Journal of King Saud University – Science 32 (2020) 3018–3025

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

On Jakimovski-Leviatan-Păltănea approximating operators involving Boas-Buck-type polynomials

Khursheed J. Ansari [a], M.A. Salman [b], M. Mursaleen [c,d,*], A.H.H. Al-Abied [e]

[a] Department of Mathematics, College of Science, King Khalid University, 61413 Abha, Saudi Arabia
[b] Math & Sciences Department, Community College of Qatar P.O. Box 7344, Doha, Qatar
[c] Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
[d] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[e] Department of Mathematics, Dhamar University, Dhamar, Yemen

ARTICLE INFO

Article history:
Received 3 May 2020
Revised 7 August 2020
Accepted 9 August 2020
Available online 18 August 2020

Keywords:
Szász operators
Appell polynomials
Phillips operators
Modulus of continuity
Korovkin's theorem
Boas-Buck-type polynomials

ABSTRACT

A sequence of approximating operators is constructed in the present article with the help of Boas-Buck-type polynomials (BB-polynomials). We called this constructed operator as Jakimovski-Leviatan-Păltănea operators (JLP-operators) involving BB-polynomials. We establish some approximation properties of approximating operators converging towards the function to be approximated. We investigate versatile Korovkin-type property and also demonstrate the rate of convergence. Moreover, some approximation results are given in the weighted spaces. Furthermore, a Voronoskaja type theorem is also proved as well as approximation result when functions belong to the Lipschitzian class.

1. Introduction and preliminaries

In the theory of approximation, our main task is to provide the arithmetic representation of non-arithmetic quantities or functions which are difficult to handle to simple functions. Korovkin (Korovkin, 1953) found out the simplest criterion for positive approximation processes at the beginning of the second half of the last century. This concept has affected to a great extent not only traditional approximation theory but also diverse section of mathematics, e.g. orthogonal polynomials, several types of differential equations, in particular partial differential equations, wavelet and harmonic analysis etc. Szász operator (Szász, YYYY) was modified by Mazhar and Totik (1985) as

$$\mathsf{F}_m(f;u) = me^{-mu}\sum_{i=0}^{\infty}\frac{(mu)^i}{i!}\int_0^{\infty}e^{-mv}\frac{(mv)^i}{i!}f(v)dv, \tag{1}$$

for the exponential type function f. A new type of operators with the help of Appell polynomials were constructed by Büyükyazıcı et al. (2014) as follows:

$$J_m(f;u) = \frac{e^{-mu}}{g(1)}\sum_{i=0}^{\infty}p_i(mu)f\left(\frac{i}{m}\right), \tag{2}$$

where Appell polynomials are denoted by p_k in the above equation and generating functions for this are outlined by

$$g(u)e^{u\bar{x}} = \sum_{k=0}^{\infty}p_k(\bar{x})u^k, \tag{3}$$

where $g(\bar{y}) = \sum_{k=0}^{\infty}a_k\bar{y}^k (a_0 \neq 0)$ is an analytic function in the disk $|z| < \widetilde{R}$, provided $\widetilde{R} > 1$ and suppose that $g(1) \neq 0$.

Some remarkable results analogous to Szász (YYYY) were obtained by them and if we take $g(v) = 1$, by using above generating functions, we get Szász operators (Szász, YYYY). Based on a parameter $\bar{\rho} > 0$, Păltănea (2008) generalized the Phillips operators (Phillips, 1954) which provides the connection with Szász

* Corresponding author at: Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan.
E-mail addresses: ansari.jkhursheed@gmail.com (K.J. Ansari), mohammed.salman@ccq.edu.qa (M.A. Salman), mursaleenm@gmail.com (M. Mursaleen), abeid1979@gmail.com (A.H.H. Al-Abied).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.jksus.2020.08.007
1018-3647/

operators as $\bar{\rho} \to \infty$. Verma and Gupta (2015) modified the operator given in Eq. (2) as follows:

$$J^{*}_{m,\bar{\rho}}(f;u) = \sum_{i=1}^{\infty} L_{m,i}(u) \int_0^{\infty} Q^{\bar{\rho}}_{m,i}(v) f(v) dv + L_{m,0}(u) f(0), \quad (4)$$

where $L_{m,i}(u) = \frac{e^{-mu}}{g(1)} p_i(mu)$ and $Q^{\bar{\rho}}_{m,i}(v) = \frac{m\bar{\rho}}{\Gamma(i\bar{\rho})} e^{-m\bar{\rho}v} (m\bar{\rho}v)^{i\bar{\rho}-1}$. The approximating operators (4) reduces to the Phillips operators if we take $g(z) = 1$ and $\bar{\rho} = 1$.

Ismail (1974) generalized the well-known Szász operators. Ansari et al. (2019), Mursaleen et al. (2019), Mursaleen et al. (2018), Mursaleen et al. (2019) also introduced different generalizations of Szász operators with the concept of Durrmeyer, Pălta nea and Sheffer operators and sequences. For more literature on such type generalization of operators and its approximation properties, one is suggested to refer Alotaibi and Mursaleen (2020), Ansari et al. (2018), Ansari et al. (2019), Kilicman et al. (2020), Mohiuddine et al. (2017), Mursaleen et al. (2019), Verma and Gupta (2015).

Recently, Sucu et al. (2012) constructed linear positive operators with the assistance of BB-polynomials. BB-polynomials (Ismail, 2005) have generating functions of the form

$$\varrho(u)\psi(\bar{x}\varsigma(u)) = \sum_{j=0}^{\infty} p_j(\bar{x}) u^j, \quad (5)$$

where ϱ, ψ and ς are analytic functions such as

$$\varsigma(u) = \sum_{i=1}^{\infty} h_i u^i, \; h_1 \neq 0 (i \geqslant 0), \quad (6)$$

and have the explicit relation as follows:

$$p_j(\bar{x}) = \sum_{i=0}^{j} a_{j-i} b_i \bar{x}^i, \; j = 0, 1, 2, \cdots. \quad (7)$$

Circumscribe to the BB-polynomials satisfying:

(i) $\psi : \mathbb{R} \to (0, \infty)$,
(ii) $\varrho(1) \neq 0, \varsigma'(1) = 1, \; p_j(\bar{x}) \geqslant 0, \;\; j = 0, 1, 2, \ldots$,
(iii) The power series (1.5)–(1.8) converges for $|u| < \tilde{R}$ provided $\tilde{R} > 1$.

The following sequence of positive linear operators involving the BB-polynomials was introduced by Sucu et al. (2012)

$$B_n(f;\bar{x}) := \frac{1}{\varrho(1)\psi(n\bar{x}\varsigma(1))} \sum_{j=0}^{\infty} p_j(n\bar{x}) f\left(\frac{j}{n}\right) \quad (8)$$

where $\bar{x} \geqslant 0$ and $n \in \mathbb{N}$.

Lemma 1. *From (5), we obtain*

(i) $\sum_{j=0}^{\infty} p_j\left(n\bar{x}\right) = \varrho(1)\psi\left(n\bar{x}\varsigma(1)\right);$

(ii) $\sum_{j=0}^{\infty} j p_j\left(n\bar{x}\right) = n\bar{x}\varrho(1)\psi'\left(n\bar{x}\varsigma(1)\right) + \varrho'(1)\psi\left(n\bar{x}\varsigma(1)\right);$

(iii) $\sum_{j=0}^{\infty} j^2 p_j\left(n\bar{x}\right) = \left(n\bar{x}\right)^2 \varrho(1)\psi''\left(n\bar{x}\varsigma(1)\right) + n\bar{x}[\varrho(1) + 2\varrho'(1) + \varrho(1)\varsigma''(1)]\psi'\left(n\bar{x}\varsigma(1)\right) + [\varrho''(1) + \varrho'(1)]\psi\left(n\bar{x}\varsigma(1)\right);$

(iv) $\sum_{j=0}^{\infty} j^3 p_j\left(n\bar{x}\right) = \left(n\bar{x}\right)^3 \varrho(1)\psi'''\left(n\bar{x}\varsigma(1)\right) + 3\left(n\bar{x}\right)^2[\varrho(1) + \varrho'(1) + \varrho(1)\varsigma''(1)]\psi''\left(n\bar{x}\varsigma(1)\right) + n\bar{x}[\varrho(1) + 3\varrho(1)\varsigma''(1) + \varrho(1)\varsigma'''(1) + 3\varrho'(1)\varsigma''(1) + 6\varrho'(1) + 3\varrho''(1)]\psi'\left(n\bar{x}\varsigma(1)\right) + [\varrho'(1) + 3\varrho''(1) + \varrho'''(1)]\psi\left(n\bar{x}\varsigma(1)\right);$

(v) $\sum_{j=0}^{\infty} j^4 p_j\left(n\bar{x}\right) = \left(n\bar{x}\right)^4 \varrho(1)\psi^{(iv)}\left(n\bar{x}\varsigma(1)\right) + [6\varrho(1) + 5\varrho(1)\varsigma''(1) + 4\varrho'(1)]\left(n\bar{x}\right)^3\psi'''\left(n\bar{x}\varsigma(1)\right) + \left[7\varrho(1) + 3\varrho(1)\varsigma''(1)^2 + 18\varrho(1)\varsigma''(1) + 4\varrho(1)\varsigma'''(1) + 18\varrho'(1) + 12\varrho'(1)\varsigma''(1) + 6\varrho''(1)\right] \times \left(n\bar{x}\right)^2\psi''\left(n\bar{x}\varsigma(1)\right) + [\varrho(1) + 7\varrho(1)\varsigma''(1) + 6\varrho(1)\varsigma'''(1) + \varrho(1)\varsigma^{(iv)}(1) + 14\varrho'(1) + 18\varrho'(1)\varsigma''(1) + 4\varrho'(1)\varsigma'''(1) + 18\varrho''(1) + 6\varrho''(1)\varsigma''(1) + 4\varrho'''(1)]\left(n\bar{x}\right)\psi'\left(n\bar{x}\varsigma(1)\right) + [\varrho'(1) + 7\varrho''(1) + 6\varrho'''(1) + \varrho^{(iv)}(1)]\psi\left(n\bar{x}\varsigma(1)\right).$

Proof. One can find the proof of (i)–(iii) in Sucu et al. (2012). Here we will provide the proof of (iv) and (v).

