x-y\|_{PCB} \\ &\leqslant \tfrac{KK_1(2+LL_h)te^{\Omega t}}{(\Omega-\omega)}\|x-y\|_{PCB}.\end{aligned}$$

Hence,

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|e^{-\Omega t} \leqslant \frac{KK_1(2+LL_h)t_1}{(\Omega-\omega)}\|x-y\|_{PCB}.$$

Similarly, for $t \in (t_i, s_i], i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| &\leqslant L_{J_i^1}\Big(Ke^{(t_i-s_{i-1})\omega}L_{J_i^1}\|x(t_{i-1}^-) - y(t_{i-1}^-)\| + K(t_i-s_{i-1})e^{(t_i-s_{i-1})\omega}L_{J_i^2}\|x(t_{i-1}^-) - y(t_{i-1}^-)\| \\ &+KK_1(2+LL_h)\int_{s_{i-1}}^{t_i}(t_i-s)e^{(t_i-s)\omega}\|x(s)-y(s)\|ds\Big) \\ &\leqslant L_{J_i^1}\Big(Ke^{(t_i-s_{i-1})\omega+\Omega t_{i-1}}L_{J_i^1}\|x-y\|_{PCB} + K(t_i-s_{i-1})e^{(t_i-s_{i-1})\omega+\Omega t_{i-1}}L_{J_i^2}\|x-y\|_{PCB} \\ &+KK_1(2+LL_h)\int_{s_{i-1}}^{t_i}(t_i-s)e^{(t_i-s)\omega+\Omega s}ds\|x-y\|_{PCB}\Big) \\ &\leqslant L_{J_i^1}\Big(Ke^{(t_i-s_{i-1})\omega+\Omega t_{i-1}}L_{J_i^1}\|x-y\|_{PCB} + K(t_i-s_{i-1})e^{(t_i-s_{i-1})\omega+\Omega t_{i-1}}L_{J_i^2}\|x-y\|_{PCB} + \tfrac{KK_1(2+LL_h)t_ie^{\Omega t_i}}{(\Omega-\omega)}\|x-y\|_{PCB}\Big).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\|e^{-\Omega t} &\leqslant L_{J_i^1}\Big(Ke^{(t_i-s_{i-1})\omega+(t_{i-1}-t_i)\Omega}L_{J_i^1} + K(t_i-s_{i-1}) \\ &\times e^{(t_i-s_{i-1})\omega+(t_{i-1}-t_i)\Omega}L_{J_i^2} + \tfrac{KK_1(2+LL_h)t_i}{(\Omega-\omega)}\Big)\|x-y\|_{PCB} \\ &\leqslant L_{J_i^1}\Big(Ke^{(t_{i-1}-s_{i-1})\Omega}L_{J_i^1} + Ks_ie^{(t_{i-1}-s_{i-1})\Omega}L_{J_i^2} \\ &+\tfrac{KK_1(2+LL_h)t_i}{(\Omega-\omega)}\Big)\|x-y\|_{PCB}.\end{aligned}$$

After summarizing the above inequalities, we have the following

$$\|(\mathcal{F}x) - (\mathcal{F}y)\|_{PCB} \leqslant L_F\|x-y\|_{PCB},$$

where

$$\begin{aligned}L_F = \max_{1\leqslant i\leqslant m}\Bigg\{&\bigg(Ke^{\Omega(t_i-s_i)}L_{J_i^1} + Kt_{i+1}e^{\Omega(t_i-s_i)}L_{J_i^2} + \frac{KK_1(2+LL_h)t_{i+1}}{(\Omega-\omega)}\bigg), \\ &\frac{KK_1(2+LL_h)t_1}{(\Omega-\omega)}, L_{J_i^1}\Big(Ke^{(t_{i-1}-s_{i-1})\Omega}L_{J_i^1} + Ks_ie^{(t_{i-1}-s_{i-1})\Omega}L_{J_i^2} \\ &+\frac{KK_1(2+LL_h)t_i}{(\Omega-\omega)}\Big)\Bigg\}.\end{aligned}$$

Hence, $\mathcal{F}$ is a strict contraction mapping for sufficiently large $\Omega > \omega$. Application of Banach fixed point theorem immediately gives a unique mild solution to the problem Eq. (1.5). □

4. Nonlocal problems

The nonlocal condition is a generalization of the classical initial condition. The study of nonlocal initial value problems are important because they appear in many physical systems. Byszewski (1991) was the first author who studied the existence and uniqueness of mild solutions to the Cauchy problems with nonlocal conditions. In this section, we investigate the existence and uniqueness of mild solution Eq. (1.5) with nonlocal conditions.