(iv) Differentiating the generating function (5) thrice with respect to u, we get

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j(j-1)(j-2)p_j(n\bar{x})u^{j-3} \\
&= (n\bar{x})^2\left\{n\bar{x}\varrho(u)(\varsigma'(u))^3\psi'''(n\bar{x}\varsigma(u)) + \left(\varrho'(u)(\varsigma'(u))^2 + 2\varrho(u)\varsigma'(u)\varsigma''(u)\right)\psi''(n\bar{x}\varsigma(u))\right\} \\
&\quad + n\bar{x}\Big\{n\bar{x}\varrho(u)\varsigma'(u)\varsigma''(u)\psi''(n\bar{x}\varsigma(u)) + n\bar{x}\varrho'(u)(\varsigma'(u))^2\psi''(n\bar{x}\varsigma(u)) \\
&\quad + (2\varrho'(u)\varsigma''(u) + \varrho(u)\varsigma'''(u) + \varrho''(u)\varsigma'(u))\psi'(n\bar{x}\varsigma(u)) \\
&\quad + n\bar{x}\varrho'(u)(\varsigma'(u))^2\psi''(n\bar{x}\varsigma(u)) + (\varrho''(u)\varsigma'(u) + \varrho'(u)\varsigma''(u))\psi'(n\bar{x}\varsigma(u)) \\
&\quad + \varrho'''(u)\psi(n\bar{x}\varsigma(u)) + n\bar{x}\varrho''(u)\varsigma'(u)\psi'(n\bar{x}\varsigma(u))\Big\}.
\end{aligned} \tag{9}
$$

Put $u = 1$ in the above equation and then using $\varsigma'(1) = 1$, Lemma 1 (ii)–(iii), finally we get

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j^3 p_j(n\bar{x}) \\
&= (n\bar{x})^3\varrho(1)\psi'''(n\bar{x}\varsigma(1)) + (n\bar{x})^2\{\varrho'(1) + 2\varrho(1)\varsigma''(1) + \varrho(1)\varsigma''(1) + 2\varrho'(1)\}\psi''(n\bar{x}\varsigma(1)) \\
&\quad + n\bar{x}\{3\varrho'(1)\varsigma''(1) + 3\varrho''(1) + \varrho(1)\varsigma'''(1)\}\psi'(n\bar{x}\varsigma(1)) + \varrho'''(1)\psi(n\bar{x}\varsigma(1)) \\
&\quad + 3\left\{(n\bar{x})^2\varrho(1)\psi''(n\bar{x}\varsigma(1)) + n\bar{x}(\varrho(1) + 2\varrho'(1) + \varrho(1)\varsigma''(1))\psi'(n\bar{x}\varsigma(1))\right. \\
&\quad \left. + (\varrho''(1) + \varrho'(1))\psi(n\bar{x}\varsigma(1))\right\} - 2\{n\bar{x}\varrho(1)\psi'(n\bar{x}\varsigma(1)) + \varrho'(1)\psi(n\bar{x}\varsigma(1))\} \\
&= (n\bar{x})^3\varrho(1)\psi'''(n\bar{x}\varsigma(1)) + 3\{\varrho(1) + \varrho'(1) + \varrho(1)\varsigma''(1)\}(n\bar{x})^2\psi''(n\bar{x}\varsigma(1)) \\
&\quad + \{\varrho(1) + 3\varrho(1)\varsigma''(1) + \varrho(1)\varsigma'''(1) + 3\varrho'(1)\varsigma''(1) + 6\varrho'(1) + 3\varrho''(1)\}n\bar{x}\psi'(n\bar{x}\varsigma(1)) \\
&\quad + \{\varrho'(1) + 3\varrho''(1) + \varrho'''(1)\}\psi(n\bar{x}\varsigma(1))
\end{aligned}
$$

(v) Now writing a simplified form of Eq.(9), we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j(j-1)(j-2)p_j(n\bar{x})u^{j-3} \\
&= (n\bar{x})^3\varrho(u)(\varsigma'(u))^3\psi'''(n\bar{x}\varsigma(u)) \\
&\quad + (n\bar{x})^2\left\{3\varrho'(u)(\varsigma'(u))^2 + 3\varrho(u)\varsigma'(u)\varsigma''(u)\right\}\psi''(n\bar{x}\varsigma(u)) \\
&\quad + n\bar{x}\{3\varrho'(u)\varsigma''(u) + 3\varrho''(u)\varsigma'(u) + \varrho(u)\varsigma'''(u)\}\psi'(n\bar{x}\varsigma(u)) + \varrho'''(u)\psi(n\bar{x}\varsigma(u)).
\end{aligned}
$$

Differentiating the above equation with respect to u, and then using $\varsigma'(1) = 1$, we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j(j-1)(j-2)(j-3)p_j(n\bar{x}) \\
&= (n\bar{x})^4\varrho(1)\psi^{(iv)}(n\bar{x}\varsigma(1)) + (n\bar{x})^3\{6\varrho(1)\varsigma''(1) + 4\varrho'(1)\}\psi'''(n\bar{x}\varsigma(1)) \\
&\quad + (n\bar{x})^2\left\{12\varrho'(1)\varsigma''(1) + 6\varrho''(1) + 4\varrho(1)\varsigma'''(1) + 3\varrho(1)(\varsigma''(1))^2\right\}\psi''(n\bar{x}\varsigma(1)) \\
&\quad + n\bar{x}\{4\varrho'''(1) + 4\varrho'(1)\varsigma'''(1) + 6\varrho''(1)\varsigma''(1) + \varrho(1)\varsigma^{(iv)}(1)\}\psi'(n\bar{x}\varsigma(1)) + \varrho^{(iv)}(1)\psi(n\bar{x}\varsigma(1)).
\end{aligned}
$$

Now using Lemma 1 (ii)–(iv), finally we have

$$
\begin{aligned}
&\sum_{j=0}^{\infty} j^4 p_j(n\bar{x}) \\
&= (n\bar{x})^4\varrho(1)\psi^{(iv)}(n\bar{x}\varsigma(1)) + (n\bar{x})^3\{6\varrho(1)\varsigma''(1) + 4\varrho'(1)\}\psi'''(n\bar{x}\varsigma(1)) \\
&\quad + (n\bar{x})^2\left\{12\varrho'(1)\varsigma''(1) + 6\varrho''(1) + 4\varrho(1)\varsigma'''(1) + 3\varrho(1)(\varsigma''(1))^2\right\}\psi''(n\bar{x}\varsigma(1)) \\
&\quad + n\bar{x}\{4\varrho'''(1) + 4\varrho'(1)\varsigma'''(1) + 6\varrho''(1)\varsigma''(1) + \varrho(1)\varsigma^{(iv)}(1)\}\psi'(n\bar{x}\varsigma(1)) + \varrho^{(iv)}(1)\psi(n\bar{x}\varsigma(1)) \\
&\quad + 6\Big[(n\bar{x})^3\varrho(1)\psi'''(n\bar{x}\varsigma(1)) + 3(n\bar{x})^2\{\varrho(1) + \varrho'(1) + \varrho(1)\varsigma''(1)\}\psi''(n\bar{x}\varsigma(1)) \\
&\quad + n\bar{x}\{\varrho(1) + 3\varrho(1)\varsigma''(1) + \varrho(1)\varsigma'''(1) + 3\varrho'(1)\varsigma''(1) + 6\varrho'(1) + 3\varrho''(1)\}\psi'(n\bar{x}\varsigma(1)) \\
&\quad + \{\varrho'(1) + 3\varrho''(1) + \varrho'''(1)\}\psi(n\bar{x}\varsigma(1))\Big] \\
&\quad - 11\Big[(n\bar{x})^2\varrho(1)\psi''(n\bar{x}\varsigma(1)) + n\bar{x}\{\varrho(1) + 2\varrho'(1) + \varrho(1)\varsigma''(1)\}\psi'(n\bar{x}\varsigma(1)) \\
&\quad + \{\varrho''(1) + \varrho'(1)\}\psi(n\bar{x}\varsigma(1))\Big] \\
&\quad + 6[n\bar{x}\varrho(1)\psi'(n\bar{x}\varsigma(1)) + \varrho'(1)\psi(n\bar{x}\varsigma(1))] \\
&= (n\bar{x})^4\varrho(1)\psi^{(iv)}(n\bar{x}\varsigma(1)) + [6\varrho(1) + 5\varrho(1)\varsigma''(1) + 4\varrho'(1)](n\bar{x})^3\psi'''(n\bar{x}\varsigma(1)) \\
&\quad + \left[7\varrho(1) + 3\varrho(1)\varsigma''(1)^2 + 18\varrho(1)\varsigma''(1) + 4\varrho(1)\varsigma'''(1) + 18\varrho'(1) + 12\varrho'(1)\varsigma''(1) + 6\varrho''(1)\right] \\
&\quad \times (n\bar{x})^2\psi''(n\bar{x}\varsigma(1)) + [\varrho(1) + 7\varrho(1)\varsigma''(1) + 6\varrho(1)\varsigma'''(1) + \varrho(1)\varsigma^{(iv)}(1) + 14\varrho'(1) \\
&\quad + 18\varrho'(1)\varsigma''(1) + 4\varrho'(1)\varsigma'''(1) + 18\varrho''(1) + 6\varrho''(1)\varsigma''(1) + 4\varrho'''(1)](n\bar{x})\psi'(n\bar{x}\varsigma(1)) \\
&\quad + [\varrho'(1) + 7\varrho''(1) + 6\varrho'''(1) + \varrho^{(iv)}(1)]\psi(n\bar{x}\varsigma(1)).
\end{aligned}
$$

2. Construction of operators and auxiliary results

Considering the revised form of Sucu et al. (2012) positive linear operators involving the BB-polynomials, we construct the JLP-operators including BB-polynomials as

$$
B^*_{n,\bar{\rho}}(f;\bar{x}) = \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,j}(t)f(t)dt + L_{n,0}(\bar{x})f(0), \tag{10}
$$

where $L_{n,j}(\bar{x}) = \frac{p_j(n\bar{x})}{\varrho(1)\psi(n\bar{x}\varsigma(1))}$ and $Q^{\bar{\rho}}_{n,j}(t) = \frac{n\bar{\rho}}{\Gamma(j\bar{\rho})}e^{-n\bar{\rho}t}(n\bar{\rho}t)^{j\bar{\rho}-1}$.

Remark 1. Let M be the space of polynomials. For $g \in M$, we have

$$
\lim_{\bar{\rho}\to\infty} B^*_{n,\bar{\rho}}(g;\bar{x}) = B_n(g;\bar{x}); \tag{11}
$$

for all $\bar{x} \in [0,\infty)$.

For $r \in \mathbb{N}^0$, we have

$$
\begin{aligned}
\int_0^{\infty} Q^{\bar{\rho}}_{n,j}(t)t^r dt = &\int_0^{\infty} \frac{n\bar{\rho}}{\Gamma(j\bar{\rho})}e^{-n\bar{\rho}t}(n\bar{\rho}t)^{j\bar{\rho}-1}t^r dt \\
&= \frac{\Gamma(j\bar{\rho}+r)}{(n\bar{\rho})^r\Gamma(j\bar{\rho})},
\end{aligned}
$$

where

$$
\lim_{\bar{\rho}\to\infty} \frac{\Gamma(j\bar{\rho}+r)}{(n\bar{\rho})^r\Gamma(j\bar{\rho})} = \left(\frac{j}{n}\right)^r. \tag{12}
$$

Here, we will give some auxiliary definitions as well as necessary lemmas followed by our main result. We will assume throughout the paper that the sequence of operators $B^*_{n,\bar{\rho}}$ are positive and also we consider

$$
\lim_{z\to\infty} \frac{\psi^{(k)}(z)}{\psi(z)} = 1 \text{ for } k \in \{1,2,3,\ldots,r\}. \tag{13}
$$

$\psi(z) = e^z$ is such an example satisfying relation (13).

Lemma 2. *$B^*_{n,\bar{\rho}}$ satisfy the following equalities*

$$
\begin{aligned}
(i)\quad & B^*_{n,\bar{\rho}}(1;\bar{x}) = 1; \\
(ii)\quad & B^*_{n,\bar{\rho}}(t;\bar{x}) = \tfrac{\psi'(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x} + \tfrac{\varrho'(1)}{n\varrho(1)}; \\
(iii)\quad & B^*_{n,\bar{\rho}}(t^2;\bar{x}) = \tfrac{\psi''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^2 + \left(\tfrac{2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1)}{\varrho(1)} + \tfrac{1}{\bar{\rho}}\right)\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n\psi(n\bar{x}\varsigma(1))} \\
& \quad + \tfrac{(1+\bar{\rho})\varrho'(1)+\bar{\rho}\varrho''(1)}{n^2\bar{\rho}\varrho(1)}; \\
(iv)\quad & B^*_{n,\bar{\rho}}(t^3;\bar{x}) = \tfrac{\psi'''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^3 + 3[\varrho(1) + \bar{\rho}(\varrho(1) + \varrho(1)\varsigma''(1) + \varrho'(1))]\tfrac{\bar{x}^2\psi''(n\bar{x}\varsigma(1))}{n\bar{\rho}\varrho(1)\psi(n\bar{x}\varsigma(1))} \\
& \quad + ([\varrho(1) + 3\varrho(1)\varsigma''(1) + \varrho(1)\varsigma'''(1) + 6\varrho'(1) + 3\varrho'(1)\varsigma''(1) + 3\varrho''(1)]\bar{\rho}^2 \\
& \quad + 3[\varrho(1) + \varrho(1)\varsigma''(1) + 2\varrho'(1)]\bar{\rho} + 2\varrho(1))\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n^2\bar{\rho}^2\varrho(1)\psi(n\bar{x}\varsigma(1))} \\
& \quad + ([\varrho'(1) + 3\varrho''(1) + \varrho'''(1)]\bar{\rho}^2 + 3[\varrho'(1) + \varrho''(1)]\bar{\rho} + 2\varrho'(1))\tfrac{1}{n^3\bar{\rho}^2\varrho(1)};
\end{aligned}
$$

$$
\begin{aligned}
(v)\quad & B^*_{n,\bar{\rho}}(t^4;\bar{x}) = \tfrac{\psi^{iv}(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^4 + [(\bar{\rho}(6 + 5\varsigma''(1)) + 6)\varrho(1) + 4\bar{\rho}\varrho'(1)]\tfrac{\bar{x}^3\psi'''(n\bar{x}\varsigma(1))}{n\bar{\rho}\varrho(1)\psi(n\bar{x}\varsigma(1))} \\
& \quad + \Big\{\Big[\bar{\rho}^2\left(7 + 3\varsigma''(1)^2 + 18\varsigma''(1) + 4\varsigma'''(1)\right) + 18\bar{\rho}(1 + \varsigma''(1)) + 11\Big]\varrho(1) \\
& \quad + 6\bar{\rho}[\bar{\rho}(3 + 2\varsigma''(1)) + 3]\varrho'(1) + 6\varrho''(1)\Big\}\tfrac{\bar{x}^2\psi''(n\bar{x}\varsigma(1))}{n^2\bar{\rho}^2\varrho(1)\psi(n\bar{x}\varsigma(1))} \\
& \quad + \Big\{\big[\bar{\rho}^3(1 + 7\varsigma''(1) + 6\varsigma'''(1) + \varsigma^{iv}(1)) \\
& \quad + 6\bar{\rho}^2(1 + 3\varsigma''(1) + \varsigma'''(1)) + 11\bar{\rho}(1 + \varsigma''(1)) + 6\big]\varrho(1) \\
& \quad + \big[2\bar{\rho}^3(7 + 9\varsigma''(1) + 2\varsigma'''(1)) + 18\bar{\rho}^2(2\bar{\rho} + \varsigma''(1)) + 22\bar{\rho}\big]\varrho'(1) \\
& \quad + 6\bar{\rho}^2(\bar{\rho}(3 + \varsigma''(1)) + 3)\varrho''(1) + 4\bar{\rho}^3\varrho'''(1)\Big\}\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n^3\bar{\rho}^3\varrho(1)\psi(n\bar{x}\varsigma(1))} \\
& \quad + \Big\{\big[\bar{\rho}^2(\bar{\rho} + 6) + 11\bar{\rho} + 6\big]\varrho'(1) + \big[\bar{\rho}^2(7\bar{\rho} + 18) + 11\bar{\rho}\big]\varrho''(1) \\
& \quad + 6\bar{\rho}^2(\bar{\rho} + 1)\varrho'''(1) + \bar{\rho}^3\varrho^{iv}(1)\Big\}\tfrac{1}{n^4\bar{\rho}^3\varrho(1)}.
\end{aligned}
$$

Proof. We can obtain (i) easily by the fact that $\sum_{j=0}^{\infty} L_{n,j}(\bar{x})\int_0^{\infty} Q^{\bar{\rho}}_{n,j}(\bar{x})dt = 1$. Next, by using, Lemma (1) and operator (10), we have

$$
\begin{aligned}
(ii)\quad B^*_{n,\bar{\rho}}(t;\bar{x}) &= \sum_{j=1}^{\infty} L_{n,j}(\bar{x})\int_0^{\infty} Q^{\bar{\rho}}_{n,j}(t)tdt + L_{n,0}(\bar{x})f(0), \\
&= \sum_{j=1}^{\infty} L_{n,j}(\bar{x})\tfrac{\Gamma(j\bar{\rho}+1)}{n\bar{\rho}\Gamma(j\bar{\rho})} \\
&= \tfrac{1}{n\varrho(1)\psi(n\bar{x}\varsigma(1))}\sum_{j=0}^{\infty} jP_j(n\bar{x}) \\
&= \tfrac{\psi'(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x} + \tfrac{\varrho'(1)}{n\varrho(1)}.
\end{aligned}
$$