We consider the following nonlocal differential problem with deviated argument in a Banach space X:

$$\begin{aligned}&x''(t) = Ax(t) + f(t, x(t), x[h(x(t), t)]), \\ &t \in (s_i, t_{i+1}) \quad i = 0, 1, \ldots, m, \\ &x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, m, \\ &x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, 2, \ldots, m, \\ &x(0) = x_0 + p(x), \quad x'(0) = y_0 + q(x),\end{aligned} \tag{4.1}$$

where $x(t)$ be a state function, $0 = s_0 < t_1 < s_1 < t_2, \ldots, t_m < s_m < t_{m+1} = T < \infty$. The functions $J_i^1(t, x(t_i^-))$ and $J_i^2(t, x(t_i^-))$ represent non-instantaneous impulses same as in system Eq. (1.5). A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$

on X. The functions $p(x)$ and $q(x)$ will be suitably specified later.

Definition 4.1. A function $x \in C_L(J,X)$ is called a mild solution of the impulsive problem Eq. (4.1) if it satisfies the following relations:

$$x(0) = x_0 + p(x),\ x'(0) = y_0 + q(x),$$

the non-instantaneous impulse conditions

$$x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m,$$
$$x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m$$

and $x(t)$ is the solution of the following integral equations

$$x(t) = C(t)(x_0 + p(x)) + S(t)(y_0 + q(x)) + \int_0^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \quad t \in [0, t_1],$$

$$x(t) = C(t-s_i)(J_i^1(s_i, x(t_i^-))) + S(t-s_i)(J_i^2(s_i, x(t_i^-))) + \int_{s_i}^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \quad t \in [s_i, t_{i+1}],$$
$$i = 1,2,\ldots,m.$$

Further, we need assumptions on the functions p and q to show the existence and uniqueness of the solution for the problem Eq. (4.1)

(A6) The functions $p, q : C(J,X) \to X$are continuous and there exist positive constants c_p and c_q such that

(i) $\|p(x_1) - p(x_2)\| \leqslant c_p\|x_1 - x_2\|$,
(ii) $\|q(x_1) - q(x_2)\| \leqslant c_q\|x_1 - x_2\|$.

Theorem 4.1. *Let $x_0 \in D(A)$, $y_0 \in E$. If all the assumptions (A1)–(A6) are satisfied, then the second order nonlocal problem Eq. (4.1) has a unique mild solution $x \in C_L(J,X)$ provided that*

$$Ke^{\omega t_1}c_p + K\frac{(e^{\omega t_1} - 1)}{\omega}c_q < 1.$$

Proof. By choosing

$$\delta' = \max_{1\leqslant i\leqslant m}\left\{\left(Ke^{\omega T}C_{J_i^1} + KTe^{\omega T}C_{J_i^2} + \frac{NKT}{\omega}e^{\omega T}\right)e^{-\Omega s_i},\right.$$
$$\left. Ke^{\omega T}\|x_0 + p(x)\| + KTe^{\omega T}\|y_0 + q(x)\| + \frac{NKT}{\omega}e^{\omega T},\ e^{-\Omega t_i}C_{J_i^1}\right\},$$

we set

$$\mathcal{W}' = \{x \in C_L(J,X) : \|x\|_{PCB} \leqslant \delta'\}.$$

We define a map $\mathcal{F} : \mathcal{W}' \to \mathcal{W}'$ given by

$$(\mathcal{F}x)(t) = J_i^1\big(t, C(t_i - s_{i-1})(J_i^1(s_{i-1}, x(t_{i-1}^-))) + S(t_i - s_{i-1})(J_i^2(s_{i-1}, x(t_{i-1}^-))) + \int_{s_{i-1}}^{t_i} S(t_i - s)f(s, x(s), x[h(x(s), s)])ds\big),\ t \in (t_i, s_i]\ i = 1,2,\ldots,m;$$

$$(\mathcal{F}x)(t) = C(t)(x_0 + p(x)) + S(t)(y_0 + q(x)) + \int_0^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \quad \forall\, t \in [0, t_1];$$

$$(\mathcal{F}x)(t) = C(t-s_i)(J_i^1(s_i, x(t_i^-))) + S(t-s_i)(J_i^2(s_i, x(t_i^-))) + \int_{s_i}^t S(t-s)f(s, x(s), x[h(x(s), s)])ds,$$
$$t \in (s_i, t_{i+1}]\ i = 1,2,\ldots,m.$$