Now, we will compute (iii),

$$
\begin{aligned}
(iii)\quad B^*_{n,\bar{\rho}}(t^2;\bar{x}) &= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \int_0^\infty Q^{\bar{\rho}}_{n,j}(t) t^2 dt + L_{n,0}(\bar{x}) f(0),\\
&= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \tfrac{\Gamma(j\bar{\rho}+2)}{(n\bar{\rho})^2\Gamma(j\bar{\rho})}\\
&= \tfrac{1}{n^2\varrho(1)\psi(n\bar{x}\varsigma(1))}\left(\sum_{j=0}^{\infty} j^2 P_j(n\bar{x}) + \tfrac{1}{\bar{\rho}}\sum_{j=0}^{\infty} jP_j(n\bar{x})\right)\\
&= \tfrac{\psi''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^2 + \left(\tfrac{2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1)}{\varrho(1)} + \tfrac{1}{\bar{\rho}}\right)\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n\psi(n\bar{x}\varsigma(1))}\\
&\quad + \tfrac{(1+\bar{\rho})\varrho'(1)+\bar{\rho}\varrho''(1)}{n^2\bar{\rho}\varrho(1)}.
\end{aligned}
$$

On the one hand, we will compute $B^*_{n,\bar{\rho}}(t^3;\bar{x})$, we have

$$
\begin{aligned}
&(iv)\quad B^*_{n,\bar{\rho}}(t^3;\bar{x})\\
&= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \int_0^\infty Q^{\bar{\rho}}_{n,j}(t) t^3 dt + L_{n,0}(\bar{x}) f(0),\\
&= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \tfrac{\Gamma(j\bar{\rho}+3)}{(n\bar{\rho})^3\Gamma(j\bar{\rho})}\\
&= \tfrac{1}{n^3\varrho(1)\psi(n\bar{x}\varsigma(1))}\left(\sum_{j=0}^{\infty} j^3 P_j(n\bar{x}) + \tfrac{3}{\bar{\rho}}\sum_{j=0}^{\infty} j^2 P_j(n\bar{x}) + \tfrac{2}{\bar{\rho}^2}\sum_{j=0}^{\infty} jP_j(n\bar{x})\right)\\
&= \tfrac{\psi'''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^3 + 3[\varrho(1)+\bar{\rho}(\varrho(1)+\varrho(1)\varsigma''(1)+\varrho'(1))]\tfrac{\bar{x}^2\psi''(n\bar{x}\varsigma(1))}{n\bar{\rho}\varrho(1)\psi(n\bar{x}\varsigma(1))}\\
&\quad + ([\varrho(1)+3\varrho(1)\varsigma''(1)+\varrho(1)\varsigma'''(1)+6\varrho'(1)+3\varrho'(1)\varsigma''(1)+3\varrho''(1)]\bar{\rho}^2\\
&\quad + 3[\varrho(1)+\varrho(1)\varsigma''(1)+2\varrho'(1)]\bar{\rho}+2\varrho(1))\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n^2\bar{\rho}^2\varrho(1)\psi(n\bar{x}\varsigma(1))}\\
&\quad + ([\varrho'(1)+3\varrho''(1)+\varrho'''(1)]\bar{\rho}^2 + 3[\varrho'(1)+\varrho''(1)]\bar{\rho}+2\varrho'(1))\tfrac{1}{n^3\bar{\rho}^2\varrho(1)}.
\end{aligned}
$$

Using Lemma 1 and operator (10), in the similar way, we have

$$
\begin{aligned}
(v) B^*_{n,\bar{\rho}}(t^4;\bar{x}) &= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \int_0^\infty Q^{\bar{\rho}}_{n,j}(t) t^4 dt + L_{n,0}(\bar{x}) f(0)\\
&= \sum_{j=1}^{\infty} L_{n,j}(\bar{x}) \tfrac{\Gamma(j\bar{\rho}+4)}{(n\bar{\rho})^4\Gamma(j\bar{\rho})}\\
&= \tfrac{1}{n^4\varrho(1)\psi(n\bar{x}\varsigma(1))}\left(\sum_{j=0}^{\infty} j^4 P_j(n\bar{x}) + \tfrac{6}{\bar{\rho}}\sum_{j=0}^{\infty} j^3 P_j(n\bar{x})\right.\\
&\quad \left. + \tfrac{11}{\bar{\rho}^2}\sum_{j=0}^{\infty} j^2 P_j(n\bar{x}) + \tfrac{6}{\bar{\rho}^3}\sum_{j=0}^{\infty} jP_j(n\bar{x})\right)\\
&= \tfrac{\psi^{iv}(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^4 + [(\bar{\rho}(6+5\varsigma''(1))+6)\varrho(1)\\
&\quad + 4\bar{\rho}\varrho'(1)]\tfrac{\bar{x}^3\psi'''(n\bar{x}\varsigma(1))}{n\bar{\rho}\varrho(1)\psi(n\bar{x}\varsigma(1))} + \left\{\left[\bar{\rho}^2\left(7+3\varsigma''(1)^2+18\varsigma''(1)\right.\right.\right.\\
&\quad \left.+4\varsigma'''(1)\right)+18\bar{\rho}(1+\varsigma''(1))+11]\varrho(1)+6\bar{\rho}[\bar{\rho}(3+2\varsigma''(1))\\
&\quad +3]\varrho'(1)+6\varrho''(1)\}\tfrac{\bar{x}^2\psi''(n\bar{x}\varsigma(1))}{n^2\bar{\rho}^2\varrho(1)\psi(n\bar{x}\varsigma(1))} + \{[\bar{\rho}^3(1+7\varsigma''(1)+6\varsigma'''(1)\\
&\quad +\varsigma^{iv}(1))+6\bar{\rho}^2(1+3\varsigma''(1)+\varsigma'''(1))+11\bar{\rho}(1+\varsigma''(1))+6]\varrho(1)\\
&\quad +[2\bar{\rho}^3(7+9\varsigma''(1)+2\varsigma'''(1))+18\bar{\rho}^2(2\bar{\rho}+\varsigma''(1))+22\bar{\rho}]\varrho'(1)\\
&\quad +6\bar{\rho}^2(\bar{\rho}(3+\varsigma''(1))+3)\varrho''(1)+4\bar{\rho}^3\varrho'''(1)\}\tfrac{\bar{x}\psi'(n\bar{x}\varsigma(1))}{n^3\bar{\rho}^3\varrho(1)\psi(n\bar{x}\varsigma(1))}\\
&\quad +\{[\bar{\rho}^2(\bar{\rho}+6)+11\bar{\rho}+6]\varrho'(1)+[\bar{\rho}^2(7\bar{\rho}+18)+11\bar{\rho}]\varrho''(1)\\
&\quad +6\bar{\rho}^2(\bar{\rho}+1)\varrho'''(1)+\bar{\rho}^3\varrho^{iv}(1)\}\tfrac{1}{n^4\bar{\rho}^3\varrho(1)}.
\end{aligned}
$$

Lemma 3. *We have*

$$
\begin{aligned}
B^*_{n,\bar{\rho}}((t-\bar{x});\bar{x}) &= \left(\tfrac{\psi'(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}-1\right)\bar{x} + \tfrac{\varrho'(1)}{n\varrho(1)};\\
B^*_{n,\bar{\rho}}\left((t-\bar{x})^2;\bar{x}\right) &= \left(\tfrac{\psi''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))} - \tfrac{2\psi'(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}+1\right)\bar{x}^2\\
&\quad + \left(\tfrac{(2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1))\psi'(n\bar{x}\varsigma(1))}{n\varrho(1)\psi(n\bar{x}\varsigma(1))} + \tfrac{\psi'(n\bar{x}\varsigma(1))}{n\bar{\rho}\psi(n\bar{x}\varsigma(1))}\right.\\
&\quad \left. - \tfrac{2\varrho'(1)}{n\varrho(1)}\right)\bar{x} + \tfrac{\varrho''(1)+\varrho'(1)}{n^2\varrho(1)} + \tfrac{\varrho'(1)}{n^2\bar{\rho}\varrho(1)};
\end{aligned}
$$

$$
\begin{aligned}
B^*_{n,\bar{\rho}}\left((t-\bar{x})^4;\bar{x}\right) = &\left(\psi^{iv}(n\bar{x}\varsigma(1)) - 4\psi'''(n\bar{x}\varsigma(1)) + 6\psi''(n\bar{x}\varsigma(1)) - 4\psi'(n\bar{x}\varsigma(1))\right.\\
&\left. +\psi(n\bar{x}\varsigma(1))\right)\tfrac{\bar{x}^4}{\psi(n\bar{x}\varsigma(1))} + \{[(\bar{\rho}(6+5\varsigma''(1))+6)\varrho(1)\\
&+4\bar{\rho}\varrho'(1)]\psi'''(n\bar{x}\varsigma(1)) - 12[\varrho(1)+\bar{\rho}(\varrho(1)+\varrho(1)\varsigma''(1)\\
&+\varrho'(1))]\psi''(n\bar{x}\varsigma(1)) + 6[(2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1))\bar{\rho}\\
&+\varrho(1))\psi'(n\bar{x}\varsigma(1)) - 4\bar{\rho}\varrho'(1)\psi(n\bar{x}\varsigma(1))\}\tfrac{\bar{x}^3}{n\bar{\rho}\varrho(1)\psi(n\bar{x}\varsigma(1))}\\
&+\left\{\left(\left[\bar{\rho}^2\left(7+3\varsigma''(1)^2+18\varsigma''(1)+4\varsigma'''(1)\right)+18\bar{\rho}(1+\varsigma''(1))\right.\right.\right.\\
&+11]\varrho(1)+6\bar{\rho}[\bar{\rho}(3+2\varsigma''(1))+3]\varrho'(1)+6\varrho''(1))\psi''(n\bar{x}\varsigma(1))\\
&-4([\varrho(1)+3\varrho(1)\varsigma''(1)+\varrho(1)\varsigma'''(1)+6\varrho'(1)+3\varrho'(1)\varsigma''(1)\\
&+3\varrho''(1)]\bar{\rho}^2+3[\varrho(1)+\varrho(1)\varsigma''(1)+2\varrho'(1)]\bar{\rho}+2\varrho(1))\psi'(n\bar{x}\varsigma(1))\\
&+6\bar{\rho}[(\varrho''(1)+\varrho'(1))\bar{\rho}+\varrho'(1)]\psi'(n\bar{x}\varsigma(1))\}\tfrac{\bar{x}^2}{n^2\bar{\rho}^2\varrho(1)\psi(n\bar{x}\varsigma(1))}\\
&+\{([\bar{\rho}^3(1+7\varsigma''(1)+6\varsigma'''(1)+\varsigma^{iv}(1))+6\bar{\rho}^2(1+3\varsigma''(1)\\
&+\varsigma'''(1))+11\bar{\rho}(1+\varsigma''(1))+6]\varrho(1)\\
&+[2\bar{\rho}^3(7+9\varsigma''(1)+2\varsigma'''(1))+18\bar{\rho}^2(2\bar{\rho}+\varsigma''(1))+22\bar{\rho}]\varrho'(1)\\
&+6\bar{\rho}^2(\bar{\rho}(3+\varsigma''(1))+3)\varrho''(1)+4\bar{\rho}^3\varrho'''(1))\psi'(n\bar{x}\varsigma(1))\\
&-4\bar{\rho}([\varrho'(1)+3\varrho''(1)+\varrho'''(1)]\bar{\rho}^2+3[\varrho'(1)+\varrho''(1)]\bar{\rho}\\
&+2\varrho'(1))\psi(n\bar{x}\varsigma(1))\}\tfrac{\bar{x}}{n^3\bar{\rho}^3\varrho(1)\psi(n\bar{x}\varsigma(1))} + \{[\bar{\rho}^2(\bar{\rho}+6)+11\bar{\rho}+6]\varrho'(1)\\
&+[\bar{\rho}^2(7\bar{\rho}+18)+11\bar{\rho}]\varrho''(1)+6\bar{\rho}^2(\bar{\rho}+1)\varrho'''(1)+\bar{\rho}^3\varrho^{iv}(1)\}\tfrac{1}{n^4\bar{\rho}^3\varrho(1)}.
\end{aligned}
$$

Now, we will prove well-known Korovkin type approximation theorem for the introduced operators. Suppose $UC_B[0,\infty)$ is the space of bounded and uniformly continuous functions on $[0,\infty)$.

Theorem 1. *For a given continuous function $f \in UC_B[0,\infty), B^*_{n,\bar{\rho}}$ converges uniformly to f on $[0,A]$.*

Proof. By considering the equality (13) given as in Lemma 2, we deduce that

$$\lim_{n\to\infty} B^*_{n,\bar{\rho}}(t^i;\bar{x}) = \bar{x}^i, \qquad i = 0,1,2. \tag{14}$$

On each subset of $[0,\infty)$ which must be compact, this convergence is satisfied uniformly. Applying Korovkin's theorem (Altomare and Campiti, 1994), we conclude to our desired result.

3. Weighted approximation properties of $B^*_{n,\bar{\rho}}$ operators

It was Gadzhiev who demonstrated weighted Korovkin-type theorems (Gadjiev, 1974). Let $B_{\bar{x}^2}[0,\infty)$ denote the set of all those functions g which satisfy growth condition $|g(\bar{x})| \leqslant M_g(1+\bar{x}^2)$, defined on the positive real axis where M_g is a constant which depends only on g. Let $C_{\bar{x}^2}[0,\infty)$ be the subspace of all those functions which are continuous and also belong to $B_{\bar{x}^2}[0,\infty)$. Also, $C^*_{\bar{x}^2}[0,\infty)$ be the subspace of $C_{\bar{x}^2}[0,\infty)$, for which the limit $\lim_{\bar{x}\to\infty}(g(\bar{x})/1+\bar{x}^2)$ exists, $g \in C_{\bar{x}^2}[0,\infty)$. It is clear that $C^*_{\bar{x}^2}[0,\infty) \subset C_{\bar{x}^2}[0,\infty) \subset B_{\bar{x}^2}[0,\infty)$. $C^*_{\bar{x}^2}[0,\infty)$ is equipped with

$$\|g\|_{\bar{x}^2} = \sup_{\bar{x}\in[0,\infty)} \frac{|g(\bar{x})|}{1+\bar{x}^2}.$$

Lemma 4. *Let $\gamma(\bar{x}) = 1+\bar{x}^2$. For $f \in C_{\bar{x}^2}[0,\infty)$,*

$$\|B^*_{n,\bar{\rho}}(\gamma;\bar{x})\|_{\bar{x}^2} \leqslant M.$$

Proof. By Lemma 2, part (i) and (iii), we will have

$$
\begin{aligned}
B^*_{n,\bar{\rho}}(\gamma;\bar{x}) = 1 &+ \tfrac{\psi''(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}^2 + \tfrac{1}{n}\left(\tfrac{2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1)}{\varrho(1)} + \tfrac{1}{\bar{\rho}}\right)\tfrac{\psi'(n\bar{x}\varsigma(1))}{\psi(n\bar{x}\varsigma(1))}\bar{x}\\
&+ \tfrac{1}{n^2}\tfrac{(1+\bar{\rho})\varrho'(1)+\bar{\rho}\varrho''(1)}{\bar{\rho}\varrho(1)}.
\end{aligned}
$$

Then, we obtain

$$\begin{aligned}
&\|B^*_{n,\bar\rho}(\gamma;\bar x)\|_{\bar x^2}\\
&=\sup_{\bar x\geqslant 0}\left\{\frac{1}{1+\bar x^2}\left(1+\frac{\psi''(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x^2+\frac{1}{n}\left[\frac{2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1)}{\varrho(1)}+\frac{1}{\bar\rho}\right]\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x\right.\right.\\
&\left.\left.+\frac{1}{n^2}\frac{(1+\bar\rho)\varrho'(1)+\bar\rho\varrho''(1)}{\bar\rho\varrho(1)}\right)\right\}\\
&\leqslant 1+\sup_{\bar x\geqslant 0}\frac{\psi''(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}+\frac{1}{n}\left(\frac{2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1)}{\varrho(1)}+\frac{1}{\bar\rho}\right)\sup_{\bar x\geqslant 0}\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\\
&+\frac{1}{n^2}\frac{(1+\bar\rho)\varrho'(1)+\bar\rho\varrho''(1)}{\bar\rho\varrho(1)}.
\end{aligned}$$

Because $\lim_{n\to\infty}\frac{1}{n}=0$ and using condition given in Eq. (13), there is $M>0$ such that

$$\|B^*_{n,\bar\rho}(\gamma;\bar x)\|_{\bar x^2}\leqslant M.$$

It can be seen that $B^*_{n,\bar\rho}$ defined by Eq.(10) acts from $C_{\bar x^2}[0,\infty)$ to $B_{\bar x^2}[0,\infty)$ by using Lemma 4.

Now we will give some theorems based on the weighted approximation.

Theorem 2. *Let $B^*_{n,\bar\rho}$ verifies the condition (13). Then for each $f\in C^*_{\bar x^2}[0,\infty)$,*

$$\lim_{n\to\infty}\|B^*_{n,\bar\rho}(f)-f\|_{\bar x^2}=0.$$

Proof. As in Gadzhiev (1975), it is enough to prove that

$$\lim_{n\to\infty}\|B^*_{n,\bar\rho}(t^r;\bar x)-\bar x^r\|_{\bar x^2}=0,\qquad r=0,1,2. \tag{15}$$

The first condition of Eq. (15) is verified for $r=0$ as $B^*_{n,\bar\rho}(1;\bar x)=1$. Now, from Lemma 2 part (ii), we have

$$\begin{aligned}
\|B^*_{n,\bar\rho}(t;\bar x)-\bar x\|_{\bar x^2}=&\sup_{\bar x\in[0,\infty)}\frac{|B^*_{n,\bar\rho}(t;\bar x)-\bar x|}{1+\bar x^2}\\
&=\left|\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-1\right|\sup_{\bar x\in[0,\infty)}\frac{\bar x}{1+\bar x^2}+\left|\frac{\varrho'(1)}{n\varrho(1)}\right|\sup_{\bar x\in[0,\infty)}\frac{1}{1+\bar x^2}\\
&\leqslant\left|\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-1\right|+\frac{1}{n}\left|\frac{\varrho'(1)}{n\varrho(1)}\right|
\end{aligned}$$

$$\lim_{n\to\infty}\|B^*_{n,\bar\rho}(t;\bar x)-\bar x\|_{\bar x^2}=0,$$

concludes that the condition given in Eq. (15) holds for $r=1$. In the same fashion, from Lemma 2 (iii), we have

$$\begin{aligned}
&\|B^*_{n,\bar\rho}(t^2;\bar x)-\bar x^2\|_{\bar x^2}\\
&=\sup_{\bar x\in[0,\infty)}\frac{|B^*_{n,\bar\rho}(t^2;\bar x)-\bar x^2|}{1+\bar x^2}\\
&=\left|\frac{\psi''(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-\frac{2\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}+1\right|\sup_{\bar x\in[0,\infty)}\frac{\bar x^2}{1+\bar x^2}\\
&+\left|\left(\frac{(2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1))\psi'(n\bar x\varsigma(1))}{n\varrho(1)\psi(n\bar x\varsigma(1))}+\frac{\psi'(n\bar x\varsigma(1))}{n\bar\rho\psi(n\bar x\varsigma(1))}-\frac{2\varrho'(1)}{n\varrho(1)}\right)\right|\sup_{\bar x\in[0,\infty)}\frac{\bar x}{1+\bar x^2}\\
&+\left|\frac{\varrho''(1)+\varrho'(1)}{n^2\varrho(1)}+\frac{\varrho'(1)}{n^2\bar\rho\varrho(1)}\right|\sup_{\bar x\in[0,\infty)}\frac{1}{1+\bar x^2}\\
&\leqslant\left|\frac{\psi''(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-\frac{2\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}+1\right|\\
&+\left|\left(\frac{(2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1))\psi'(n\bar x\varsigma(1))}{n\varrho(1)\psi(n\bar x\varsigma(1))}+\frac{\psi'(n\bar x\varsigma(1))}{n\bar\rho\psi(n\bar x\varsigma(1))}-\frac{2\varrho'(1)}{n\varrho(1)}\right)\right|\\
&+\left|\frac{\varrho''(1)+\varrho'(1)}{n^2\varrho(1)}+\frac{\varrho'(1)}{n^2\bar\rho\varrho(1)}\right|,
\end{aligned}$$

which gives

$$\lim_{n\to\infty}\|B^*_{n,\bar\rho}(t^2;\bar x)-\bar x^2\|_{\bar x^2}=0,$$

So Eq. (15) holds for $r=2$.