We have,

$$\|(\mathcal{F}x) - (\mathcal{F}y)\|_{PCB} \leqslant L_F'\|x - y\|_{PCB},$$

where

$$L_F' = \max_{1\leqslant i\leqslant m}\left\{\left(Ke^{\Omega(t_i - s_i)}L_{J_i^1} + Kt_{i+1}e^{\Omega(t_i - s_i)}L_{J_i^2} + \frac{KK_1(2 + LL_h)t_{i+1}}{(\Omega - \omega)}\right),\right.$$
$$Ke^{\omega t_1}c_p + K\frac{(e^{\omega t_1} - 1)}{\omega}c_q + \frac{KK_1(2 + LL_h)t_1}{(\Omega - \omega)},$$
$$\left. L_{J_i^1}\left(Ke^{(t_{i-1} - s_{i-1})\Omega}L_{J_i^1} + Ks_ie^{(t_{i-1} - s_{i-1})\Omega}L_{J_i^2} + \frac{KK_1(2 + LL_h)t_i}{(\Omega - \omega)}\right)\right\}.$$

Thus, $\mathcal{F}$ is a strict contraction mapping for sufficiently large $\Omega > \omega$. Application of Banach fixed point theorem immediately gives a unique mild solution to the problem Eq. (4.1). The proof of this theorem is the consequence of Theorem 3.1. □

5. Ulam's type stability

In this section, we show Ulam's type stability for the system Eq. (1.5).

Let $\epsilon > 0$, $\psi \geqslant 0$ and $\phi \in PC(J, \mathbb{R}^+)$ be the nondecreasing. We consider the following inequalities

$$\begin{cases} \|y''(t) - Ay(t) - f(t, y(t), y[h(y(t), t)])\| \leqslant \epsilon, \quad t \in (s_i, t_{i+1}) \\ \quad i = 0,1,\ldots,m, \\ \|y(t) - J_i^1(t, y(t_i^-))\| \leqslant \epsilon, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ \|y'(t) - J_i^2(t, y(t_i^-))\| \leqslant \epsilon, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m \end{cases} \tag{5.1}$$

and

$$\begin{cases} \|y''(t) - Ay(t) - f(t, y(t), y[h(y(t), t)])\| \leqslant \phi(t), \\ \quad t \in (s_i, t_{i+1}) \quad i = 0,1,\ldots,m, \\ \|y(t) - J_i^1(t, y(t_i^-))\| \leqslant \psi, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ \|y'(t) - J_i^2(t, y(t_i^-))\| \leqslant \psi, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m \end{cases} \tag{5.2}$$

and

$$\begin{cases} \|y''(t) - Ay(t) - f(t, y(t), y[h(y(t), t)])\| \leqslant \epsilon\phi(t), \\ \quad t \in (s_i, t_{i+1}) \quad i = 0,1,\ldots,m, \\ \|y(t) - J_i^1(t, y(t_i^-))\| \leqslant \epsilon\psi, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m. \\ \|y'(t) - J_i^2(t, y(t_i^-))\| \leqslant \epsilon\psi, \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m. \end{cases} \tag{5.3}$$

Now, we take the vector space

$$Z = C_L(J,X)\cap_{i=0}^m C^2((s_i, t_{i+1}), X)\cap_{i=0}^m C((s_i, t_{i+1}), D(A)).$$

The following definitions are inspired by Wang et al. Wang and Fečkan, 2015.

Definition 5.1. The Eq. (1.5) is Ulam-Hyers stable with if there exists $c(K_1L_hL_Jm) > 0$ such that for each $\epsilon > 0$ and for each solution $y \in Z$ of the inequality Eq. (5.1), there exists a mild solution $x \in C_L(J, X)$ of the Eq. (1.5) with

$$\|y(t) - x(t)\| \leqslant c(K_1 L_h L_J m)\epsilon, \quad t \in J. \tag{5.4}$$

Definition 5.2. The Eq. (1.5) is generalized Ulam-Hyers stable if there exists $\theta_{K_1,L_h,L_J,m} \in C(\mathbb{R}^+, \mathbb{R}^+), \theta(0) = 0$ such that for each $\epsilon > 0$ and for each solution $y \in Z$ of the inequality Eq. (5.1), there exists a mild solution $x \in C_L(J, X)$ of the Eq. (1.5) with

$$\|y(t) - x(t)\| \leqslant \theta_{K_1,L_h,L_J,m}\epsilon, \quad t \in J. \tag{5.5}$$

Definition 5.3. The Eq. (1.5) is Ulam-Hyers-Rassias stable with respect to (ϕ, ψ) if there exists $c(K_1 L_h L_J m\phi) > 0$ such that for each $\epsilon > 0$ and for each solution $y \in Z$ of the inequality Eq. (5.3), there exists a mild solution $x \in C_L(J, X)$ of the Eq. (1.5) with

$$\|y(t) - x(t)\| \leqslant c(K_1 L_h L_J m\phi)\epsilon(\psi + \phi(t)), \quad t \in J. \tag{5.6}$$

Definition 5.4. The Eq. (1.5) is generalized Ulam-Hyers-Rassias stable with respect to (ϕ, ψ) if there exists $c(K_1 L_h L_J m\phi) > 0$ such that for each solution $y \in Z$ of the inequality Eq. (5.2), there exists a mild solution $x \in C_L(J, X)$ of the Eq. (1.5) with

$$\|y(t) - x(t)\| \leqslant c(K_1 L_h L_J m\phi)(\psi + \phi(t)), \quad t \in J. \tag{5.7}$$

Remark 5.1. A function $y \in Z$ is a solution of the inequality Eq. (5.3) if and only if there is $G \in \cap_{i=0}^m C^2((s_i,\ t_{i+1}), X)\cap_{i=0}^m C((s_i, t_{i+1}), D(A)), g_1 \in \cap_{i=1}^m C([t_i, s_i], X)$ and $g_2 \in \cap_{i=1}^m C^1([t_i, s_i], X)$ such that:

(a) $\|G(t)\| \leqslant \epsilon\phi(t), t \in \cap_{i=0}^m (s_i, t_{i+1}), \|g_1(t)\| \leqslant \epsilon\psi$ and $\|g_2(t)\| \leqslant \epsilon\psi, t \in \cap_{i=0}^m [t_i, s_i]$;
(b) $y''(t) = Ay(t) + f(t, y(t), y[h(y(t), t)]) + G(t),\ t \in (s_i,\ t_{i+1})$ $i = 0, 1, \ldots, m$;
(c) $y(t) = J_i^1(t, y(t_i^-)) + g_1(t),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m.$
(d) $y'(t) = J_i^2(t, y(t_i^-)) + g_2(t),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m.$

Easily, we can have similar remarks for the inequalities Eqs. (5.1) and (5.2).

Remark 5.2. A function $y \in Z$ is a solution of the inequality Eq. (5.3) then y is a solution of the following integral inequality

$$\begin{cases} \|y(t) - J_i^1(t, y(t_i^-))\| \leqslant \epsilon\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ \|y'(t) - J_i^2(t, y(t_i^-))\| \leqslant \epsilon\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ \|y(t) - C(t)x_0 - S(t)y_0 - \int_0^t S(t-s)f(s, y(s), y[h(y(s), s)])ds\| \\ \qquad \leqslant \frac{\epsilon K}{\omega}\int_0^t [e^{\omega(t-s)} - 1]\phi(s)ds, \quad t \in [0, t_1]; \\ \|y(t) - C(t-s_i)(J_i^1(s_i, y(t_i^-))) - S(t-s_i)(J_i^2(s_i, y(t_i^-))) \\ \qquad - \int_{s_i}^t S(t-s)f(s, y(s), y[h(y(s), s)])ds\| \leqslant \epsilon\psi K e^{\omega(t-s_i)} \\ \qquad + \frac{\epsilon\psi K}{\omega}[e^{\omega(t-s_i)} - 1] + \frac{\epsilon K}{\omega}\int_{s_i}^t [e^{\omega(t-s)} - 1]\phi(s)ds, \\ \qquad t \in [s_i, t_{i+1}],\quad i = 1, 2, \ldots, m. \end{cases} \tag{5.8}$$

By Remark 5.1, we have

$$\begin{cases} y''(t) = Ay(t) + f(t, y(t), y[h(y(t), t)]) + G(t), \\ \qquad t \in (s_i, t_{i+1})\ i = 0, 1, \ldots, m; \\ y(t) = J_i^1(t, y(t_i^-)) + g_1(t),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ y'(t) = J_i^2(t, y(t_i^-)) + g_2(t), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m. \end{cases} \tag{5.9}$$

The solution $y \in Z$ with $y(0) = x_0$ and $y'(0) = y_0$ of the Eq. (5.9) is given by

$$\begin{cases} y(t) = J_i^1(t, y(t_i^-)) + g_1(t), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ y'(t) = J_i^2(t, y(t_i^-)) + g_2(t), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ y(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[f(s, y(s), \\ \qquad y[h(y(s), s)]) + G(s)]ds, \quad t \in [0, t_1]; \\ y(t) = C(t-s_i)((J_i^1(s_i, y(t_i^-))) + g_1(s_i)) \\ \qquad + S(t-s_i)((J_i^2(s_i, y(t_i^-))) + g_2(s_i)) \\ + \int_{s_i}^t S(t-s)[f(s, y(s), y[h(y(s), s)]) + G(s)]ds, \\ \qquad t \in [s_i, t_{i+1}],\quad i = 1, 2, \ldots, m. \end{cases} \tag{5.10}$$