For r = 0, 1, 2, we have

$$\lim_{n\to\infty}\|B^*_{n,\bar\rho}(t^r;\bar x)-\bar x^r\|_{\bar x^2}=0.$$

The proof completes here.

Theorem 3. *Let α be a positive constant, $B^*_{n,\bar\rho}$ be the positive linear operators sequence defined by Eq. (10). Then, we get*

$$\lim_{n\to\infty}\sup_{0\leqslant\bar x<\infty}\frac{|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}=0,$$

$f\in C^*_{\bar x^2}[0,\infty)$.

Proof. Let $0\leqslant\bar x_0<\infty$ be arbitrary but fixed. Then

$$\begin{aligned}
&\sup_{0\leqslant\bar x<\infty}\frac{|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}\\
&\leqslant\sup_{\bar x\leqslant\bar x_0}\frac{|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}+\sup_{\bar x>\bar x_0}\frac{|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}\\
&\leqslant\|B^*_{n,\bar\rho}(f)-f\|_{C[0,\bar x_0]}+\|f\|_{\bar x^2}\sup_{\bar x>\bar x_0}\frac{|B^*_{n,\bar\rho}(1+t^2;\bar x)|}{(1+\bar x^2)^{1+\alpha}}+\sup_{\bar x>\bar x_0}\frac{|f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}.
\end{aligned} \tag{16}$$

Since $|f(\bar x)|\leqslant\|f\|_{\bar x^2}(1+\bar x^2)$, we have $\sup_{\bar x>\bar x_0}\frac{|f(\bar x)|}{(1+\bar x^2)^{1+\alpha}}\leqslant\frac{\|f\|_{\bar x^2}}{(1+\bar x_0^2)^{\alpha}}$. For an arbitrary $\varepsilon>0$, we can opt $\bar x_0$ to be remarkably large that

$$\frac{\|f\|_{\bar x^2}}{(1+\bar x_0^2)^{\alpha}}<\frac{\varepsilon}{3}. \tag{17}$$

In the light of Theorem 1, we will get

$$\begin{aligned}
\|f\|_{\bar x^2}\lim_{n\to\infty}\frac{|B^*_{n,\bar\rho}(1+t^2;\bar x)|}{(1+\bar x^2)^{1+\alpha}}=&\frac{1+\bar x^2}{(1+\bar x^2)^{1+\alpha}}\|f\|_{\bar x^2}\\
&=\frac{\|f\|_{\bar x^2}}{(1+\bar x^2)^{\alpha}}\\
&\leqslant\frac{\|f\|_{\bar x^2}}{(1+\bar x_0^2)^{\alpha}}<\frac{\varepsilon}{3}.
\end{aligned} \tag{18}$$

It can be seen the first term of the inequality (16) brings that

$$\|B^*_{n,\bar\rho}(f)-f\|_{C[0,\bar x_0]}<\frac{\varepsilon}{3},\qquad\text{as}\qquad n\to\infty. \tag{19}$$

Combining (17) and (19), we get the desired result.

The modulus of continuity of $h\in UC_B[0,\infty)$ is

$$\omega(h,\delta)=\max_{|b-a|\leqslant\delta}|h(b)-h(a)|,\ \ a,b\in[0,\infty).$$

It is well known that $\lim_{\delta\to0+}\omega(h,\delta)=0$ for any $h\in UC_B[0,\infty)$ and

$$|h(b)-h(a)|\leqslant\left(\frac{|b-a|}{\delta}+1\right)\omega(h,\delta),\qquad\delta>0. \tag{20}$$

Theorem 4. *For $f\in UC_B[0,\infty)$*

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|\leqslant 2\omega\left(f;\left(\sqrt{\delta_n(\bar x)}\right)\right),$$

where $\delta_n(\bar x)$ is as follows:

$$\begin{aligned}
\delta_n(\bar x)=\ &B^*_{n,\bar\rho}\left((t-\bar x)^2;\bar x\right)\\
&=\left(\frac{\psi''(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-\frac{2\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}+1\right)\bar x^2\\
&+\left(\frac{(2\varrho'(1)+\varrho(1)+\varrho(1)\varsigma''(1))\psi'(n\bar x\varsigma(1))}{n\varrho(1)\psi(n\bar x\varsigma(1))}+\frac{\psi'(n\bar x\varsigma(1))}{n\bar\rho\psi(n\bar x\varsigma(1))}-\frac{2\varrho'(1)}{n\varrho(1)}\right)\bar x\\
&+\frac{\varrho''(1)+\varrho'(1)}{n^2\varrho(1)}+\frac{\varrho'(1)}{n^2\bar\rho\varrho(1)}.
\end{aligned}$$

Proof. Applying triangular inequality, we get

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)| = |\sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)(f(t)-f(\bar x))dt| \leqslant \sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)|f(t)-f(\bar x)|dt.$$

Now using inequality (20), Hölder's inequality and Lemma 2, we get

$$\begin{aligned}|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)| &= \omega(f,\delta)\sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)\left(\tfrac{|t-\bar x|}{\delta}+1\right)dt\\ &\leqslant \omega(f,\delta)\sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)dt\\ &+\tfrac{\omega(f,\delta)}{\delta}\sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)|t-\bar x|dt\\ &=\omega(f,\delta)+\tfrac{\omega(f,\delta)}{\delta}\left(\sum_{j=0}^{\infty} L_{n,k}(\bar x)\int_0^\infty Q^{\bar\rho}_{n,k}(t)(t-\bar x)^2dt\right)^{\frac12}\\ &=\omega(f,\delta)+\tfrac{\omega(f,\delta)}{\delta}\left(B^*_{n,\bar\rho}\left((t-\bar x)^2;\bar x\right)\right)^{\frac12}.\end{aligned}$$

Now choosing $\delta=\delta_n(\bar x)$, we have

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)| \leqslant 2\omega\left(f;\left(\sqrt{\delta_n(\bar x)}\right)\right).$$

Hence, the desired result is obtained.

Now, we will denote by $C^2_B[0,\infty)=\{h\in C_B[0,\infty): h',h''\in C_B[0,\infty)\}$. Let

$$K_2(h,\delta)=\inf_{h_1\in C^2_B[0,\infty)}\{\|h-h_1\|+\delta\|h''_1\|\},$$

$$\omega_2\left(h,\sqrt\delta\right)=\sup_{0<\mu\leqslant\sqrt\delta}\ \sup_{\bar x,\bar x+\mu\in[0,\infty)}|h(\bar x+2\mu)-2h(\bar x+\mu)+h(\bar x)|$$

denote the classical Peetre's K-functional and the second modulus of smoothness of $h\in C_B[0,\infty)$, where $\delta>0$ and $h,h_1,h'_1,h''_1\in C^2_B[0,\infty)$. By Theorem 2.4 of Devore and Lorentz (1993),

$$K_2(h,\delta)\leqslant C\omega_2\left(h,\sqrt\delta\right),\quad C>0. \tag{21}$$

Theorem 5. *Suppose $f\in UC_B[0,\infty)$. Then for every non-negative $\bar x$, there exists $C>0$ such that*

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|\leqslant C\omega_2(f,\delta_n(\bar x))+\omega(f,\alpha_n(\bar x)),$$

where

$$\delta_n(\bar x)=\sqrt{B^*_{n,\bar\rho}\left((t-\bar x)^2;\bar x\right)+(\alpha_n(\bar x))^2},\quad \alpha_n(\bar x)=\left(\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}-1\right)\bar x+\frac{\varrho'(1)}{n\varrho(1)}.$$

Proof. For $0\leqslant\bar x<\infty$, we define

$$\widehat{\mathscr B}^*_{n,\bar\rho}(f;\bar x)=B^*_{n,\bar\rho}(f;\bar x)+f(\bar x)-f\left(\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\frac{\varrho'(1)}{n\varrho(1)}\right).$$

From Lemma 2 part (i) & (ii) and Lemma 3 part (i), we have

$$\begin{aligned}\widehat{\mathscr B}^*_{n,\bar\rho}(1;\bar x)&=B^*_{n,\bar\rho}(1;\bar x)+1-1=1\\ \widehat{\mathscr B}^*_{n,\bar\rho}(t;\bar x)&=B^*_{n,\bar\rho}(t;\bar x)+\bar x-\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}\right)=\bar x\\ \widehat{\mathscr B}^*_{n,\bar\rho}((t-\bar x);\bar x)&=\widehat{\mathscr B}^*_{n,\bar\rho}(t;\bar x)-\bar x\widehat{\mathscr B}^*_{n,\bar\rho}(1;\bar x)=0.\end{aligned}$$