Easily, we can have similar remarks for the solution of the inequalities Eqs. (5.1) and (5.2). In order to discuss the stability of the problem Eq. (1.5), we need the following additional assumption:

(A7) Let $\phi \in C(J, \mathbb{R}^+)$ be a nondecreasing function. There exists $c_\phi > 0$ such that

$$\int_0^t \phi(s)ds \leqslant c_\phi\phi(t),\ \forall\ t \in J.$$

Lemma 5.1 (*Impulsive Gronwall inequality*). *(see Theorem 16.4, Bainov and Simeonov, 1992). Let $\mathbb{M}_0 = \mathbb{M} \cup \{0\}$, where $\mathbb{M} = \{1, \ldots, m\}$ and the following inequality holds*

$$u(t) \leqslant a(t) + \int_0^t b(s)u(s)ds + \sum_{0<t_k<t}\beta_k u(t_k^-), \quad t \geqslant 0, \tag{5.11}$$

where $u, a, b \in PC(\mathbb{R}^+, \mathbb{R}^+)$, a is nondecreasing and $b(t) > 0,\ \beta_k > 0,\ k \in \mathbb{M}$. Then for $t \in \mathbb{R}^+$,

$$u(t) \leqslant a(t)(1+\beta)^k \exp\left(\int_0^t b(s)ds\right), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0, \tag{5.12}$$

where $\beta = \sup_{k\in\mathbb{M}}\{\beta_k\}$ and $t_0 = 0$.

Theorem 5.1. *Let $x_0 \in D(A)$, $y_0 \in E$. If the assumptions* (A1)–(A4) and (A7) *are satisfied. Then, the Eq. (1.5) is Ulam-Hyers-Rassias stable with respect to (ϕ, ψ).*

Proof. Let $y \in C_L(J, D(A)) \cap C^2((s_i, t_{i+1}], X)$ be a solution of the inequality Eq. (5.3) and x is the unique mild solution of the problem Eq. (1.5) which is given by

$$\begin{cases} x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \\ \qquad t \in [0, t_1]; \\ x(t) = C(t-s_i)(J_i^1(s_i, x(t_i^-))) + S(t-s_i)(J_i^2(s_i, x(t_i^-))) \\ + \int_{s_i}^t S(t-s)f(s, x(s), x[h(x(s), s)])ds, \quad t \in [s_i, t_{i+1}], \\ \qquad i = 1, 2, \ldots, m. \end{cases} \tag{5.13}$$

For $t \in [s_i, t_{i+1}],\ \ i = 1, 2, \ldots, m$. By inequality Eq. (5.8), we have

$$\begin{aligned} &\|y(t) - C(t-s_i)(J_i^1(s_i, y(t_i^-))) - S(t-s_i)(J_i^2(s_i, y(t_i^-))) \\ &\quad - \int_{s_i}^t S(t-s)f(s, y(s), y[h(y(s), s)])ds\| \\ &\leqslant \epsilon\psi K e^{\omega(t-s_i)} + \frac{\epsilon\psi K}{\omega}[e^{\omega(t-s_i)} - 1] + \frac{\epsilon K}{\omega}\int_{s_i}^t [e^{\omega(t-s)} - 1]\phi(s)ds \\ &\leqslant \epsilon\psi K e^{\omega T} + \frac{\epsilon\psi K}{\omega}e^{\omega T} + \frac{\epsilon K}{\omega}e^{\omega T}\int_0^t \phi(s)ds \\ &\leqslant \epsilon\psi K e^{\omega T} + \frac{\epsilon K}{\omega}e^{\omega T}(\psi + c_\phi\phi(t)). \end{aligned}$$

For $t \in (t_i, s_i],\ i = 1, 2, \ldots, m$, we have

$$\|y(t) - J_i^1(t, y(t_i^-))\| \leqslant \epsilon\psi.$$

For $t \in [0, t_1]$, we have

$$\begin{aligned} &\|y(t) - C(t)x_0 - S(t)y_0 - \int_0^t S(t-s)f(s, y(s), y[h(y(s), s)])ds\| \\ &\leqslant \frac{\epsilon K}{\omega}\int_0^t [e^{\omega(t-s)} - 1]\phi(s)ds \leqslant \frac{\epsilon K}{\omega}e^{\omega T}c_\phi\phi(t). \end{aligned}$$