Let $0\leqslant\bar x<\infty$and $\sigma\in C^2_B[0,\infty)$. Using Taylor's formula

$$\sigma(t)=\sigma(\bar x)+\sigma'(\bar x)(t-\bar x)+\int_{\bar x}^{t}(t-u)\sigma''(u)\mathrm{d}u.$$

Applying $\widehat{\mathscr B}^*_{n,\bar\rho}$, we get

$$\begin{aligned}\widehat{\mathscr B}^*_{n,\bar\rho}(\sigma;\bar x)-\sigma(\bar x)&=\sigma'(\bar x)\widehat{\mathscr B}^*_{n,\bar\rho}((t-\bar x);\bar x)+\widehat{\mathscr B}^*_{n,\bar\rho}\left(\int_{\bar x}^t(t-u)\sigma''(u)\mathrm{d}u;\bar x\right)\\ &=B^*_{n,\bar\rho}\left(\int_{\bar x}^t(t-u)\sigma''(u)\mathrm{d}u;\bar x\right)\\ &-\int_{\bar x}^{\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\frac{\varrho'(1)}{n\varrho(1)}}\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}-u\right)\sigma''(u)\mathrm{d}u.\end{aligned}$$

Since

$$\begin{aligned}|\int_{\bar x}^t(t-u)\sigma''(u)\mathrm{d}u|&\leqslant\int_{\bar x}^t|t-u||\sigma''(u)|\mathrm{d}u\\ &\leqslant\|\sigma''\||\int_{\bar x}^t|t-u|\mathrm{d}u|\leqslant(t-\bar x)^2\|\sigma''\|\end{aligned}$$

and

$$\begin{aligned}&|\int_{\bar x}^{\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\frac{\varrho'(1)}{n\varrho(1)}}\left(\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\frac{\varrho'(1)}{n\varrho(1)}-u\right)\sigma''(u)\mathrm{d}u|\\ &\leqslant\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}-\bar x\right)^2\|\sigma''\|,\end{aligned}$$

we conclude that

$$\begin{aligned}|\widehat{\mathscr B}^*_{n,\bar\rho}(\sigma;\bar x)-\sigma(\bar x)|&\leqslant \|B^*_{n,\bar\rho}\left(\int_{\bar x}^t(t-u)\sigma''(u)\mathrm{d}u;\bar x\right)\\ &-\int_{\bar x}^{\frac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\frac{\varrho'(1)}{n\varrho(1)}}\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}-u\right)\sigma''(u)\mathrm{d}u\|\\ &\leqslant\|\sigma''\|B^*_{n,\bar\rho}\left((t-\bar x)^2;\bar x\right)+\|\sigma''\|\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}-\bar x\right)^2\\ &=\|\sigma''\|\delta_n^2(\bar x).\end{aligned}$$

From Lemma 2 (i), we have

$$|\widehat{\mathscr B}^*_{n,\bar\rho}(f;\bar x)|\leqslant|B^*_{n,\bar\rho}(f;\bar x)|+2\|f\|\leqslant 3\|f\|,$$

i.e.

$$\begin{aligned}|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|&\leqslant|\widehat{\mathscr B}^*_{n,\bar\rho}(f-\sigma;\bar x)-(f-\sigma)(\bar x)|\\ &+|f\left(\tfrac{\psi'(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\bar x+\tfrac{\varrho'(1)}{n\varrho(1)}\right)-f(\bar x)|+|\widehat{\mathscr B}^*_{n,\bar\rho}(\sigma;\bar x)-\sigma(\bar x)|\\ &\leqslant 4\|f-\sigma\|+\omega(f,\alpha_n(\bar x))+\delta_n^2(\bar x)\|\sigma''\|.\end{aligned}$$

Hence,

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|\leqslant 4K_2(f,\delta_n^2(\bar x))+\omega(f,\alpha_n(\bar x)).$$

So that

$$|B^*_{n,\bar\rho}(f;\bar x)-f(\bar x)|\leqslant C\omega_2(f,\delta_n(\bar x))+\omega(f,\alpha_n(\bar x)).$$

4. Voronovskaja type theorem

In order to study the Voronovskaja type theorem for Jakimovski-Leviatan-Pal̆tănea operators including BB-Polynomials, we consider the following assumptions on the analytic functions ϱ,ψ and ς:

$$\lim_{n\to\infty}n\left[\frac{\psi'(n\bar x\varsigma(1))-\psi(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\right]=\sigma_1(\bar x) \tag{22}$$

$$\lim_{n\to\infty}n\left[\frac{\psi''(n\bar x\varsigma(1))-2\psi'(n\bar x\varsigma(1))+\psi(n\bar x\varsigma(1))}{\psi(n\bar x\varsigma(1))}\right]=\sigma_2(\bar x). \tag{23}$$

Using the assumptions (22), (23) and Lemma 3 the following result can be obtained.

Lemma 5. *For $B^*_{n,\bar\rho}$ operators, we have*

$$\lim_{n\to\infty}nB^*_{n,\bar\rho}((\upsilon-\bar x);\bar x)=\bar x\sigma_1(\bar x)+\frac{\varrho'(\bar x)}{\varrho(1)};$$

$$\lim_{n\to\infty}nB^*_{n,\bar\rho}\left((\upsilon-\bar x)^2;\bar x\right)=\bar x^2\sigma_2(\bar x)+\bar x\left(1+\frac{1}{\bar\rho}+\varsigma''(1)\right).$$

Theorem 6. *Let $f \in C_{\bar{x}^2}[0,\infty)$ such that $f', f'' \in C_{\bar{x}^2}[0,\infty)$. Then*

$$\lim_{n\to\infty} n\Big(B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})\Big) = \Big(\bar{x}\sigma_1(\bar{x}) + \tfrac{\varrho'(\bar{x})}{\varrho(1)}\Big)f'(\bar{x}) + \Big(\bar{x}^2\sigma_2(\bar{x}) + \bar{x}\Big(1 + \tfrac{1}{\bar{\rho}} + \varsigma''(1)\Big)\Big)\tfrac{f''(\bar{x})}{2},$$

uniformly for $\bar{x} \in [0, A]$.

Proof. Suppose $f, f', f'' \in C_{\bar{x}^2}[0,\infty)$ and $0 \leqslant \bar{x} < \infty$ be fixed. We can write by Taylor's formula that

$$f(\upsilon) = f(\bar{x}) + (\upsilon - \bar{x})f'(\bar{x}) + \frac{(\upsilon-\bar{x})^2}{2!}f''(\bar{x}) + r(\upsilon,\bar{x})(\upsilon-\bar{x})^2,$$

where $r(\upsilon,\bar{x})$ denotes the Peano's form of the remainder, $r(\upsilon,\bar{x}) \in C_B[0,\infty)$, and $\lim_{\upsilon\to\bar{x}} r(\upsilon,\bar{x}) = 0$. Applying $B^*_{n,\bar{\rho}}$, we will have

$$n\Big[B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})\Big] = nf'(\bar{x})B^*_{n,\bar{\rho}}(\upsilon-\bar{x};\bar{x}) + \tfrac{nf''(\bar{x})}{2!}B^*_{n,\bar{\rho}}\Big((\upsilon-\bar{x})^2;\bar{x}\Big) + nB^*_{n,\bar{\rho}}\Big(r(\upsilon,\bar{x})(\upsilon-\bar{x})^2;\bar{x}\Big).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} n\Big[B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})\Big] &= f'(\bar{x})\lim_{n\to\infty} nB^*_{n,\bar{\rho}}(\upsilon-\bar{x};\bar{x}) + \tfrac{f''(\bar{x})}{2!}\lim_{n\to\infty} nB^*_{n,\bar{\rho}}\Big((\upsilon-\bar{x})^2;\bar{x}\Big) \\ &\quad + \lim_{n\to\infty} nB^*_{n,\bar{\rho}}\Big(r(\upsilon,\bar{x})(\upsilon-\bar{x})^2;\bar{x}\Big) \\ &= \Big(\bar{x}\sigma_1(\bar{x}) + \tfrac{\varrho'(\bar{x})}{\varrho(1)}\Big)f'(\bar{x}) + \Big(\bar{x}^2\sigma_2(\bar{x}) + \bar{x}\Big(1 + \tfrac{1}{\bar{\rho}} + \varsigma''(1)\Big)\Big)\tfrac{f''(\bar{x})}{2} \\ &\quad + \lim_{n\to\infty} nB^*_{n,\bar{\rho}}\Big(r(\upsilon,\bar{x})(\upsilon-\bar{x})^2;\bar{x}\Big) \\ &= \Big(\bar{x}\sigma_1(\bar{x}) + \tfrac{\varrho'(\bar{x})}{\varrho(1)}\Big)f'(\bar{x}) + \Big(\bar{x}^2\sigma_2(\bar{x}) + \bar{x}\Big(1 + \tfrac{1}{\bar{\rho}} + \varsigma''(1)\Big)\Big)\tfrac{f''(\bar{x})}{2} + E.\end{aligned}$$

With the help of Cauchy–Schwarz inequality, we have

$$|E| \leqslant \lim_{n\to\infty} nB^*_{n,\bar{\rho}}\big(r^2(\upsilon,\bar{x});\bar{x}\big)^{\frac{1}{2}}B^*_{n,\bar{\rho}}\Big((\upsilon-\bar{x})^4;\bar{x}\Big)^{\frac{1}{2}}. \tag{24}$$

Observe that $r^2(\bar{x},\bar{x}) = 0$ and $r^2(\cdot,\bar{x}) \in UC_B[0,\infty)$. Then, from Theorem 1

$$\lim_{n\to\infty} B^*_{n,\bar{\rho}}\big(r^2(\upsilon,\bar{x});\bar{x}\big) = r^2(\bar{x},\bar{x}) = 0, \tag{25}$$

uniformly for $\bar{x} \in [0, A]$. And from Lemma 3, we can see that

$$B^*_{n,\bar{\rho}}\Big((\upsilon-\bar{x})^4;\bar{x}\Big)^{\frac{1}{2}} = O\left(\frac{1}{n^2}\right),$$

which gives

$$\lim_{n\to\infty} n.B^*_{n,\bar{\rho}}\Big((\upsilon-\bar{x})^4;\bar{x}\Big)^{\frac{1}{2}} = 0. \tag{26}$$

Hence, from Eq.(25) and (26), we have $E = 0$. Thus,

$$\lim_{n\to\infty} n\Big[B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})\Big] = \Big(\bar{x}\sigma_1(\bar{x}) + \tfrac{\varrho'(\bar{x})}{\varrho(1)}\Big)f'(\bar{x}) + \Big(\bar{x}^2\sigma_2(\bar{x}) + \bar{x}\Big(1 + \tfrac{1}{\bar{\rho}} + \varsigma''(1)\Big)\Big)\tfrac{f''(\bar{x})}{2},$$

which completes the proof.

5. Rate of convergence

Let $f \in C_B[0,\infty), 0 < \gamma \leqslant 1$, and $M > 0$. We say that a function $f \in Lip_M(\gamma)$ if

$$|f(\upsilon) - f(\bar{x})| \leqslant M|\upsilon - \bar{x}|^{\gamma} \;\; \upsilon, \bar{x} \in [0,\infty)$$

is satisfied.

Theorem 7. *For $f \in Lip_M(\gamma)$, we have*

$$|B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})| \leqslant M(\delta_n(\bar{x}))^{\frac{\gamma}{2}}$$

where

$$\delta_n(\bar{x}) = B^*_{n,\bar{\rho}}\Big((t-\bar{x})^2;\bar{x}\Big).$$

Proof. For $f \in Lip_M(\gamma)$,

$$\begin{aligned}|B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})| &= |\sum_{j=0}^{\infty} L_{n,k}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,k}(t)f(t)(f(t) - f(\bar{x}))|dt \\ &\leqslant \sum_{j=0}^{\infty} L_{n,k}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,k}(t)f(t)|f(t) - f(\bar{x})|dt \\ &\leqslant M\sum_{j=0}^{\infty} L_{n,k}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,k}(t)f(t)|t - \bar{x}|^{\gamma}dt.\end{aligned}$$

By Hölder's inequality with the values $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get following inequality,

$$|B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})| \leqslant M\left(\sum_{j=0}^{\infty} L_{n,k}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,k}(t)f(t)(t-\bar{x})^2dt\right)^{\frac{\gamma}{2}}\left(\sum_{j=0}^{\infty} L_{n,k}(\bar{x}) \int_0^{\infty} Q^{\bar{\rho}}_{n,k}(t)f(t)dt\right)^{\frac{2-\gamma}{2}}.$$

From Lemma 2 we get

$$\begin{aligned}&= M\Big(B^*_{n,\bar{\rho}}\Big((t-\bar{x})^2;\bar{x}\Big)\Big)^{\frac{\gamma}{2}}\Big(B^*_{n,\bar{\rho}}(1;\bar{x})\Big)^{\frac{2-\gamma}{2}} \\ &= M\Big(B^*_{n,\bar{\rho}}\Big((t-\bar{x})^2;\bar{x}\Big)\Big)^{\frac{\gamma}{2}}.\end{aligned}$$

Choosing $\delta : \delta_n(\bar{x}) = B^*_{n,\bar{\rho}}\Big((t-\bar{x})^2;\bar{x}\Big)$, we obtain

$$|B^*_{n,\bar{\rho}}(f;\bar{x}) - f(\bar{x})| \leqslant M(\delta_n(\bar{x}))^{\frac{\gamma}{2}}.$$

6. Conclusions and further remarks

Here, a sequence of Jakimovski-Leviatan-Păltănea operators is constructed involving Boas-Buck-type polynomials (BB-polynomials). We have developed some approximation properties of this operator and investigated versatile Korovkin-type property and also obtained the rate of convergence. Moreover, some approximation results are given in the weighted spaces. Furthermore, a Voronovskaja type theorem is also proved as well as approximation result when functions belong to the Lipschitzian class.

We tried to construct positive linear operators with the help of a function to approximate that function which is difficult to be studied. In this regard, we introduced a very novel operator not studied to date which improves and generalizes an existing operator, like BB-polynomials (Büyükyazıcı et al., 2014) and JLP operators (Sucu et al., 2012) which are already studied. We tried to introduce to a new JLP operator involving BB-polynomials and this operator is a more generalized form of the previous. All the necessary calculations and results are given which will be helpful for those who are going to study different variations and generalizations of JLP operators involving BB-polynomials. There is further scope that these operators can be extended to q and (p,q)-analogues which will be more general in nature and in particular, the q-analogue gives better rate of approximation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The author (Khursheed J. Ansari) extends his appreciation to the "Deanship of Scientific Research at King Khalid University" for funding this work through research groups program under Grant No. R.G.P.1/198/41.

References

Alotaibi, A., Mursaleen, M., 2020. Approximation of Jakimovski-Leviatan-Beta type integral operators via q-calculus. AIMS Math. 5 (4), 3019–3034.
Altomare, F., Campiti, M., 1994. Korovkin-type Approximation Theory and its Applications. Walter de Gruyter, Berlin, New York.
Ansari, K.J., Ahmad, I., Mursaleen, M., Hussain, I., 2018. On some statistical approximation by (p, q))Bleimann, Butzer and Hahn operators, Symmetry 10 (12), Article No. 731..
Ansari, K.J., Mursaleen, M., Rahman, S., 2019. Approximation by Jakimovski-Leviatan operators of Durrmeyer type involving multiple Appell polynomials. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM. 113 (2), 1007–1024.
Ansari, K.J., Rahman, S., Mursaleen, M., 2019. Approximation and error estimation by modified Păltănea operators associating Gould-Hopper polynomials. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM. 113 (3), 2827–2851.
Büyükyazící, İ., Tanberkan, H., Serenbay, S. Kirci, Atakut, Ç., 2014. Approximation by Chlodowsky type Jakimovski-Leviatan operators, J. Comput. Appl. Math. 259, 153–163.
Devore, R.A., Lorentz, G.G., 1993. Constructive Approximation. Springer, Berlin.
Gadjiev, A.D., 1974. The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P.P. Korovkin, Dokl. Akad. Nauk SSSR 218 (5), Transl. in Soviet Math. Dokl. 15 (5), 1433–1436..
Gadzhiev, A.D., 1975–1976. Theorems of the type of P.P. Korovkin's theorems (Russian), presented at the international conference on the theory of approximation of functions (Kaluga, 1975). Mat. Zametki 1976, 20 (5), 781–786..
Ismail, M.E.H., 1974. On a generalization of Szász operators. Mathematica (Cluj) 39, 259–267.
Ismail, M.E.H., 2005. Classical and Quantum Orthogonal Polynomials in One Variable. Cambridge University Press, Cambridge, UK.
Korovkin, P.P., 1953. On convergence of linear positive operators in the space of continuous functions (Russian). Doklady Akad. Nauk. SSSR (NS) 90, 961–964.
Kilicman, A., Mursaleen, M.A., Al-Abied, A.A.H.A., 2020. Stancu type Baskakovâ Durrmeyer operators and approximation properties. Mathematics 8. https://doi.org/10.3390/math8071164. Article No. 1164.
Mazhar, S.M., Totik, V., 1985. Approximation by modified Szá sz operators. Acta Sci. Math. 49, 257–269.
Mohiuddine, S.A., Acar, T., Alotaibi, A., 2017. Construction of a new family of Bernstein-Kantorovich operators. Math. Methods Appl. Sci. 40, 7749–7759.
Mursaleen, M., Al-Abeid, A.A.H., Ansari, K.J., 2019. Approximation by Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM. 113 (2), 1251–1265..
Mursaleen, M., Rahman, S., Ansari, K.J., 2018. Approximation by generalized Stancu type integral operators involving Sheffer polynomials. Carpathian J. Math. 34 (2), 215–228.
Mursaleen, M., Rahman, S., Ansari, K.J., 2019. Approximation by Jakimovski-Leviatan-Stancu-Durrmeyer type operators. Filomat. 33 (6), 1517–1530.
Mursaleen, M., Rahman, S., Ansari, K.J., 2019. On the approximation by Bézier-Pălta ˘nea operators based on Gould-Hopper polynomials. Math. Commun. 24, 147–164.
Păltănea, R., 2008. Modified Szász-Mirakjan operators of integral form. Carpathian J. Math. 24 (3), 378–385.
Phillips, R.S., 1954. An inversion formula for Laplace transforms and semi-groups of linear operators. Ann. Math. 59, 325–356.
Sucu, S., Içöz, G., Varma, S., 2012. On some extensions of Szász operators including Boas-Buck-type polynomials. Abst. Appl, Anal.
Szász, O. Generalization of S. Bernstein's polynomials to the infinite interval, J. Res. Nat. Bur. Stand. 45, 239–245..
Verma, D.K., Gupta, V., 2015. Approximation for Jakimovski-Leviatan-Păltănea operators. Ann. Univ. Ferrara 61, 367–380.