Hence, for $t \in [s_i, t_{i+1}],\ \ i = 1, 2, \ldots, m$, we have

$$\begin{aligned} &\|y(t) - x(t)\| \\ &\quad\leqslant \|y(t) - C(t-s_i)(J_i^1(s_i, y(t_i^-))) - S(t-s_i)(J_i^2(s_i, y(t_i^-))) \\ &\qquad - \int_{s_i}^t S(t-s)f(s, y(s), y[h(y(s), s)])ds\| + Ke^{\omega T}\|J_i^1(s_i, y(t_i^-)) \\ &\qquad - J_i^1(s_i, x(t_i^-))\| + KTe^{\omega T}\|J_i^2(s_i, y(t_i^-)) - J_i^2(s_i, x(t_i^-))\| \\ &\qquad + KTe^{\omega T}\int_{s_i}^t \|f(s, y(s), y[h(y(s), s)]) - f(s, x(s), \\ &\qquad x[h(x(s), s)])ds\| \leqslant \epsilon\psi K e^{\omega T} + \frac{\epsilon K}{\omega}e^{\omega T}(\psi + c_\phi\phi(t)) \\ &\qquad + (Ke^{\omega T}L_{J_i^1} + KTe^{\omega T}L_{J_i^2})\|(y(t_i^-) - (x(t_i^-)\| \\ &\qquad + K_1(2 + LL_h)KTe^{\omega T}\int_0^t \|y(s) - x(s)\|ds \\ &\quad\leqslant \frac{\epsilon K}{\omega}e^{\omega T}[(2 + c_\phi)(\psi + \phi(t))] \\ &\qquad + K_1(2 + LL_h)KTe^{\omega T}\int_0^t \|y(s) - x(s)\|ds \\ &\qquad + \sum_{j=1}^i (Ke^{\omega T}L_{J_j^1} + KTe^{\omega T}L_{J_j^2})\|(y(t_j^-) - (x(t_j^-)\|. \end{aligned} \tag{5.14}$$

For $t \in (t_i, s_i],\ i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant \|y(t) - J_i^1(t, y(t_i^-))\| + \|J_i^1(t, y(t_i^-)) - J_i^1(t, x(t_i^-))\| \\ &\leqslant \epsilon\psi + \sum_{j=1}^i L_{J_j^1}\|(y(t_j^-) - (x(t_j^-)\| \\ &\leqslant \frac{\epsilon K}{\omega}e^{\omega T}[(2 + c_\phi)(\psi + \phi(t))] \\ &\quad + K_1(2 + LL_h)KTe^{\omega T}\int_0^t \|y(s) - x(s)\|ds \\ &\quad + \sum_{j=1}^i (Ke^{\omega T}L_{J_j^1} + KTe^{\omega T}L_{J_j^2})\|(y(t_j^-) - (x(t_j^-)\|. \end{aligned} \tag{5.15}$$

Now, for $t \in [0, t_1]$, we have

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant \frac{\epsilon K}{\omega}e^{\omega T}c_\phi\phi(t) + K_1(2 + LL_h)KTe^{\omega T} \\ &\quad \times \int_0^t \|y(s) - x(s)\|ds \\ &\leqslant \frac{\epsilon K}{\omega}e^{\omega T}[(2 + c_\phi)(\psi + \phi(t))] + K_1(2 + LL_h)KTe^{\omega T} \\ &\quad \times \int_0^t \|y(s) - x(s)\|ds \\ &\quad + \sum_{j=1}^i (Ke^{\omega T}L_{J_j^1} + KTe^{\omega T}L_{J_j^2})\|(y(t_j^-) - (x(t_j^-)\|. \end{aligned} \tag{5.16}$$

We observe that inequalities Eqs. (5.14)–(5.16) give together an impulsive Gronwall inequality of a form of Eq. (5.11) on J. Therefore, we can apply impulsive Gronwall inequality Eq. (5.12) for $t \in J$, since $t \in (t_i, t_{i+1}]$ for some $i \in \mathbb{M}_0$.

Consequently, we have

$$\begin{aligned} \|y(t) - x(t)\| &\leqslant \frac{K}{\omega}e^{\omega T}(2 + c_\phi)(1 + Ke^{\omega T}L_J)^i e^{K_1(2+LL_h)KTe^{\omega T}t} \\ &\quad \times \epsilon(\psi + \phi(t)) \\ &\leqslant \frac{K}{\omega}e^{\omega T}(2 + c_\phi)(1 + Ke^{\omega T}L_J)^m e^{K_1(2+LL_h)KTe^{\omega T}T} \\ &\quad \times \epsilon(\psi + \phi(t)) \\ &\leqslant c(K_1L_hL_Jm\phi)\epsilon(\psi + \phi(t)), \end{aligned}$$

for any $t \in J$, where $L_J = \sup_{i\in\mathbb{M}}\{L_{J_i^1} + TL_{J_i^2}\}$ and $c(K_1L_hL_Jm\phi)$ is a constant depending on K_1, L_h, L_J, m, ϕ. Hence, the Eq. (1.5) is Ulam-Hyers-Rassias stable with respect to (ϕ, ψ). □

Theorem 5.2. *If the assumptions (A1)- (A4) and (A7) are satisfied. Then, the Eq. (1.5) is generalized Ulam-Hyers-Rassias stable with respect to* (ϕ, ψ).

Proof. It can be easily proved by applying same procedure of Theorem 5.1 and taking inequality Eq. (5.2).

Theorem 5.3. *If the assumptions (A1)–(A4) and (A7) are satisfied. Then, the Eq. (1.5) is Ulam-Hyers stable.*

Proof. It can be easily proved by applying same procedure of Theorem 5.1 and taking inequality Eq. (5.1).

6. Application

Example 6.1. Let $X = L^2(0,\pi)$. We consider the following partial differential equations with deviated argument

$$\begin{cases} \partial_{tt}Z(t,y) = \partial_{yy}Z(t,y) + f_2(y, Z(h(t),y)) + f_3(t,y,Z(t,y)), \\ \quad y \in (0,\pi), \quad t \in (2i, 2i+1], \quad i \in \{0\} \cup \mathbb{N}, \\ Z(t,0) = Z(t,\pi) = 0, \quad t \in [0,T],\ 0 < T < \infty, \\ Z(0,y) = x_0, \quad y \in (0,\pi), \\ \partial_t Z(0,y) = y_0, \quad y \in (0,\pi), \\ Z(t)(y) = (\sin it) Z((2i-1)^-, y), \quad y \in (0,\pi), \\ \quad t \in (2i-1, 2i], \quad i \in \mathbb{N}, \\ \partial_t Z(t)(y) = (i \cos it) Z((2i-1)^-, y), \quad y \in (0,\pi), \\ \quad t \in (2i-1, 2i], \quad i \in \mathbb{N}, \end{cases} \tag{6.1}$$

where $0 = s_0 < t_1 < s_1 < t_2, \ldots, t_m < s_m < t_{m+1} = T < \infty$ with $t_i = 2i-1, s_i = 2i$ and

$$f_3(t,y,Z(t,y)) = \int_0^y \overline{K}(y,s)(a_1|Z(t,s)| + b_1|Z(t,s)|)ds.$$

We assume that $a_1, b_1 \geqslant 0, (a_1,b_1) \neq (0,0), h : J_1 \to [0,T]$ is locally Hölder continuous in t with $h(0) = 0$ and $\overline{K} : [0,\pi] \times [0,\pi] \to \mathbb{R}$.

We define an operator A, as follows,

$$Ax = x'' \quad \text{with} \quad D(A) = \{x \in X :\ x'' \in X \text{ and } x(0) = x(\pi) = 0\}. \tag{6.2}$$

Here, clearly the operator A is the infinitesimal generator of a strongly continuous cosine family of operators on X. A has infinite series representation

$$Ax = \sum_{n=1}^{\infty} -n^2(x,x_n)x_n, \quad x \in D(A),$$

where $x_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1,2,3\ldots$is the orthonormal set of eigenfunctions of A. Moreover, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)_{t\in\mathbb{R}}$ on X which is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos nt (x,x_n)x_n, \quad x \in X,$$

and the associated sine family $S(t)_{t\in\mathbb{R}}$ on X which is given by

$$S(t)x = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt (x,x_n)x_n, \quad x \in X.$$

The Eq. (6.1) can be reformulated as the following abstract differential equation in X:

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x[h(x(t),t)]), \quad t \in (s_i, t_{i+1}), \\ \quad i \in \{0\} \cup \mathbb{N}, \\ x(t) = J_i^1(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}, \\ x'(t) = J_i^2(t, x(t_i^-)), \quad t \in (t_i, s_i], \quad i \in \mathbb{N}, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{6.3}$$

where $x(t) = Z(t,.)$, that is $x(t)(y) = Z(t,y), y \in (0,\pi)$. Functions $J_i^1(t, x(t_i^-)) = (\sin it)Z((2i-1)^-, y)$ and $J_i^2(t, x(t_i^-)) = i(\cos it)Z((2i-1)^-, y)$ represent noninstantaneous impulses during intervals $(t_i, s_i]$. The operator A is same as in Eq. (6.2).

The function $f : J_1 \times X \times X \to X$, is given by

$$f(t,\psi,\xi)(y) = f_2(y,\xi) + f_3(t,y,\psi),$$

where $f_2 : [0,\pi] \times X \to H_0^1(0,\pi)$ is given by

$$f_2(y,\xi) = \int_0^y \overline{K}(y,x)\xi(x)dx,$$

and

$$\|f_3(t,y,\psi)\| \leqslant V(y,t)(1 + \|\psi\|_{H^2(0,\pi)})$$

with $V(.,t) \in X$ and V is continuous in its second argument. For more details see (Sakthivel et al., 2009; Gal, 2007). Thus, Theorem 3.1 can be applied to the problem Eq. (6.1). We can choose the functions $p(x)$ and $q(x)$ as given below

$$p(x) = \sum_k^n \alpha_k\, x(t_k), \quad t_k \in J \text{ for all } k = 1,2,3,\cdots,n,$$

$$q(x) = \sum_k^n \beta_k\, x(t_k), \quad t_k \in J \text{ for all } k = 1,2,3,\cdots,n,$$

where α_k and β_k are constants.

Example 6.2. We consider particular linear case of the abstract differential Eq. (6.3) in the space $X = \mathbb{R}$. A forced string equation

$$\begin{cases} x''(t) + a_1 x(t) + a_2 \sin x(c_1 t) = g(t), \quad t \in (s_i, t_{i+1}) \\ \quad i = 0,1,\ldots,m, \\ x(t) = a_3 \tanh(x(t_i^-)) r(t), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ x'(t) = a_3 \tanh(x(t_i^-)) r'(t), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{6.4}$$

where $a_1 \in \mathbb{R}^+, a_2, a_3 \in \mathbb{R}, c_1 \in (0,1], g \in C(J_1,\mathbb{R})$ and $r \in C^1(J_2,\mathbb{R})$ for $J_2 = \cup_{i=1}^m I_i$. We define A, as follows

$$Ax = -a_1 x \quad \text{with} \quad D(A) = \mathbb{R}.$$

Here, clearly the value $-a_1$ behaves like infinitesimal generator of a strongly continuous cosine family $C(t) = \cos\sqrt{a_1}t$. The associated sine family is given by $S(t) = \frac{1}{\sqrt{a_1}}\sin\sqrt{a_1}t$. Deviated argument in the abstract differential Eq. (6.3) is represented by the term $a_2 x(c_1 t)$ of the differential Eq. (6.4). Noninstantaneous impulses $a_3\tanh(x(t_i^-))r(t)$ and $a_3\tanh(x(t_i^-))r'(t)$ are created when bob of the string is extremely pushed on each interval $(t_i, s_i]$.

Example 6.3. We generalize the above example to consider a coupled system of strings or pendulums

$$\begin{aligned} & x_n''(t) + a_{n1}\sin x_n(t) + a_{n2}\sin x_n(c_n t) = b_{n1}x_{n-1}(t) + b_{n2}x_{n+1}(t) + g_n(t), \\ & \quad t \in (s_i, t_{i+1}), i = 0,1,\ldots,m,\ n \in \mathbb{Z}, \\ & x_n(t) = a_{n3}\tanh(x_n(t_i^-))r_n(t), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ & x_n'(t) = a_{n3}\tanh(x_n(t_i^-))r_n'(t), \quad t \in (t_i, s_i], \quad i = 1,2,\ldots,m, \\ & x_n(0) = x_{n0}, \quad x_n'(0) = y_{n0}, \end{aligned} \tag{6.5}$$

where $a_{n1}, a_{n2}, a_{n3}, b_{n1}, b_{n2} \in \mathbb{R}$, $c_n \in (0,1], g_n \in C(J_1, \mathbb{R})$ and $r_n \in C^1(J_2, \mathbb{R})$. Moreover we suppose $\sup_n |a_{nk}| < \infty, k = 1,2,3$, $\sup_n(|b_{n1}| + |b_{n2}|) < \infty$ and $\sup_n(\|g_n\| + \|r_n\| + \|r_n{}'\|) < \infty$. Then we consider Eq. (6.5) on ℓ_∞ and use Exercise 1 on p. 39 from Fattorini, 1985. The lattice ODE Eq. (6.5) is a generalization of the discrete sine-Gordon equation Scott, 2003 and $x_n(c_n t)$ represents pantograph-like terms Derfel and Iserles, 1997.

7. Conclusion

The research presented in this paper focuses on the existence, uniqueness and stability of solutions to the impulsive systems represented by second order nonlinear differential equations with noninstantaneous impulses and deviated argument. We used strongly continuous cosine family of bounded linear operators and Banach's fixed point theorem to get the existence and uniqueness of the solutions. Moreover, Ulam's type stability is established using impulsive Gronwall inequality.

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